Unfolding schematic formal systems: from non-finitist to feasible arithmetic

Thomas Strahm

Institut für Informatik und angewandte Mathematik, Universität Bern

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Unfolding schematic formal systems (Feferman '96)

Given a schematic formal system S, which operations and predicates, and which principles concerning them, ought to be accepted if one has accepted S ?

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Example (Non-finitist arithmetic NFA)

Logical operations: $\neg$, $\wedge$, $\forall$.

(1) $x' \neq 0$

(2) $\mathrm{Pd}(x') = x$

(3) $P(0) \wedge (\forall x)(P(x) \rightarrow P(x')) \rightarrow (\forall x)P(x)$.

Schematic formal systems

- The informal philosophy behind the use of schemata is their open-endedness
- Implicit in the acceptance of a schemata is the acceptance of any meaningful substitution instance
- Schematas are applicable to any language which one comes to recognize as embodying meaningful notions

Background and previous approaches

General background: Implicitness program (Kreisel '70)

Various means of extending a formal system by principles which are implicit in its axioms.

- Reflection principles, transfinite recursive progressions (Turing '39, Feferman '62)
- Autonomous progressions and predicativity (Feferman, Schütte '64)
- Reflective closure based on self-applicative truth (Feferman '91)

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- Operations are not bound to any specific mathematical domain

The full unfolding $\mathcal{U}(\mathsf{S})$

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- Each relation symbol R of S together with $\mathsf{U_S}$ determines a predicate $R^\star$ of our partial combinatory algebra with $R(x_1, \ldots, x_n)$ if and only if $(x_1, \ldots, x_n) \in R^\star$.

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- Operations on predicates, such as e.g. conjunction, are just special kinds of operations. Each logical operation l of S determines a corresponding operation $l^\star$ on predicates.
- Families or sequences of predicates given by an operation f form a new predicate $Join(f)$, the disjoint union of the predicates from f.

The substitution rule

Substitution rule (Subst)

$$\frac{A[\bar{P}]}{A[\bar{B}/\bar{P}]} \qquad \text{(Subst)}$$

$\bar{P} = P_1, \ldots, P_m$: sequence of free predicate symbols

$\bar{B} = B_1, \ldots, B_m$: sequence of formulas

$A[\bar{B}/\bar{P}]$ denotes the formula $A[\bar{P}]$ with P_i replace by B_i $(1 \leq i \leq n)$

The three unfolding systems

Definition ($\mathcal{U}(\mathsf{S})$, $\mathcal{U}_0(\mathsf{S})$, $\mathcal{U}_1(\mathsf{S})$)

- $\mathcal{U}(\mathsf{S})$: full (predicate) unfolding of S
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Remark: The original formulation of unfolding made use of a background theory of typed operations with general Least Fixed Point operator. The present formulation is a simplification of this approach.

The proof theory of the three unfolding systems for NFA

Theorem (Feferman, Str.)

We have the following proof-theoretic characterizations.

1. $\mathcal{U}_0(\mathsf{NFA})$ *is proof-theoretically equivalent to* PA.
2. $\mathcal{U}_1(\mathsf{NFA})$ *is proof-theoretically equivalent to* $\mathsf{RA}_{<\omega}$.
3. $\mathcal{U}(\mathsf{NFA})$ *is proof-theoretically equivalent to* $\mathsf{RA}_{<\Gamma_0}$.

In each case we have conservation with respect to arithmetic statements of the system on the left over the system on the right.

Finitist arithmetic

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Example (Finitist arithmetic FA)

Logical operations: $\wedge$, $\vee$, $\exists$.

(1) $x' = 0 \rightarrow \bot$

(2) $\mathrm{Pd}(x') = x$

(3) $$\frac{\Gamma \rightarrow P(0) \qquad \Gamma, P(x) \rightarrow P(x')}{\Gamma \rightarrow P(x)}.$$

Note that the statements proved are sequents Σ of the form $\Gamma \rightarrow A$, where Γ is a finite sequence (possibly empty) of formulas. The logic is formulated in Gentzen-style.

Generalization of the substitution rule (Subst)

We have to generalize the substitution rule (Subst) to rules of inference:

Substitution rule (Subst')

Given that the rule of inference

$$\frac{\Sigma_1, \Sigma_2, \dots, \Sigma_n}{\Sigma}$$

is *derivable*, we can adjoin each of its substitution instances

$$\frac{\Sigma_1[\bar{B}/\bar{P}],\ \Sigma_2[\bar{B}/\bar{P}], \dots, \Sigma_n[\bar{B}/\bar{P}]}{\Sigma[\bar{B}/\bar{P}]}$$

as a new rule of inference.

The proof theory of the three unfolding systems for FA

The full unfolding of FA includes the basic logical operations as operations on predicates as well as *Join*.

Theorem (Feferman, Str.)

All three unfolding systems for finitist arithmetic, $\mathcal{U}_0(\mathsf{FA})$, $\mathcal{U}_1(\mathsf{FA})$ and $\mathcal{U}(\mathsf{FA})$ are proof-theoretically equivalent to Skolem's Primitive Recursive Arithmetic PRA.

Support of Tait's informal analysis of finitism.

Extended finitism and the bar rule

In the following

- We will study a natural bar rule BR leading to extensions $\mathcal{U}_0(\mathsf{FA}+\mathsf{BR})$, $\mathcal{U}_1(\mathsf{FA}+\mathsf{BR})$ and $\mathcal{U}(\mathsf{FA}+\mathsf{BR})$ of our unfolding systems for finitism

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- The so-obtained extensions will all have the strength of Peano arithmetic PA
- This shows one way how Kreisel's analysis of extended finitism fits in our framework

Defining $\mathcal{U}_0(\mathsf{FA} + \mathsf{BR})$: Formulating the bar rule

- The rule $\mathsf{NDS}(\mathsf{f}, \prec)$ says that for each possibly infinite descending chain f w.r.t. $\prec$ there is an x such that $\mathsf{f}x = 0$, where f denotes a new constant of our applicative language.

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- We must modify $\mathsf{TI}(\prec, P)$, since its standard formulation for a unary predicate P is of the form:

$$(\forall x)[(\forall u \prec x)P(u) \rightarrow P(x)] \rightarrow (\forall x)P(x).$$

The idea is to treat this as a rule of the form:

$$\text{from} \quad (\forall u)[u \prec x \rightarrow P(u)] \rightarrow P(x) \quad \text{infer} \quad P(x).$$

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- But we still need an additional step to reformulate the hypothesis of this rule in the language of FA, the basic idea being to use a skolemized form of the universal quantifier.

The key observation

Theorem

Assume that $\mathsf{NDS}(\mathsf{f}, \prec)$ *is derivable in* $\mathcal{U}_0(\mathsf{FA} + \mathsf{BR})$. *Then* $\mathcal{U}_0(\mathsf{FA} + \mathsf{BR})$ *justifies nested recursion along* $\prec$.

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- Use one direction of Tait's famous result, i.e. that nested recursion on $\omega\alpha$ entails ordinary recursion on ω^α or, more useful in our setting, nested recursion on $\omega\alpha$ entails $\mathsf{NDS}(\mathsf{f}, \omega^\alpha)$

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- Tait's argument can be directly formalized in $\mathcal{U}_0(\mathsf{FA} + \mathsf{BR})$

The proof theory of the three unfolding systems for FA with bar rule

Theorem (Feferman, Str.)

All three unfolding systems for finitist arithmetic with bar rule, $\mathcal{U}_0(\mathsf{FA} + \mathsf{BR})$, $\mathcal{U}_1(\mathsf{FA} + \mathsf{BR})$ and $\mathcal{U}(\mathsf{FA} + \mathsf{BR})$ are proof-theoretically equivalent to Peano arithmetic PA.

Support of Kreisel's analysis of extended finitism.

The language of feasible arithmetic

- The basic schematic system FEA of feasible arithmetic is based on a language for binary words generated from the empty word by the two binary successors S_0 and S_1; in addition, it includes some natural basic operations on the binary words like, for example, word concatenation and multiplication

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- The logical operations of FEA are conjunction ($\wedge$), disjunction ($\vee$), and the bounded existential quantifier ($\exists^{\leq}$)
- FEA is formulated as a system of sequents in this language: apart from the defining axioms for basic operations on words, its heart is a schematically formulated, i.e. open-ended induction rule along the binary words, using a free predicate letter P.

The basic schematic system FEA

Example (Feasible arithmetic FEA)

Logical operations: $\wedge$, $\vee$, $\exists^{\leq}$.

(1) defining equations for the function symbols of the language of FEA

(2) $$\frac{\Gamma \rightarrow P(\epsilon) \qquad \Gamma, P(\alpha) \rightarrow P(\mathsf{S}_i(\alpha)) \quad (i = 0, 1)}{\Gamma \rightarrow P(\alpha)}$$

The strength of the unfoldings of FEA

Theorem (Eberhard, Str.)

The provably total functions of $\mathcal{U}_0(\mathsf{FEA})$ *and* $\mathcal{U}(\mathsf{FEA})$ *are exactly the polynomial time computable functions.*

Remarks on the upper bound computation

- A suitable upper bound for $\mathcal{U}$(FEA) is obtained via the weak truth theory T_{PT} introduced by Eberhard and Strahm
- The involved proof-theoretic analysis of T_{PT} using a novel realizability interpretation is due to Eberhard
- To be precise, we consider a slight (conservative) extension of T_{PT} which facilitates the treatment of the generalized substitution rule

Formulating the full unfolding with a truth predicate

The axioms of $\mathcal{U}_{\mathsf{T}}(\mathsf{FEA})$ extend those of $\mathcal{U}_0(\mathsf{FEA})$ by the following axioms about the truth predicate T:

Truth unfolding

$$
\begin{array}{rcl}
\mathsf{T}(x \doteq y) & \leftrightarrow & x = y \\
\mathsf{T}(x \mathbin{\dot{\wedge}} y) & \leftrightarrow & \mathsf{T}(x) \wedge \mathsf{T}(y) \\
\mathsf{T}(x \mathbin{\dot{\vee}} y) & \leftrightarrow & \mathsf{T}(x) \vee \mathsf{T}(y) \\
\mathsf{T}(\dot{\exists}\alpha x) & \leftrightarrow & (\exists \beta \leq \alpha)\mathsf{T}(x\beta) \\
\mathsf{T}(\pi_i(\bar{x})) & \leftrightarrow & P_i(\bar{x})
\end{array}
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The end

Thank you very much for your attention.

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