Determinacy of Infinite Games and Reverse Mathematics

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abstract

This thesis consists of two parts. The first part treats determinacy in (classical) reverse mathematics and the second part treats determinacy in intuitionistic mathematics.

The first part investigates the logical strength of the determinacy of infinite games in the Cantor space in terms of second order arithmetic. We define new determinacy schemata inspired by the Wadge classes of Polish spaces and show that the following equivalences hold over the system RCA_0^*, which consists of the axioms of discrete ordered semi-ring with exponentiation, Δ^0_1 comprehension and Σ^0_0 induction, and which is known as a weaker system than the popular base theory RCA_0:

- Δ^0_1-$\mathsf{Det}^* \leftrightarrow \Sigma^0_1$-$\mathsf{Det}^* \leftrightarrow \mathsf{WKL}_0^*$;
- $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)$-$\mathsf{Det}^* \leftrightarrow \mathsf{WKL}_0$;
- $(\Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}_0^* \leftrightarrow \mathsf{ACA}_0$;
- Δ^0_2-$\mathsf{Det}^* \leftrightarrow \Sigma^0_2$-$\mathsf{Det}^* \leftrightarrow \mathsf{ATR}_0$;
- $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_2)$-$\mathsf{Det}^* \leftrightarrow \mathsf{ATR}_0 + \Sigma^1_1$ induction;
- $\mathsf{Bisep}(\Sigma^0_1, \Sigma^0_2)$-$\mathsf{Det}^* \leftrightarrow \mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)$-$\mathsf{Det}^* \leftrightarrow \Pi^1_1$-$\mathsf{CA}_0$;
- $\mathsf{Bisep}(\Delta^0_2, \Sigma^0_2)$-$\mathsf{Det}^* \leftrightarrow \Pi^1_1$-$\mathsf{TR}_0$;

where Det^* stands for the determinacy of infinite games in the Cantor space.

We also work on complete determinacy, which asserts that, for a given game G, we have the set W with the following properties:

- $s \in W \rightarrow$ player I wins $\varphi(f)$ if the game starts at s,
- $s \notin W \rightarrow$ player II wins $\varphi(f)$ if the game starts at s.

We investigates the logical strength of the complete determinacy as well. We show the following equivalence over RCA_0^*:

- $\mathsf{WKL}_0^* \leftrightarrow \Delta^0_1$-$\mathsf{comp.Det}^*$;
- $\mathsf{ACA}_0 \leftrightarrow \Sigma^0_1$-$\mathsf{comp.Det}^* \leftrightarrow (\Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{comp.Det}^*$;
- $\mathsf{ATR}_0 \leftrightarrow \Delta^0_2$-$\mathsf{comp.Det}^* \leftrightarrow \Delta^0_1$-$\mathsf{comp.Det}$;

- Π^1_1-$\mathsf{CA}_0 \leftrightarrow \Sigma^0_2$-$\mathsf{comp.Det}^* \leftrightarrow \mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)$-$\mathsf{Det}^* \leftrightarrow \Sigma^0_1$-$\mathsf{Det} \leftrightarrow (\Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}$;
- Π^1_1-$\mathsf{CA}_0 \leftrightarrow \mathsf{Bisep}(\Delta^0_2, \Sigma^0_2)$-$\mathsf{comp.Det}^* \leftrightarrow \Delta^0_2$-$\mathsf{Det}$;

where comp.Det stands for complete determinacy in the Baire space and where comp.Det* stands for complete determinacy in the Cantor space. In particular, we see that determinacy does not always imply complete determinacy.

The second part investigates determinacy in Brouwerian intuitionistic mathematics. We give some examples of games such that the character of this mathematical setting—the lack of the law of excluded middle and the adoption of continuity principle—makes the behavior of determinacy drastically different from that in the classical setting.

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Contents

Chapter 1

General introduction

The aim of the author's research is to clarify the essential difference of the following mathematical settings:

Classical mathematics The usual mathematics based on classical logic. We can prove statements by way of contradiction.

Bishop's constructive mathematics The mathematics based on intuitionistic logic. An important character of this logic is that, the statement "A" means "there is a proof of A" and so neither the law of excluded middle (LEM) nor the proof by way of contradiction is available. Classical mathematics is equivalent to this mathematics plus LEM.

Intuitionistic mathematics Mathematics based on intuitionistic logic with some special axioms, called "intuitionistic axioms" here, which are based on the philosophy of the pioneer, L. E. J. Brouwer, and which are not included in Bishop's constructive mathematics. The most famous theorem in this mathematics is "every function on real numbers field is continuous."

Hilbert's finitistic mathematics Mathematics based on classical logic, but only "finitistic" reasoning about finitistically-defined mathematical objects (e. g., natural numbers) is allowed. Especially, the use of mathematical induction is restricted to finitistic assertions.

Predicative mathematics Mathematics based on classical logic, but impredicative definitions such as "the least element of a set" is not allowed in general.

The relationship among these five settings is as follows. Hilbert's finitistic mathematics accepts only those axioms that both classical and predicative mathematics accept. Bishop's constructive mathematics accepts only those axioms that both classical and intuitionistic mathematics accept. Nevertheless, Hilbert's finitistic mathematics and Bishop's constructive mathematics are incomparable. The former accepts LEM which is not accepted by the latter, the latter accepts the general induction scheme which is not accepted by the former.

The difference among these settings appears most characteristically in the treatment of continuum, and so the difference appearing in theorems on continuum is of the author's main interest.

For technical purposes, the continuum can be identified with the set of infinite sequences of natural numbers, called reals. The author's medium-term topic is, via this identification, the investigation, on different mathematical settings, of the following descriptive-set-theoretic properties of sets of reals:

Lebesgue measurability Lebesgue measure is a generalized notion of "volume."

Baire property A set S has the Baire property if there exists an open set A such that $(S - A) \cup (A - S)$ is a meager set, i. e., a countable union of nowhere dense sets.

Perfect set property A set S has the perfect set property if it is either countable or it contains a perfect set, i. e., a closed set without isolated points.

Determinacy For a given set S of infinite sequences of natural numbers, consider the following game: Two players, called players I and II, alternately choose natural numbers to form an infinite sequence f. If f is in S, then player I wins; otherwise player II wins. A set S is determinate if one of the players has a winning strategy in this game.

While there have been many works on each of the settings, and on other settings (including intermediate settings between the above settings), it seems reasonable to look for a framework on which researches of these different settings can be practiced uniformly. For this purpose, a promising way is to consider the following type of questions: To prove a given mathematical theorem, what axioms are necessary and sufficient? Once we have the answer, we can easily determine whether the theorem hold on any given setting.

This kind of research is called reverse mathematics and actually two reverse mathematics programs are on-going. One is classical reverse mathematics, which is based on classical logic. Its main interest is what set existence axiom (usually formalized in the language of second order arithmetic) is necessary and sufficient to prove each mathematical statement over Hilbert's finitistic mathematics (cf. [30]). Although there are some controversies on what exactly predicative mathematics is, it seems to be respected that this mathematics has strong connection to the system called ATR_0, one of the most important systems in classical reverse mathematics. Therefore classical mathematics and predicative mathematics, as well as Hilbert's finitistic mathematics, are in the scope of classical reverse mathematics. The other program is constructive reverse mathematics, which is based on intuitionistic logic. Its main interest is what non-constructive axioms, e. g., fragments of LEM or intuitionistic axioms, is necessary and sufficient to prove each mathematical statement over Bishop's constructive mathematics (cf. [12]). Classical mathematics and intuitionistic mathematics, as well as Bishop's constructive mathematics, are in its scope.

In classical reverse mathematics, on one hand, there have been many results for theorems from descriptive set theory. It was shown that determinacy statements for open sets and G_δ sets (countable intersections of open sets) characterize certain set existence axioms (cf. [30] and [33]). These axioms are only relatively strong in the sense of classical reverse mathematics (technically, stronger than or equal to ATR_0). Though a sufficient set existence axioms to prove the determinacy of $G_{\delta\sigma}$ sets (countable unions of G_δ sets) has already been known, the necessary set existence axiom has not yet been found. It is already known that the determinacy of $G_{\delta\sigma\delta}$ sets (countable intersections of $F_{\sigma\delta}$ sets) is not implied by any natural set existence axiom formalized in the language of second order arithmetic.

In constructive reverse mathematics, on the other hand, the situation is different. There are not so many researches on descriptive set theory in intuitionistic mathematics or in Bishop's constructive mathematics. In the former, although there exist some results on Baire property and perfect set property, it is only recently that determinacy was formalized and investigated (cf. [34]). In Bishop's constructive mathematics, there are some results on the definability of Lebesgue measure (cf. [3]) and on Baire's categorization of sets (cf. [4]). The reverse mathematical aspects of all of these remain to be clarified.

As the first step of the author's research, she worked on the determinacy of infinite games. In classical mathematics, although the axiom of choice implies the existence of a non-determinate set, all of Lebesgue measurability, Baire property and perfect set property are implied by determinacy. Therefore she considered that determinacy is a good beginning point for her research.

During the first half of her Ph.D study, the author worked on classical reverse mathematics. In [20] and [24], she showed that almost all of important set existence axioms (including those weaker than ATR_0) are characterized by determinacy statements of 0-1 games, in which both players can choose only 0 or 1 at each turn. In other words, she showed that the strength of determinacy statements actually varies quite widely, depending on kinds of games. During the second half, she worked on intuitionistic mathematics and constructive reverse mathematics, staying in the Netherlands, one of the research centers of the subject. In [22], she considered some variations of games to look into the role of intuitionistic axioms and non-constructive axioms and showed that a certain kind of non-constructive axiom is crucial for determinacy statements. She also showed, in [23], the equivalence between a weaker version of König's lemma and a property of uniformly continuous functions in constructive reverse mathematics.

Clearly studies in the two reverse mathematics will help her to achieve the research aim, and she aspires to continue the researches in this direction.

The author's long-term agenda is to consider the above two reverse mathematics uniformly. Since classical logic proves many non-constructive axioms, classical reverse mathematics does not pay attention to non-constructive axioms used in proofs. On the other hand, nor does constructive reverse mathematics pay attention to set existence axioms and induction axioms. For the author's aim, it is needed to compare the five settings mentioned at the beginning over the same system—the same base theory with the same logic, i. e., to combine these two reverse mathematics into a new big reverse mathematics.

This thesis is an accumulation of the results of authors works on determinacy.

The first part considers determinacy of 0-1 games in classical reverse mathematics. Chapter 2 is the introduction to Part 1. Chapter 3 is a preliminary for the first part. The basic knowledges for classical reverse mathematics, determinacy in second order arithmetic will be provided in this chapter. Chapter 4 considers the relationship between determinacy of 0-1 games and set existence axioms in second order arithmetic. Chapter 5 considers a uniform version of determinacy—complete determinacy. Chapter 6 contains brief comments on related works to Part 1.

The second part considers determinacy in intuitionistic mathematics. Chapter 7 is the introduction to Part 2. Chapter 8 is a preliminary for the second part. The basic knowledge for intuitionistic mathematics and the notion of *predeterminacy*, a formalization of determinacy in intuitionistic mathematics, will be provided in this chapter. Following the introduction for Part 2, Chapter 9 considers various games in intuitionistic mathematics. Chapter 10 reformalizes and investigates predeterminacy in classical mathematics.

The last chapter closes this thesis, giving several interesting problems waiting to be solved.

Part I

Determinacy in classical reverse mathematics

Chapter 2

Introduction to Part 1

In this chapter, we consider games in classical reverse mathematics.

Classical reverse mathematics is a program based on the following fundamental question: To prove mathematical theorems, what set existence axioms are necessary and sufficient? In 1970s, such research based on classical logic started in the context of the language of second order arithmetic, that formalizes the natural numbers and sets thereof.

In this research, equivalences between mathematical theorems and set existence axioms are investigated. The usual base theory RCA_0 consists of the axioms of discrete ordered semi-ring for $(\mathbb{N}, +, \cdot, 0, 1, <)$ plus the schemata of recursive comprehension and induction for formulae of the form $\exists n\varphi(n)$, where $\varphi(x)$ does not contain any unbounded number quantifier or set quantifier. This system is, in a sense, a formalized version of Hilbert's finitistic mathematics, in which only "finitistic" reasoning about finitistically-defined mathematical objects is allowed.

Actually, it has been shown that most of theorems in non-abstract mathematics, such as classical analysis or countable algebra, formalized in the language of second order arithmetic are proved in RCA_0 or equivalent to some set existence axioms in second order arithmetic over RCA_0.

Classical descriptive set theory, which investigates the properties of definable sets in Polish space, is also in the scope of classical reverse mathematics. Until now, necessary and sufficient set existence axioms for Souslin's theorem, Lusin's separation theorem, Kondo's uniformization theorem, etc. are known.

Determinacy of infinite games is one important topic in descriptive set theory in classical reverse mathematics. The games we consider is as follows:

Two players, say player I and player II, alternately choose an element of X to form an infinite sequence f. Player I wins if and only if a given formula $\varphi(f)$ holds. Player II wins if and only if player I does not win. The formula $\varphi(f)$ is called a winning condition for player I. A determinacy statement asserts that one of the players has a winning strategy in such games. Within the framework of second order arithmetic, the strength of determinacy of games in the Baire space $\mathbb{N}^{\mathbb{N}}$, i.e., the above games with $X = \mathbb{N}$, has been previously investigated in [17], [18], [30], [31], [32] and [33].

After the introduction of basic notions of determinacy and classical reverse mathematics in Chapter 3, Chapter 4 investigates the determinacy strength of games in the Cantor space $2^{\mathbb{N}}$, i.e., the games with $X = 2 = \{0, 1\}$ by investigating necessary and sufficient set comprehension axioms, combining two papers [20] and [24]. Games in the Cantor space have the following properties:

- **Games are played in a compact space.** The Cantor space is a compact space, while the Baire space is not. This means that we can use compactness to analyze subsets of the Cantor space.
- **Games in the Baire space can be translated to those in the Cantor space.** In proofs of several theorems in this part, we use the translation of games in the Baire space into ones in the Cantor space, i.e., for a given game G in the Baire space, we can define a game G' in the Cantor space such that a player who has a winning strategy in G' has a winning strategy also in G. In particular, G is determinate if and only if G' is determinate. Together with the above character, we can analyze games in the Baire space via ones in the Cantor space with compactness.
- **Determinacy in the Cantor space are used to analyze properties of ω-language.** In automata theory, determinacy of games are used to analyze properties of accepted language (cf. [26]). In almost all cases, the games are ones in the Cantor space.

The results of previous works on the determinacy of games in the Baire space and of [20] are summarized in Table 2.1, where a subsystem of second order arithmetic and types of determinacy in the same line are pairwise equivalent over a suitable base theory (RCA_0; except for the last line), and where $(\Sigma^0_n)_k$

Subsystem of S. O. A.	Determinacy in $2^{\mathbb{N}}$	Determinacy in $\mathbb{N}^{\mathbb{N}}$
WKL_0	Δ^0_1 Σ^0_1	
ACA_0	$(\Sigma^0_1)_2$	
ATR_0	Δ^0_2 Σ^0_2	Δ^0_1 Σ^0_1
Π^1_1-CA_0		$(\Sigma^0_1)_2$
Π^1_1-TR_0		Δ^0_2
Σ^1_1-ID_0	$(\Sigma^0_2)_2$	Σ^0_2
$\vdots$	$\vdots$	$\vdots$
$[\Sigma^1_1]^k$-ID_0	$(\Sigma^0_2)_{k+1}$	$(\Sigma^0_2)_k$
$\vdots$	$\vdots$	$\vdots$
$[\Sigma^1_1]^{\mathrm{TR}}$-$\mathsf{ID}_0$	Δ^0_3	Δ^0_3

Table 2.1: Results of earlier researches and [20]

is the class of formulae built from Σ^0_n formulas by applying the difference operator $k-1$ times.

In Table 2.1, we can find a large gap between Σ^0_2 and $(\Sigma^0_2)_2$ determinacy in the Cantor space. Then following two questions arose: (1) Is there a class whose determinacy in the Cantor space is equivalent to either Π^1_1-CA_0 or Π^1_1-TR_0 ? and (2) Is there a finer hierarchy of determinacy in the Cantor space?

To find such a hierarchy, we consider the determinacy schemata which formalize, in the language of second order arithmetic, the determinacies whose winning conditions are in the Borel Wadge classes in Polish spaces and also investigate the strengths of the schemata. The Borel Wadge classes, which are defined as those classes of the Borel sets that are closed under continuous pre-images, are known to form a finer hierarchy than the Hausdorff difference hierarchy (cf. [15] and [36]). Since continuous pre-images preserve Boolean operations, this hierarchy seems to be the finest among those hierarchies to which we can define corresponding determinacy schemata at least in an obvious manner.

If we continue to work over RCA_0, we will find that weak König's lemma and $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)$, Σ^0_1 and Δ^0_1 determinacies in the Cantor space are pairwise equivalent over RCA_0, where $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)$ determinacy formalizes the determinacy of the Wadge class immediately above the class $\boldsymbol{\Sigma}^0_1$ of open sets (see Figure 3.2 and [15]). However, the proof of $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)$ determinacy essentially needs Σ^0_1 induction, while that of Σ^0_1 and Δ^0_1 determinacies does not (in [20]). To shed light on the differences among the strengths of the determinacies of those classes, we replace the base theory with RCA_0^*, which lacks Σ^0_1 induction. Although almost all equivalences in Table 2.1 hold over RCA_0, it is routine to check that the base theory can be replaced with RCA_0^*.

As a result, Table 2.1 is enhanced into Table 2.2. Table 2.2 indicates that a subsystem of second order arithmetic and types of determinacy in the same line are pairwise equivalent over RCA_0^* (except for the last line).

Subsystem S. O. A.	Determinacy in $2^{\mathbb{N}}$	Determinacy in $\mathbb{N}^{\mathbb{N}}$
WKL_0^*	Δ^0_1 Σ^0_1	-
WKL_0	$\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)$	-
ACA_0	$(\Sigma^0_1)_2$	-
ATR_0	Δ^0_2 Σ^0_2	Δ^0_1 Σ^0_1
$\mathsf{ATR}_0 + \Sigma^1_1$ induction	$\mathsf{Bisep}(\Delta^0_1, \Sigma^0_2)$	?
$\Pi^1_1\text{-}\mathsf{CA}_0$	$\mathsf{Bisep}(\Sigma^0_1, \Sigma^0_2)$ $\vdots$ $\mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)$	$\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)$ $(\Sigma^0_1)_2$
$\Pi^1_1\text{-}\mathsf{TR}_0$	$\mathsf{Bisep}(\Delta^0_2, \Sigma^0_2)$	Δ^0_2
$\Sigma^1_1\text{-}\mathsf{ID}_0$	$(\Sigma^0_2)_2$	Σ^0_2
$\vdots$	$\vdots$	$\vdots$
$[\Sigma^1_1]^k\text{-}\mathsf{ID}_0$	$(\Sigma^0_2)_{k+1}$	$(\Sigma^0_2)_k$
$\vdots$	$\vdots$	$\vdots$
$[\Sigma^1_1]^{\mathrm{TR}}\text{-}\mathsf{ID}_0$	Δ^0_3	Δ^0_3

Table 2.2: Results of the present study

The proof of each equivalence is as follows. When we prove Γ determinacy in some system Sys, we reduce a Γ game $\varphi(f)$ to an "easier" one $\varphi^*(f)$ such that the determinacy of $\varphi^*(f)$ can be proved in Sys. The key point in this direction is how to reduce the game within restricted comprehension axioms. Conversely, we prove that Γ determinacy implies the set comprehension axiom that characterizes Sys as follows. For any formula $\psi(k)$ in a given class for which comprehension axiom characterizes the system Sys, we construct the following game whose determinacy is implied by Γ determinacy:

- Player I chooses k and asks whether $\psi(k)$.
- Player II answers yes or no.
- Player II wins if and only if her answer is correct.

In such a game, player I cannot win if player II answers correctly. By Γ determinacy, player II has a winning strategy, which provides the desired set $\{n : \varphi(n)\}$.

Chapter 5 treats a variant of determinacy.

For a game in $X^{\mathbb{N}}$, *positions* are finite sequences from X. In contrast to the usual set theoretic context, in the context of weak arithmetic, determinacy of a game $\varphi(f)$ does not seem to yield the *winning set for player I* in $\varphi(f)$, i. e., the set W of positions such that

- $s \in W \rightarrow$ player I wins $\varphi(f)$ if the game starts at s,
- $s \notin W \rightarrow$ player II wins $\varphi(f)$ if the game starts at s.

W can be regarded as a labeling on all positions by "win" and "lose." In other words, determinacy statement does not seem to require determinateness to be decidable. If the essence of determinacy is that we can decide who is the winner uniformly, the appropriate formulation of determinacy in weak arithmetic might be the existence of the winning sets, contrasting the usual determinacy.

The topic of Chapter 5 is *complete determinacy*, which asserts the existence of the winning set.

Actually, for applications in the context of weak arithmetic, complete determinacy is more useful. For example, the proof in [32] of the implication of Σ^1_1 choice from ATR_0, and the investigation in [19] of the comparison of systems between ACA_0 and Π^1_1-CA_0 used complete determinacy. Furthermore, complete determinacy is a strong tool to analyze winning strategies in a game. Consider a game of the form: $\exists n(f[n] \in A) \vee \eta(f)$, where $f[n] = \langle f(0), f(1), ..., f(n-1)\rangle$. Then, player I wins if and only if he reach a position $s \in A \cap \{t : \text{player I wins } \eta(f) \text{ at } t\}$.

We will measure the strength of complete determinacy statements by investigating necessary and sufficient set comprehension axioms and will addresses the question: Do the two formulations, i. e., the usual determinacy and complete determinacy differ? In other words, is complete determinacy really stronger than determinacy?

Together with results of [18], [20], [24], [32] and [33] on determinacy strength in the same setting, we will have such Table 2.3, where the statements are equivalent to the system on the same row over RCA_0. Our contribution is the establishment of the new two columns for complete determinacy

Subsystem of S. O. A.	Determinacy in $2^{\mathbb{N}}$	Complete det. in $2^{\mathbb{N}}$	Determinacy in $\mathbb{N}^{\mathbb{N}}$	Complete det. $\mathbb{N}^{\mathbb{N}}$
WKL_0^*	Δ^0_1 Σ^0_1	Δ^0_1		
WKL_0	$\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)$			
ACA_0	$(\Sigma^0_1)_2$	Σ^0_1 $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)$ $(\Sigma^0_1)_2$		
ATR_0	Δ^0_2 Σ^0_2	Δ^0_2	Δ^0_1 Σ^0_1	Δ^0_1
$\mathsf{ATR}_0 + \Sigma^1_1\text{-}\mathsf{Ind}.$	$\mathsf{Bisep}(\Delta^0_1, \Sigma^0_2)$			
$\Pi^1_1\text{-}\mathsf{CA}_0$	$\mathsf{Bisep}(\Sigma^0_1, \Sigma^0_2)$ $\mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)$	Σ^0_2 $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_2)$ $\mathsf{Bisep}(\Sigma^0_1, \Sigma^0_2)$ $\mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)$	$\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)$ $(\Sigma^0_1)_2$	Σ^0_1 $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)$ $(\Sigma^0_1)_2$
$\Pi^1_1\text{-}\mathsf{TR}_0$	$\mathsf{Bisep}(\Delta^0_2, \Sigma^0_2)$	$\mathsf{Bisep}(\Delta^0_2, \Sigma^0_2)$	Δ^0_2	Δ^0_2

Table 2.3: Determinacy and complete determinacy

in $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ and this shows that complete determinacy statement is sometimes strictly stronger than the corresponding determinacy statement: we see that Σ^0_1-$\mathsf{comp.Det}^*$, Σ^0_2-$\mathsf{comp.Det}^*$ and Σ^0_1-$\mathsf{comp.Det}$ are strictly stronger than Σ^0_1-Det^*, Σ^0_2-Det^* and Σ^0_1-Det, respectively.

In Chapter 6, we overview related works on the results in Part1. In the first section, we see results on stronger determinacy schemata. In the second section, we see the method of analyzing winning strategies, labeling by elements of a well ordering, which was suggested in [2] on the setting of set theory. We will see a close relationship between complete determinacy and this method.

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Chapter 3

Preliminaries for Part 1

In this chapter, we recall some basic definitions and facts about second order arithmetic and about determinacy. We also define various determinacy schemata.

3.1 Basic definitions and notations in second order arithmetic

The language L_1 of first order arithmetic consists of $+, \cdot, 0, 1, =, <$, number variables $x, y, ...$, propositional connectives and number quantifiers. The language L_2 of second order arithmetic consists of L_1 plus set variables $X, Y, ...$, set quantifiers and $\in$. Terms and formulae are defined in the usual way. A formula is Π^0_0, Σ^0_0 or Δ^0_0 if it is built up from atomic formulae by propositional connectives and bounded number quantifiers $\forall x < t$ and $\exists x < t$. A Σ^0_n (resp. Π^0_n) formula is one consisting of n number quantifiers beginning with an existential (resp. universal) one followed by a Π^0_0 formula. A formula is Σ^1_0, Π^1_0, or *arithmetical* if it does not contain set quantifiers. A Σ^1_n (resp. Π^1_n) formula is one consisting of n set quantifier beginning with an existential (resp. universal) one followed by a Π^1_0 formula.

In what follows, Γ is a class of formulae, e. g., Σ^i_n and Π^i_n.

Definition 3.1.1. We use the following schemata.

Γ comprehension: $\exists Y \forall x(x \in Y \leftrightarrow \varphi(x))$, where $\varphi(x)$ is a Γ formula in which Y does not occur freely.

Bounded Γ comprehension: $\forall n \exists Y \forall i (i \in Y \leftrightarrow (i < n \wedge \varphi(i)))$, where $\varphi(x)$ is a Γ formula in which Y does not occur freely.

Γ induction: $(\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(n+1))) \to \forall n \varphi(n)$, where $\varphi(x)$ is a Γ formula.

Γ transfinite induction: $\mathrm{WO}(Z, <_Z) \wedge \forall j(\forall i(i <_Z j \to \varphi(i)) \to \varphi(j)) \to \forall j \varphi(j)$, where $\varphi(x)$ is a Γ formula and where $\mathrm{WO}(Z, <_Z)$ is the Π^1_1 formula which asserts that $(Z, <_Z)$ is well-ordered, i. e., $(Z, <_Z)$ is a linear ordering without infinite $<_Z$-descending chains.

Definition 3.1.2 (RCA_0)**.** RCA_0 is the formal system in the language L_2 which consists of the axioms of discrete ordered semi-ring for $(\mathbb{N}, +, \cdot, 0, 1, <)$ plus the schemata of Σ^0_1 induction and of Δ^0_1 comprehension:

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \to \exists Y \forall n(\varphi(n) \leftrightarrow n \in Y),$$

where $\varphi(x)$ is a Σ^0_1 formula without free occurrences of Y, and where $\psi(x)$ is a Π^0_1 formula.

Over RCA_0, Γ comprehension implies not only Γ induction but also Γ transfinite induction.

Lemma 3.1.3. *Over* RCA_0, Γ *comprehension implies* Γ *transfinite induction.*

Proof. Let $(Y, \prec_Y)$ be a well-ordering and $\varphi(x)$ a Γ formula. Assume that $\forall j(\forall i(i \prec_Y j \to \varphi(i)) \to \varphi(j))$ and $\exists k \neg\varphi(k)$ hold. By Γ comprehension and Δ^0_1 comprehension, we have the set $Z = \{k : \neg\varphi(k)\}$. Since Z is non-empty, we can take $k \in Z$. Note that if $l \in Z$, then $m \prec_Y l$ for some $m \in Z$ by the assumption. Define $f : \mathbb{N} \to Z$ by (unbounded) primitive recursion as follows:

$$f(0) = k \quad \text{and} \quad f(n+1) = \text{the } <\text{-minimum } l \in Z \text{ such that } l \prec_Y f(n).$$

Then $f(n+1) \prec_Y f(n)$ holds for all n, which contradicts the well-orderedness of $(Y, \prec_Y)$. □

As mentioned in the previous chapter, we need a weaker base theory RCA^*_0.

Definition 3.1.4 (RCA_0^*)**.** Let $\mathrm{L}_i(\exp)$ $(i = 1, 2)$ be L_i augmented by a binary operation symbol $\exp(m, n)$, which is intended to denote exponentiation. For each $n < \omega$, Σ^i_n and Π^i_n formulae of $\mathrm{L}_2(\exp)$ are defined accordingly. RCA_0^* is the $\mathrm{L}_2(\exp)$-theory consisting of the axioms of discrete ordered semi-ring for $(\mathbb{N}, +, \cdot, 0, 1, <)$ plus exponentiation axioms, $m^0 = 1$, $m^{n+1} = m^n \cdot m$, where $t_1^{t_2}$ is an abbreviation for $\exp(t_1, t_2)$, and the schemata of Δ^0_1 comprehension and of Π^0_0 induction.

Note that the L_2-theory RCA_0 can be regarded as the $\mathrm{L}_2(\exp)$-theory RCA_0^* plus Σ^0_1 induction, since exp can be defined in RCA_0.

Lemma 3.1.5. *Let $1 \leq n < \omega$ and $i < 2$. Π^i_n comprehension is equivalent to Σ^i_n comprehension over RCA_0^*.*

Proof. Assume Σ^i_n comprehension. Let $\varphi(x)$ be a Π^i_n formula. Then we can find Σ^i_n formula $\psi(x)$ such that $\varphi(k) \leftrightarrow \neg\psi(k)$ for all $k \in \mathbb{N}$. By Σ^i_n comprehension, we have sets $X = \{k : \psi(k)\}$ and $Y = \{k : k \notin X\}$. It is easy to check that $k \in Y$ if and only if $\varphi(k)$ for all $k \in \mathbb{N}$. □

Lemma 3.1.6. *Over RCA_0^*, Γ comprehension implies Γ induction.*

Proof. Let $\varphi(x)$ be a Γ formula. Assume $\varphi(0)$ and $\forall n(\varphi(n) \to \varphi(n+1))$. By Γ comprehension and Δ^0_1 comprehension, we have the set $Z = \{k : \varphi(k)\}$. By the assumption, $0 \in Z$ and $\forall n(n \in Z \to n + 1 \in Z)$ hold, and so, by Σ^0_0 induction, $\forall n(n \in Z)$, which means $\forall n\varphi(n)$. □

Notation 3.1.7 (sequences)**.** The set of all infinite sequences from X (i. e., functions from $\mathbb{N}$ to X) is denoted by $X^{\mathbb{N}}$. We call $2^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$ the *Cantor space* and $\mathbb{N}^{\mathbb{N}}$ the *Baire space*. The set of all finite sequences from X is denoted by $X^{<\mathbb{N}}$. Note that the *empty sequence*, denoted by $\langle\rangle$, belongs to $X^{<\mathbb{N}}$.

Fix any sequences s and t in $X^{\mathbb{N}}$. $|s|$ denotes the *length* of s and $s(i)$ denotes the $(i + 1)$-th element of s for $i < |s|$. The *concatenation* of s and t, denoted by $s * t$, is $\langle s(0), s(1), ..., s(|s|-1), t(0), t(1), ..., t(|t|-1)\rangle$. If $f \in X^{\mathbb{N}}$, $s * f$ denotes $\langle s(0), s(1), ..., s(|s| - 1), f(0), f(1), ..., f(n), ...\rangle$. For $s \in X^{<\mathbb{N}}$ and $n \leq |s|$, $s[n]$ is the *n-th initial segment* of s, i. e., $\langle s(0), ..., s(n - 1)\rangle$. If f is an infinite sequence, $f[n]$ denotes $\langle f(0), ..., f(n - 1)\rangle$. If $s = t[k]$ for some $k \leq |t|$, s is called an *initial segment* of t, $s \subseteq t$ for notation. If $n \leq |s|$, $s \ominus n$ is the sequence with the first n elements removed from s, i. e., $\langle s(n), s(n + 1), ..., s(|s| - 1)\rangle$. If f is in $X^{\mathbb{N}}$, $f \ominus n$ is g defined by $g(k) =$

$f(n+k)$ for $k \in \mathbb{N}$. We also use the following abbreviations: $X^n = \{s \in X^{\mathbb{N}} : |s| = n\}$, $X^{\leq n} = \bigcup_{m \leq n} X^m$, $X^{<n} = \bigcup_{m<n} X^m$ and X^{even}(or X^{odd}) $= \bigcup_{n \in \mathbb{N}} X^{2n}$(resp. $\bigcup_{n \in \mathbb{N}} X^{2n+1}$). For $s \in X^{<\mathbb{N}}$, $(s)_X$ denotes the set $\{t \in X^{<\mathbb{N}} : s \subseteq t\}$.

Note that, since a sequence s of $X^{<\mathbb{N}}$ can be coded by a natural number in RCA_0^*, the relations $s \in X^{<\mathbb{N}}$, $s = \langle\rangle, |s| = n, s(i) = n, s \subseteq t, s = t * u, s = t \ominus n, s = t[n]$, etc., are formally defined as Π^0_0 formulae in RCA_0^*. See [30, Sections II.1-2] for more details. (Although they work in RCA_0 in [30, Sections II.1-2], the base theory can be replaced by RCA_0^*.)

A set $T \subseteq 2^{<\mathbb{N}}$ is *tree* if, for any $s \in 2^{<\mathbb{N}}$, $s \in T$ implies $t \in T$ for any $t \subseteq s$. T is *finite* if the lengths of sequences in T are bounded by some n. T is *infinite* if T is not finite.

By coding of finite sequences by natural numbers, we can define transfinite recursion and axioms of choice.

Definition 3.1.8.

Γ transfinite recursion:
$\mathrm{WO}(Y, \prec_Y) \rightarrow \exists Z \forall j \forall k (\langle k, j\rangle \in Z \leftrightarrow \theta(k, (Z)^j))$, where $\theta(x, Y)$ is a Γ formula in which Z does not occur freely, and where $(Z)^j = \{\langle m, i\rangle \in Z : i \prec_Y j\}$.

Γ axiom of choice:
$\forall n \exists Y \varphi(n, Y) \rightarrow \exists Z \forall n \varphi(n, (Z)_n)$, where $\varphi(x, Y)$ is a Γ formula φ in which Z does not occur freely, and where $(Z)_n = \{m : \langle m, n\rangle \in Z\}$.

3.2 Subsystems of second order arithmetic

Now we introduce and overview the supersystems of RCA_0^* that are widely used in the researches of second order arithmetic. For the details, see [30].

Definition 3.2.1 (WKL_0^* and WKL_0)**.** *Weak König's lemma* asserts that every infinite tree $T \subseteq 2^{<\mathbb{N}}$ has an infinite path, i. e., $f : \mathbb{N} \rightarrow \{0, 1\}$ such that $f[n] \in T$ for all n.

WKL_0^* is the system RCA_0^* plus weak König's lemma. WKL_0 is the system RCA_0 plus weak König's lemma.

In the next section, we will see the relationship among RCA_0^*, RCA_0, WKL_0^* and WKL_0.

Definition 3.2.2 (ACA_0)**.** ACA_0 is the system RCA_0^* plus Σ_1^0 comprehension.

In this system we can use number quantifiers freely. Actually, ACA_0 proves arithmetical comprehension. By Lemma 3.1.6, we can say that ACA_0 is the system RCA_0^* plus Σ_1^0 induction.

Definition 3.2.3 (ATR_0)**.** ATR_0 is the system RCA_0^* plus arithmetical transfinite recursion.

Definition 3.2.4 (Π_1^1-CA_0)**.** Π_1^1-CA_0 is the system RCA_0^* plus Π_1^1 comprehension.

Definition 3.2.5 (Π_1^1-TR_0)**.** Π_1^1-TR_0 is the system RCA_0^* plus Π_1^1 transfinite recursion.

It is known that the following inclusions hold. For detail, see [30, sections VII.2, VIII.2, VIII.6, IX.1-3 and X.4].

$$\mathsf{RCA}_0^* \subsetneq \mathsf{RCA}_0 \subsetneq \mathsf{WKL}_0, \quad \mathsf{RCA}_0^* \subsetneq \mathsf{WKL}_0^* \subsetneq \mathsf{WKL}_0$$

$$\mathsf{WKL}_0 \subsetneq \mathsf{ACA}_0 \subsetneq \mathsf{ATR}_0 \subsetneq \Pi_1^1\text{-}\mathsf{CA}_0 \subsetneq \Pi_1^1\text{-}\mathsf{TR}_0$$

3.3 The systems RCA_0^* and WKL_0^*

Here we see how RCA_0^* differs from RCA_0. We also investigate the relation between WKL_0^* and WKL_0, systems without Σ_1^0 induction.

First we overview the provable definability of functions in RCA_0^*. By Δ_1^0 comprehension, RCA_0^* proves the existence of the constant function $g_0(n) = 0$, the *successor function* $g_{\mathrm{suc}}(n) = n + 1$, the *addition* $g_+(n, m) = n + m$, the *multiplication* $g_\cdot(n, m) = n \cdot m$, the *exponentiation* $g_{\exp}(n, m) = n^m$ and the *projection functions* $p_i^n(\langle k_1, ..., k_i, ..., k_n\rangle) = k_i$. The universe of provably definable functions of RCA_0^* is closed under the operations of Lemma 3.3.1 below. Therefore every *elementary function* can be defined in RCA_0^* (cf. [29, Section 4]).

Lemma 3.3.1 ([29, 2.1. Lemma and 2.2. Lemma])**.**
Assume $k \in \omega$. Let $h(x, \vec{n}) = h(x, n_1, ...n_k)$ and $g(\vec{n}) = g(n_1, ..., n_k)$. The following are provable in RCA_0^*.

Composition *If* $g_0 : Y \to Z$ *and* $g_1 : Z \to W$*, there is* $h : Y \to W$ *defined by* $h(i) = g_1(g_0(i))$.

Bounded primitive recursion *If* $g_0 : \mathbb{N}^k \to \mathbb{N}$*,* $g_1 : \mathbb{N}^{k+2} \to \mathbb{N}$ *and* $g_2 : \mathbb{N}^{k+1} \to \mathbb{N}$*, there exists the unique* $h : \mathbb{N}^{k+1} \to \mathbb{N}$ *defined by*

$$\begin{aligned} h(0, \vec{n}) &= g_0(\vec{n}), \\ h(m+1, \vec{n}) &= \min\{g_1(m, h(m, \vec{n}), \vec{n}), g_2(m, \vec{n})\}. \end{aligned}$$

Minimization *If* $g : \mathbb{N}^{k+1} \to \mathbb{N}$ *and if all* $\langle n_1, ..., n_k \rangle \in \mathbb{N}^k$ *have* $m \in \mathbb{N}$ *with* $g(m, \vec{n}) = 1$*, there is* $h : \mathbb{N}^k \to \mathbb{N}$ *defined by*

$$h(\vec{n}) = \text{the least } m \text{ with } g(m, \vec{n}) = 1.$$

The following lemma shows that the first order part of RCA_0^* contains Σ_1^0 collection.

Lemma 3.3.2 ($\boldsymbol{\Sigma_1^0}$ collection, [29, 4.1. Lemma])**.** RCA_0^* *proves* Σ_1^0 *collection, i. e.,*

$$\forall n \exists m \varphi(n, m) \to \forall n \exists m \forall k < n \exists l < m \varphi(k, l), \quad \text{for any } \Sigma_1^0 \text{ formula } \varphi(x, y).$$

Second, we consider systems with weak König's lemma.

Clearly WKL_0 includes RCA_0 and WKL_0^* includes RCA_0^*. Actually these inclusions are strict (cf. [30, VIII.2, IX.1-3 and X.4]).

Informally, we have the following equivalences. (See [30, Section X.4])

$$\mathsf{RCA}_0 \equiv \mathsf{RCA}_0^* + \Sigma_1^0 \text{ induction}, \quad \text{and} \quad \mathsf{WKL}_0 \equiv \mathsf{WKL}_0^* + \Sigma_1^0 \text{ induction}.$$

Let us consider the first order part and the consistency strength of these systems.

$\mathrm{I}\Sigma_1$ is the L_1 theory consisting of the axioms of discrete ordered semi-ring for $(\mathbb{N}, +, \cdot, 0, 1, <)$ and of Σ_1^0 induction. $\mathrm{I}\Delta_0(\exp)$ is the $\mathrm{L}_1(\exp)$ theory consisting of the axioms of discrete ordered semi-ring with exponentiation for $(\mathbb{N}, +, \cdot, \exp, 0, 1, <)$ and of Δ_0^0 induction. $\mathrm{B}\Sigma_1(\exp)$ is the system $\mathrm{I}\Delta_0(\exp)$ plus Σ_1 collection. By the same proof as in [27] we can see that $\mathrm{I}\Delta_0(\exp)$ is strictly included in $\mathrm{B}\Sigma_1(\exp)$ although the original proof in [27] is for the language L_1.

Fact 3.3.3 (conservation theorems, [30, Sections IX.1-IX.3] and [29])**.** *The first order part of* RCA_0 *and of* WKL_0 *is* $\mathrm{I}\Sigma_1$*.* WKL_0 *is conservative over* RCA_0 *for* Π^1_1 *sentences, i. e., for any* Π^1_1 *sentence* φ*,* WKL_0 *proves* φ *if and only if* RCA_0 *proves* φ*.* WKL_0*,* RCA_0 *and* $\mathrm{I}\Sigma_1$ *have the same consistency strength as* PRA *(See [30, Section IX. 3] for its definition and basic properties.) and are conservative over* PRA *for* Π^0_2 *sentences.*

Analogously, the first order part of RCA^*_0 *and of* WKL^*_0 *is* $\mathrm{B}\Sigma_1(\exp)$*.* WKL^*_0 *is conservative over* RCA^*_0 *for* Π^1_1 *sentences.* WKL^*_0*,* RCA^*_0 *and* $\mathrm{B}\Sigma_1(\exp)$ *have the same consistency strength as* $\mathrm{I}\Delta_0(\exp)$ *and are conservative over* $\mathrm{I}\Delta_0(\exp)$ *for* Π^0_2 *sentences.*

It is shown in [30, Theorem II.8.11] that RCA_0 proves the consistency of $\mathrm{I}\Delta_0(\exp)$, and so does $\mathrm{I}\Sigma_1$. On the other hand, $\mathrm{I}\Delta_0(\exp)$ cannot prove the consistency of itself and so neither can $\mathrm{B}\Sigma_1(\exp)$, since the consistency statement is Π^0_1. Therefore $\mathrm{I}\Sigma_1$ strictly includes $\mathrm{B}\Sigma_1(\exp)$, and the implications from RCA_0 to RCA^*_0 and from WKL_0 to WKL^*_0 are also strict. Moreover the consistency strength of WKL^*_0, RCA^*_0, $\mathrm{B}\Sigma_1$ and $\mathrm{I}\Delta_0(\exp)$ is strictly weaker than that of WKL_0, RCA_0, $\mathrm{I}\Sigma_1$ and PRA.

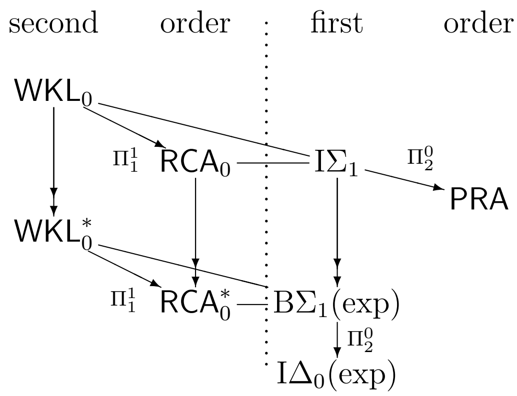


Figure 3.1: Relations among weak systems of arithmetic

The relations among these weak systems are described in Figure 3.1, where usual (one-head) arrows indicate strict implication with conservation for indicated classes of sentences, where two-head arrows indicate strict implication with different consistency strength and where lines without heads indicate full conservation between systems in $\mathrm{L}_1(\exp)$ and their counterparts in $\mathrm{L}_2(\exp)$ (e. g., for any $\mathrm{L}_1(\exp)$ sentence φ, WKL^*_0 proves φ if and only if $\mathrm{B}\Sigma_1(\exp)$ proves φ).

3.4 Determinacy in second order arithmetic

For a given formula $\varphi(f)$ with a distinguished function variable $f \in X^{\mathbb{N}}$, a *game* $\varphi(f)$ *in* $X^{\mathbb{N}}$ is defined as follows: Two players, say player I and player II, alternately choose an element x in X to form $f \in X^{\mathbb{N}}$ which is called the resulting play. Player I wins if and only if $\varphi(f)$ holds. Player II wins if and only if player I does not win. In this thesis, we assume that player I is male and that player II is female.

We regard a class Γ of formulae with a distinguished function variable as a class of games.

Remark 3.4.1. L_2 does not formally have function variables. Since the assertion "G is the graph of a function $f : \mathbb{N} \to \mathbb{N}$" is Π^0_2, this point matters in a system without arithmetical comprehension. However, we consider only the case where $X = \{0, 1\}$ in such systems, and so we can regard f as a set variable by identifying $f(n) = 0$ and $f(n) = 1$ with $n \notin f$ and $n \in f$ respectively. Therefore we do not care about this difference.

Notation 3.4.2 (strategy)**.** For a game in $X^{\mathbb{N}}$, a *strategy* σ for player I (resp. II) is a function assigning an element of X to each finite sequence from X of even (resp. odd)-length. $\mathcal{S}^X_{\mathrm{I}}$ (resp. $\mathcal{S}^X_{\mathrm{II}}$) is the set of all the strategies for player I (resp. II) in a game in $X^{\mathbb{N}}$. Note that $\mathcal{S}^X_{\mathrm{I}}$ and $\mathcal{S}^X_{\mathrm{II}}$ can be regarded as $\mathbb{N}^{\mathbb{N}}$ if $X = \mathbb{N}$ and $2^{\mathbb{N}}$ if $X = \{0, 1\}$ in RCA^*_0, by a suitable coding of finite sequences. If players I and II follow strategies σ and τ respectively, the resulting play is uniquely determined and denoted by $\sigma \otimes \tau$.

For any strategy τ for player II, k^τ is the finite play of length $2k$ in which player I plays 0 at all of his turns and in which player II plays following τ. For example, 2^τ is the sequence $\langle 0, \tau(\langle 0\rangle), 0, \tau(\langle 0, \tau(\langle 0\rangle), 0\rangle)\rangle$.

A strategy σ for a player is a *winning strategy* if the player wins $\varphi(f)$ as long as he or she plays following it. The assertion that σ is a winning strategy for player I (resp. II) in game $\varphi(f)$ in $X^{\mathbb{N}}$ can be written $\forall\tau \in \mathcal{S}^X_{\mathrm{II}}\varphi(\sigma \otimes \tau)$ (resp. $\forall\tau \in \mathcal{S}^X_{\mathrm{I}}\neg\varphi(\tau \otimes \sigma)$). A game $\varphi(f)$ is *determinate* if one of the players has a winning strategy. For a game $\varphi(f)$ in $X^{\mathbb{N}}$, we use the following abbreviation:

$$\mathsf{Det}^X[\varphi] \;\equiv\; \exists\sigma \in \mathcal{S}^X_{\mathrm{I}}\forall\tau \in \mathcal{S}^X_{\mathrm{II}}\varphi(\sigma \otimes \tau) \vee \exists\tau \in \mathcal{S}^X_{\mathrm{II}}\forall\sigma \in \mathcal{S}^X_{\mathrm{I}}\neg\varphi(\sigma \otimes \tau),$$

which asserts that $\varphi(f)$ is determinate. The following schema of Γ *determinacy* asserts that all the Γ games are determinate.

Γ **determinacy in $X^{\mathbb{N}}$:** $\mathsf{Det}^X[\varphi]$ for any Γ game $\varphi(f)$ in $X^{\mathbb{N}}$.

The following Δ^i_n *determinacy* asserts the determinacies of games both Σ^i_n and Π^i_n, in a sense.

Δ^i_n **determinacy in $X^{\mathbb{N}}$:** $\forall f \in X^{\mathbb{N}}(\varphi(f) \leftrightarrow \psi(f)) \to \mathsf{Det}^X[\varphi]$,
where $\varphi(f)$ is a Σ^i_n game and where $\psi(f)$ is a Π^i_n game in $X^{\mathbb{N}}$.

Det^* and Det abbreviate determinacy in the Cantor space and that in the Baire space, respectively.

An *s-strategy for player I* (resp. II) is a function $\sigma : (s)_X \cap \{t \in X^{\mathbb{N}} : |t| \text{ is even (resp. odd)}\} \to X$. For s-strategies σ for player I and τ for player II, $\sigma \otimes \tau$ denotes the sequence f such that $f(i) = s(i)$ for all $i < |s|$, $f(2i) = \sigma(f[2i])$ for all $2i \geq |s|$, $f(2i+1) = \tau(f[2i+1])$ for all $2i+1 \geq |s|$, in other words, the play, starting from s, in which player I follows σ and player II follows τ. Note that if $s = \langle\rangle$, the definition of $\sigma \otimes \tau$ coincides with the previous definition. For a game $\varphi(f)$ in $X^{\mathbb{N}}$, a s-strategy σ for player I (resp. II) is winning if, for every s-strategy τ for player II (resp. I), $\varphi(\sigma \otimes \tau)$ (resp. $\neg\varphi(\tau \otimes \sigma)$). *Player I (resp. II) wins at s in $\varphi(f)$* if (1) there is a winning s-strategy for player I (resp. II), or equivalently, (2) either (i) $|s|$ is even and player I (resp. II) has a winning strategy in $\varphi(s * f)$ or (ii) $|s|$ is odd and player II (resp. I) has a winning strategy in $\neg\varphi(s*f)$. Note that in (ii) of (2), the role of two players are exchanged. It is easy to check that, for any game $\varphi(f)$ in $X^{\mathbb{N}}$, if $|s|$ is even and player I wins $\varphi(f)$ at s then he wins at $s * \langle i \rangle$ for some $i \in X$ and if $|s|$ is odd and player I wins $\varphi(f)$ at s then he wins $\varphi(f)$ at $s * \langle i \rangle$ for all $i \in X$. If σ is a winning s-strategy for player I (or II) in a game $\varphi(f)$ and $t(2n) = \sigma(t[2n])$ for all $2n \geq |s|$ (resp. $t(2n) = \sigma(t[2n])$ for all $2n \geq |s|$), then player I (resp. II) wins $\varphi(f)$ also at t.

3.5 Wadge classes in descriptive set theory

Here we review Wadge classes of Polish spaces, which gives us intuitive ideas for determinacy schemata defined in the next section. Note that, in this section, we work in the usual framework of (descriptive) set theory, not in that of second order arithmetic.

In this thesis, we consider Polish space $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ with the product topology of discrete topological spaces.

The Borel sets are sets in the smallest class that includes all open sets and that is closed under complements and countable unions. Formally, it consists of all classes $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ defined along countable ordinals as follows:

- $\mathbf{\Sigma}^0_1$ is the class of open sets.
- $\mathbf{\Pi}^0_\alpha$ is the class of complements of $\mathbf{\Sigma}^0_\alpha$ sets.
- $\mathbf{\Sigma}^0_\alpha$ $(\alpha > 1)$ is the class of countable unions of sets in $\bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta$.

It is known that the Borel sets form a strict hierarchy, i. e., $\alpha < \beta$ implies $\Sigma^0_\alpha \subsetneq \Sigma^0_\beta$.

In $2^{\mathbb{N}}$ (or $\mathbb{N}^{\mathbb{N}}$), a set A is open if there exists $A' \subseteq 2^{<\mathbb{N}}$ (or $A' \subseteq \mathbb{N}^{\mathbb{N}}$) such that $f \in A \leftrightarrow \exists n(f[n] \in A')$. Note that this corresponds to our class Σ^0_1 of formulae, because, by Lemma 3.6.1, we may assume that Σ^0_1 formulae are of the form $\exists n\theta(f[n])$. We can also see such correspondence between a formula classes Σ^0_n and a set class $\mathbf{\Sigma}^0_n$ by replacing existential number quantifier $\exists n$ with countable union $\bigcup$. Since Σ^0_n formulae may contain set parameters, we can consider that classes Σ^0_n of formulae (of second order arithmetic) formalizes $\mathbf{\Sigma}^0_n$ sets (in $2^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$). Actually, in the standard model, a set is in $\mathbf{\Sigma}^0_n$ if and only if it is definable by a Σ^0_n formula with parameters from Cantor or Baire space.

It is known that the most common system of set theory ZFC proves that every Borel set is determinate.

For a pair (A, B) of subsets of a Polish space X, how can we define which set is "simpler"? Wadge reducibility gives an answer: We say that A is *Wadge reducible* to B, in symbols $A \leq_W B$, if there is a continuous map $f : X \to X$ such that $f^{-1}(B) = A$. We write $A \equiv_W B$ if and only if $A \leq_W B$ and $B \leq_W A$. Clearly $\equiv_W$ is an equivalence relation, and associated equivalence classes $\mathbf{A} = [A]_W$ are called *Wadge degrees*. For a set $A \subseteq X^{\mathbb{N}}$, the *coarse degree* $\mathbf{A}^*$ of A is $[A]_W \cup [A^c]_W$, where A^c is the complement of A. WADGE denotes the set of the coarse degrees of the Borel sets, and $\leq^*_W$ is defined by $\mathbf{A} \leq^*_W \mathbf{B} \leftrightarrow A \leq_W B \vee A \leq_W B^c$. In ZFC, $(\mathrm{WADGE}, \leq^*_W)$ is well-ordered (cf. [13, (21.15) Theorem]).

$(\mathrm{WADGE}, \leq^*_W)$ gives a finer hierarchy than *Hausdorff and Kuratowski's difference hierarchy* from [14, §37. III. Theorem], the finite level of which is defined as follows:

- $(\mathbf{\Sigma}^0_n)_1 = \mathbf{\Sigma}^0_n$

- $(\mathbf{\Sigma}^0_n)_{m+1} = \{A \setminus B : A \in \mathbf{\Sigma}^0_n \text{ and } B \in (\mathbf{\Sigma}^0_n)_m\}$

A concrete description of the Wadge hierarchy of the Borel sets is provided in [15]. The classes in Wadge hierarchy are composed from the classes of Hausdorff and Kuratowski's difference hierarchy by means of taking (countable) unions and complements. Especially, below $(\mathbf{\Sigma}^0_2)_2$, the construction principle is easier because we do not need countable unions.

Figure 3.2 is a rough sketch of Wadge hierarchy below $(\mathbf{\Sigma}^0_2)_2$, where only one of mutually dual classes is described, where $\mathbf{Sep}(\mathbf{\Delta}^0_2, \mathbf{\Sigma}^0_2)$ is the union of all Wadge classes below $(\mathbf{\Sigma}^0_2)_2$ and where $\mathbf{\Delta}^0_2$ is the union of all the Wadge classes below $\mathbf{\Sigma}^0_2$. The constructions mentioned here are as follows:

- $\mathbf{\Delta}(\mathbf{\Gamma}) = \{A : A, A^c \in \mathbf{\Gamma}\}$
- $\mathbf{Sep}(\mathbf{\Gamma}, \mathbf{\Gamma}') = \{(A \cap B^c) \cup (A^c \cap C) : A \in \mathbf{\Gamma} \text{ and } B, C \in \mathbf{\Gamma}'\}$
- $\mathbf{Bisep}(\mathbf{\Gamma}, \mathbf{\Gamma}')$
  $= \{(A \cap B^c) \cup (C \cap D) : A, C \in \mathbf{\Gamma},\ A \cap C = \emptyset \text{ and } B, D \in \mathbf{\Gamma}'\}$

Then we can easily construct corresponding schemata of determinacy; replacing $\cup$ with $\vee$ and replacing (-)^c (complement) with $\neg$. The definitions of $\mathsf{Sep}(\Gamma, \Gamma')$ and $\mathsf{Bisep}(\Gamma, \Gamma')$ determinacies in the next section are motivated by these constructions of Wadge classes in this way.

In this thesis, we consider $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)$, $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_2)$, $\mathsf{Bisep}(\Sigma^0_1, \Sigma^0_2)$, Sep-(Σ^0_1, Σ^0_2) and $\mathsf{Bisep}(\Delta^0_2, \Sigma^0_2)$ determinacies which correspond to $\mathbf{Bisep}(\mathbf{\Delta}^0_1, \mathbf{\Sigma}^0_1)$, $\mathbf{Bisep}(\mathbf{\Delta}^0_1, \mathbf{\Sigma}^0_2)$, $\mathbf{Bisep}(\mathbf{\Sigma}^0_1, \mathbf{\Sigma}^0_2)$, $\mathbf{Sep}(\mathbf{\Sigma}^0_1, \mathbf{\Sigma}^0_2)$ and $\mathbf{Bisep}(\mathbf{\Delta}^0_2, \mathbf{\Sigma}^0_2)$ respectively. In Section 4.6, we show that Π^1_1 comprehension, $\mathsf{Bisep}(\Sigma^0_1, \Sigma^0_2)$-$\mathsf{Det}^*$ and $\mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)$-$\mathsf{Det}^*$ are pairwise equivalent. Although there are ω_1 different Wadge classes between $\mathbf{Bisep}(\mathbf{\Sigma}^0_1, \mathbf{\Sigma}^0_2)$ and $\mathbf{Sep}(\mathbf{\Sigma}^0_1, \mathbf{\Sigma}^0_2)$, we will show that, in the Cantor space, the strengths of the determinacies correspond to them coincide.

3.6 Several new determinacy schemata

In this section we define, in second order arithmetic, determinacy schemata which correspond to the determinacy of Wadge classes in descriptive set theory. We only consider classes of formulae $\varphi(f)$ with a distinguished function variable $f \in X^{\mathbb{N}}$.

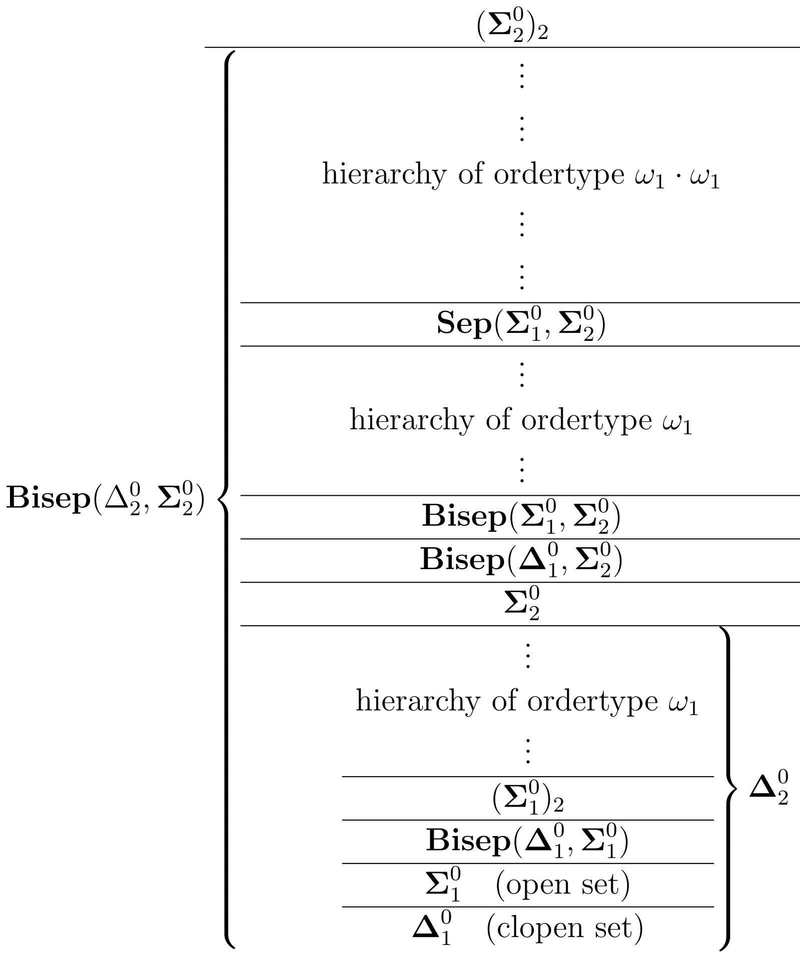


Figure 3.2: Wadge hierarchy below $(\Sigma^0_2)_2$ due to [15]

The normal form theorem for Σ^0_1 formulae over RCA_0 is proved in [30]. Here we prove it over RCA^*_0.

While we usually assume that formulae and terms may contain free variables other than explicitly displayed, only in the statement and the proof of the following lemma, we assume that all free variables of formulae and terms are among displayed ones.

Lemma 3.6.1 (normal form theorem 1)**.** *For any Σ^0_1 formula $\varphi(f, \vec{k}, \vec{Y})$ with a distinguished function variable $f \in 2^{\mathbb{N}}$, we can find a Π^0_0 formula $\theta(s, \vec{k}, \vec{Y})$, in which f does not occur freely, such that RCA^*_0 proves*

$$\forall f \in 2^{\mathbb{N}} \forall \vec{k} \forall \vec{Y} (\varphi(f, \vec{k}, \vec{Y}) \leftrightarrow \exists n \theta(f[n], \vec{k}, \vec{Y})).$$

Proof. First, we prove that, for every Π^0_0 formula $\varphi(f,\vec{k},\vec{Y})$, we can find Π^0_0 formula $\theta(s,\vec{k},\vec{Y})$ and a term $t(\vec{k})$ such that RCA^*_0 proves $\varphi(f,\vec{k},\vec{Y}) \leftrightarrow \theta(f[t(\vec{k})],\vec{k},\vec{Y})$ by induction on construction of formula. The cases of atomic formulae, Boolean operations are easy. The case of bounded universal quantifier can be proved as follows: Let $\varphi(f,\vec{k},\vec{Y})$ be Π^0_0 of the form $\forall i < t'(\vec{k})\theta'(i,f,\vec{k},\vec{Y})$, where $\theta'(x,f,\vec{k},\vec{Y})$ is Π^0_0. By induction hypothesis, we can find a Π^0_0 formula $\theta''(x,s,\vec{k},\vec{Y})$ and a term $t''(i,\vec{k})$ such that RCA^*_0 proves $\theta'(i,f,\vec{k},\vec{Y}) \leftrightarrow \theta''(i,f[t''(i,\vec{k})],\vec{k},\vec{Y})$. Then, we can check that RCA^*_0 proves

$$\begin{aligned}
& \forall i < t'(\vec{k})\theta'(i,f,\vec{k},\vec{Y}) \\
\leftrightarrow\ & \forall i < t'(\vec{k})\theta''(i,f[t''(i,\vec{k})],\vec{k},\vec{Y}) \\
\leftrightarrow\ & \forall i < t'(\vec{k})\theta''(i,(f[t''(t'(\vec{k}),\vec{k})])[t(i,\vec{k})],\vec{k},\vec{Y}),
\end{aligned}$$

by the monotony of the term. The case of bounded existential quantifier can be proved similarly.

Now we complete the proof. Let $\varphi(f,\vec{k},\vec{Y}) \equiv \exists n\theta'(n,f,\vec{k},\vec{Y})$ for some Π^0_0 formula $\theta'(x,f,\vec{k},\vec{Y})$. By the above, we can find Π^0_0 formula $\theta''(x,f,\vec{k},\vec{Y})$ and a term $t'(\vec{k})$ such that RCA^*_0 proves $\theta'(x,f,\vec{k},\vec{Y}) \leftrightarrow \theta''(x,f[t(x,\vec{k})],\vec{k},\vec{Y})$. Then RCA^*_0 proves

$$\begin{aligned}
& \varphi(f,\vec{k},\vec{Y}) \\
\leftrightarrow\ & \exists n\theta''(n,f[t(n,\vec{k})],\vec{k},\vec{Y}) \\
\leftrightarrow\ & \exists n\exists n' < n+1(n = \langle n', t(n',\vec{k})\rangle \wedge \theta''(n',(f[n])[t(n',\vec{k})],\vec{k},\vec{Y})).
\end{aligned}$$

□

Definition 3.6.2. Let Γ and Γ' be classes of formulae. $\Gamma \wedge \Gamma'$ (resp. $\Gamma \vee \Gamma'$) is the class of formulae of the form $\varphi_0 \wedge \varphi_1$ (resp. $\varphi_0 \vee \varphi_1$), where φ_0 is Γ and φ_1 is Γ'. $\neg\Gamma$ is the class of formulae of the form $\neg\varphi$, where φ is Γ.

We consider determinacy schemata also for the class defined in Definition 3.6.2.

Lemma 3.6.3. Γ *determinacy in* $X^{\mathbb{N}}$ *is equivalent to* $\neg\Gamma$ *determinacy in* $X^{\mathbb{N}}$ *over* RCA^*_0

Proof. Assume Γ determinacy in $X^{\mathbb{N}}$. Let $\varphi(f)$ be a $\neg\Gamma$ game in $X^{\mathbb{N}}$. By Γ determinacy in $X^{\mathbb{N}}$, $\neg\varphi(f')$ is determinate, where f' is defined by $f'(n) = f(n+1)$. It is easy to see that player I wins $\varphi(f)$ if and only if there exists n such that player II wins $\neg\varphi(f')$ and that player II wins $\neg\varphi(f)$ if and only if player I wins $\varphi(f')$. □

Furthermore, we consider the following determinacy schemata, which are motivated by the Borel Wadge classes, mentioned in the previous section.

Definition 3.6.4.

$\Delta(\Gamma)$ **determinacy in $X^{\mathbb{N}}$:** $\forall f \in X^{\mathbb{N}}(\psi(f) \leftrightarrow \xi(f)) \ \rightarrow\ \mathsf{Det}^X[\psi]$,
where $\psi(f)$ is Γ and where $\xi(f)$ is $\neg\Gamma$.

$\mathsf{Sep}(\Sigma^0_n, \Sigma^0_m)$ **determinacy in $X^{\mathbb{N}}$:** $\mathsf{Det}^X[(\psi \wedge \eta_0) \vee (\neg\psi \wedge \eta_1)]$,
where $\psi(f)$ is Σ^0_n, where $\eta_0(f)$ is Π^0_m and where $\eta_1(f)$ is Σ^0_m.

$\mathsf{Sep}(\Delta^0_n, \Sigma^0_m)$ **determinacy in $X^{\mathbb{N}}$:**
$\forall f \in X^{\mathbb{N}}(\psi(f) \leftrightarrow \xi(f)) \rightarrow \mathsf{Det}^X[(\psi \wedge \eta_0) \vee (\neg\psi \wedge \eta_1)]$, where $\psi(f)$ is Σ^0_n, where $\xi(f)$ is Π^0_n, where $\eta_0(f)$ is Π^0_m and where $\eta_1(f)$ is Σ^0_m.

$\mathsf{Bisep}(\Sigma^0_n, \Sigma^0_m)$ **determinacy in $X^{\mathbb{N}}$:**
$\forall f \in X^{\mathbb{N}}\neg(\psi_0(f) \wedge \psi_1(f)) \rightarrow \mathsf{Det}^X[(\psi_0 \wedge \eta_0) \vee (\psi_1 \wedge \eta_1)]$,
where $\psi_i(f)$'s are Σ^0_n, where $\eta_0(f)$ is Π^0_m and where $\eta_1(f)$ is Σ^0_m.

$\mathsf{Bisep}(\Delta^0_n, \Sigma^0_m)$ **determinacy in $X^{\mathbb{N}}$:**
$$\{\forall f \in X^{\mathbb{N}}(\neg(\psi_0(f) \wedge \psi_1(f)))\wedge \forall f \in X^{\mathbb{N}}(\psi_0(f) \leftrightarrow \xi_0(f)) \wedge \forall f \in X^{\mathbb{N}}(\psi_1(f) \leftrightarrow \xi_1(f))\} \rightarrow \mathsf{Det}^X[(\psi_0 \wedge \eta_0) \vee (\psi_1 \wedge \eta_1)],$$
where $\psi_i(f)$'s are Σ^0_n, where $\xi_i(f)$'s are Π^0_n, where $\eta_0(f)$ is Π^0_m and where $\eta_1(f)$ is Σ^0_m.

Remark 3.6.5. It is easy to check the following implications:

- Σ^0_n determinacy in $X^{\mathbb{N}}$ $\rightarrow$ Δ^0_n determinacy in $X^{\mathbb{N}}$
- $\mathsf{Bisep}(\Delta^0_m, \Sigma^0_n)$ determinacy in $X^{\mathbb{N}}$ $\rightarrow$ Σ^0_n determinacy in $X^{\mathbb{N}}$.
- $\mathsf{Bisep}(\Sigma^0_m, \Sigma^0_n)$ determinacy in $X^{\mathbb{N}}$ $\rightarrow$ $\mathsf{Bisep}(\Delta^0_m, \Sigma^0_n)$ determinacy in $X^{\mathbb{N}}$.
- $\mathsf{Sep}(\Sigma^0_m, \Sigma^0_n)$ determinacy in $X^{\mathbb{N}}$ $\rightarrow$ $\mathsf{Bisep}(\Sigma^0_m, \Sigma^0_n)$ determinacy in $X^{\mathbb{N}}$ for $m \leq n$.
- $\mathsf{Bisep}(\Delta^0_n, \Sigma^0_n)$ determinacy in $X^{\mathbb{N}}$ $\rightarrow$ $\mathsf{Sep}(\Sigma^0_m, \Sigma^0_n)$ determinacy in $X^{\mathbb{N}}$ for $m \leq n$.
- $(\Sigma^0_n)_2$ determinacy in $X^{\mathbb{N}}$ $\rightarrow$ $\mathsf{Bisep}(\Delta^0_m, \Sigma^0_n)$ determinacy in $X^{\mathbb{N}}$ for $m \leq n$.

Remark 3.6.6. $\mathbf{Bisep}(\mathbf{\Delta}^0_n, \mathbf{\Sigma}^0_{n+m}) = \mathbf{Sep}(\mathbf{\Delta}^0_n, \mathbf{\Sigma}^0_{n+m})$ in the terms of the last section. Inspired by this equality, we can see that $\mathsf{Bisep}(\Delta^0_n, \Sigma^0_{n+m})$ determinacy in $X^{\mathbb{N}}$ is equivalent to $\mathsf{Sep}(\Delta^0_n, \Sigma^0_{n+m})$ determinacy in $X^{\mathbb{N}}$ over RCA^*_0, since, for any formulae $\psi_0(f)$, $\psi_1(f)$, $\eta_0(f)$ and $\eta_1(f)$, RCA^*_0 proves the following:

$$\begin{aligned}\forall f \in X^{\mathbb{N}} \neg(\psi_0(f) \wedge \psi_1(f)) \quad &\rightarrow \\ &\{\forall f \in X^{\mathbb{N}}(((\psi_0(f) \wedge \eta_0(f)) \vee (\psi_1(f) \wedge \eta_1(f))) \\ &\qquad \leftrightarrow ((\psi_0(f) \wedge \eta_0(f)) \vee (\neg\psi_0(f) \wedge (\psi_1(f) \wedge \eta_1(f)))))\}.\end{aligned}$$

The next lemma is a reformulation of a classical separation theorem: For any disjoint pair of $\mathbf{\Pi}^0_n$ sets A and B, there exists a $\mathbf{\Delta}^0_n$ set C such that $A \subseteq C$ and $B \subseteq C^c$.

Lemma 3.6.7. *Assume $0 < n < \omega$. For any pair of Π^0_n formulae $\varphi_0(f)$ and $\varphi_1(f)$, we can find a Σ^0_n formula $\psi(f)$ and Π^0_n formula $\xi(f)$ such that RCA^*_0 proves*

$$\begin{aligned}&\forall f \in X^{\mathbb{N}} \neg(\varphi_0(f) \wedge \varphi_1(f)) \rightarrow \\ &\qquad \forall f \in X^{\mathbb{N}}((\psi(f) \leftrightarrow \xi(f)) \wedge (\varphi_0(f) \rightarrow \psi(f)) \wedge (\varphi_1(f) \rightarrow \neg\psi(f))).\end{aligned}$$

Proof. Let $\varphi_i(f)$ be Π^0_n, say $\forall m \theta_i(m, f)$, where $\theta_i(x, g)$ is Σ^0_{n-1}. Assume $\forall f \in X^{\mathbb{N}} \neg(\varphi_0(f) \wedge \varphi_1(f))$. Set $\psi(f) \equiv \exists m(\neg\theta_1(m, f) \wedge \forall l < m \theta_0(l, f))$. It is easy to check that $\psi(f)$ is equivalent to a Σ^0_n formula. We can also see that $\varphi_0(f) \rightarrow \psi(f)$ and $\varphi_1(f) \rightarrow \neg\psi(f)$. Since, for any $f \in X^{\mathbb{N}}$, either $\neg\varphi_0(f)$ or $\neg\varphi_1(f)$ holds, for any $f \in X^{\mathbb{N}}$, $\neg\psi(f)$ holds if and only if $\exists m(\neg\theta_0(m, f) \wedge \forall l \leq m \theta_1(l, f))$ holds. Therefore, $\psi(f) \leftrightarrow \forall m(\theta_0(m, f) \vee \exists l \leq m \neg\theta_1(l, f))$. □

The following lemma is useful to investigate the relationship among determinacy schemata defined in Definition 3.6.4. The motivating result is $\mathbf{Sep}(\mathbf{\Delta}^0_n, \mathbf{\Sigma}^0_n) = \mathbf{\Delta}((\mathbf{\Sigma}^0_n)_2) = \mathbf{\Delta}(\mathbf{\Sigma}^0_n \wedge \mathbf{\Pi}^0_n)$.

Lemma 3.6.8. *Let $0 < n < \omega$.*

1. *For any Σ^0_n formulae $\psi_0(f)$ and $\eta_1(f)$, for any Π^0_n formulae $\psi_1(f)$ and $\eta_0(f)$, we can find Σ^0_n formulae $\xi_0(f)$ and $\xi_1(f)$ and Π^0_n formulae $\zeta_0(f)$*

and $\zeta_1(f)$ *such that* RCA_0^* *proves*

$$\forall f \in X^{\mathbb{N}}(\psi_0(f) \leftrightarrow \psi_1(f)) \rightarrow \\ \forall f \in X^{\mathbb{N}}(((\psi_0(f) \wedge \eta_0(f)) \vee (\psi_1(f) \wedge \eta_1(f))) \\ \leftrightarrow (\xi_0(f) \wedge \zeta_0(f)) \leftrightarrow (\xi_1(f) \vee \zeta_1(f))).$$

2. *For any* Σ^0_n *formulae* $\xi_0(f)$ *and* $\xi_1(f)$ *and for any* Π^0_n *formulae* $\zeta_0(f)$ *and* $\zeta_1(f)$*, we can find* Σ^0_n *formulae* $\psi_0(f)$ *and* $\eta_1(f)$ *and* Π^0_n *formulae* $\psi_1(f)$ *and* $\eta_0(f)$ *such that* RCA_0^* *proves*

$$\forall f \in X^{\mathbb{N}}((\xi_0(f) \wedge \zeta_0(f)) \leftrightarrow (\xi_1(f) \vee \zeta_1(f))) \rightarrow \\ [\forall f \in X^{\mathbb{N}}(\psi_0(f) \leftrightarrow \psi_1(f)) \wedge \\ \{\forall f \in X^{\mathbb{N}}((\xi_0(f) \wedge \zeta_0(f)) \\ \leftrightarrow ((\psi_0(f) \wedge \eta_0(f)) \vee (\neg\psi_0(f) \wedge \eta_1(f))))\}].$$

Proof. We work in RCA_0^*.

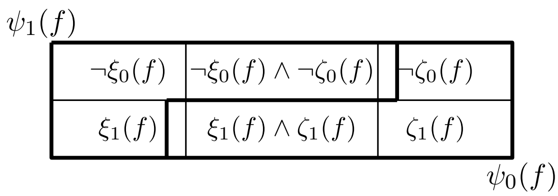
1. Assume that $\forall f \in X^{\mathbb{N}}(\psi_0(f) \leftrightarrow \psi_1(f))$ for a Σ^0_n formula $\psi_0(f)$ and a Π^0_n formula $\psi_1(f)$. Then, for any Π^0_n formula $\eta_0(f)$ and Σ^0_n formula $\eta_1(f)$, it is easy to find Σ^0_n formulae $\xi_0(f)$ and $\xi_1(f)$ and Π^0_n formulae $\zeta_0(f)$ and $\zeta_1(f)$ such that RCA_0^* proves

$$\forall f \in X^{\mathbb{N}}(\xi_0(f) \leftrightarrow (\psi_0(f) \vee (\neg\psi_0(f) \wedge \eta_1(f)))), \\ \forall f \in X^{\mathbb{N}}(\xi_1(f) \leftrightarrow (\neg\psi_0(f) \wedge \eta_1(f))), \\ \forall f \in X^{\mathbb{N}}(\zeta_0(f) \leftrightarrow ((\psi_0(f) \wedge \eta_0(f)) \vee \neg\psi_0(f))), \\ \forall f \in X^{\mathbb{N}}(\zeta_1(f) \leftrightarrow (\psi_0(f) \wedge \eta_0(f))),$$

and they enjoy the desired property.

2. Assume $\forall f \in X^{\mathbb{N}}((\xi_0(f) \wedge \zeta_0(f)) \leftrightarrow (\xi_1(f) \vee \zeta_1(f)))$ for Σ^0_n formulae $\xi_0(f)$ and $\xi_1(f)$ and Π^0_n formulae $\zeta_0(f)$ and $\zeta_1(f)$. Note that $\forall f \in X^{\mathbb{N}}((\neg\xi_0(f) \vee \neg\zeta_0(f)) \leftrightarrow \neg(\xi_1(f) \vee \zeta_1(f)))$, and so there is no $f \in X^{\mathbb{N}}$ with $\neg\xi_0(f) \wedge \zeta_1(f)$. By Lemma 3.6.7, we can find a Σ^0_n formula $\psi_0(f)$ and a Π^0_n formula $\psi_1(f)$ such that

$$\forall f \in X^{\mathbb{N}}((\psi_0(f) \leftrightarrow \psi_1(f)) \wedge (\zeta_1(f) \rightarrow \psi_0(f)) \wedge (\neg\xi_0(f) \rightarrow \neg\psi_0(f))).$$



Then we can check

$$\forall f \in X^{\mathbb{N}}((\xi_0(f) \wedge \zeta_0(f)) \leftrightarrow ((\psi_0(f) \wedge \zeta_0(f)) \vee (\neg\psi_0(f) \wedge \xi_1(f)))),$$

and so $\psi_i(f)$'s, $\eta_0(f) \equiv \zeta_0(f)$ and $\eta_1(f) \equiv \xi_1(f)$ enjoy the desired property. $\square$

By Remark 3.6.6 and Lemma 3.6.8, $\mathsf{Bisep}(\Delta^0_n, \Sigma^0_n)$-$\mathsf{Det}$, $\mathsf{Sep}(\Delta^0_n, \Sigma^0_n)$-$\mathsf{Det}$ and $\Delta(\Sigma^0_n \wedge \Pi^0_n)$-$\mathsf{Det}$ are pairwise equivalent over RCA^*_0 and $\mathsf{Bisep}(\Delta^0_n, \Sigma^0_n)$-$\mathsf{Det}^*$, $\mathsf{Sep}(\Delta^0_n, \Sigma^0_n)$-$\mathsf{Det}^*$ and $\Delta(\Sigma^0_n \wedge \Pi^0_n)$-$\mathsf{Det}^*$ are pairwise equivalent over RCA^*_0.

Chapter 4

Hierarchy of Determinacy

4.1 WKL_0^* and determinacy

In this section, we consider the relationship between weak König's lemma and determinacy.

We must notice that the class of Π^0_1 formulae is enriched by weak König's lemma.

Lemma 4.1.1. *For any tree $T \subseteq 2^{<\mathbb{N}}$, the assertion "$T$ has an infinite path" is equivalent to a Π^0_1 formula over WKL_0^*.*

Proof. For any tree $T \subseteq 2^{<\mathbb{N}}$, set $\psi(T) \equiv \forall n \exists s(|s| = n \wedge s \in T)$. It is easy to check that $\psi(T)$ is equivalent to a Π^0_1 formula over RCA_0^*. If T has an infinite path, then $\psi(T)$ holds. Conversely, if $\psi(T)$ holds, then T is an infinite tree, and so, by weak König's lemma, T has an infinite path. □

Lemma 4.1.2. *For any Π^0_1 formula $\psi(X)$, $\exists X \psi(X)$ is equivalent to a Π^0_1 formula over WKL_0^*.*

Proof. By Theorem 3.6.1, for any Π^0_1 formula $\psi(f)$, we can find a Σ^0_0 formula $\theta(x)$ such that RCA_0^* proves $\forall X(\psi(X) \leftrightarrow \forall n \theta(X[n]))$. We can check that $\exists X \psi(X)$ if and only if the binary tree $T = \{s \in 2^{<\mathbb{N}} : \forall t \subseteq s \theta(t)\}$ has an infinite path f. By Lemma 4.1.1, this assertion is equivalent to a Π^0_1 formula over WKL_0^*. □

Lemma 4.1.3. *For any Π^0_1 formula $\psi(X)$, $\forall X \psi(X)$ is equivalent to a Π^0_1 formula over RCA_0^*.*

Proof. Since we can find a Σ^0_0 formula $\theta(x)$ such that RCA^*_0 proves $\forall X(\psi(X) \leftrightarrow \forall n\theta(X[n]))$ by Theorem 3.6.1, $\forall X\psi(X)$ is equivalent to $\forall s \in 2^{<\mathbb{N}}\theta(s)$, which is Π^0_1. □

Remark 4.1.4.

1. For a Σ^0_1 game $\varphi(f)$ in the Cantor space, the assertion "σ is a winning strategy for player I (or II) in $\varphi(f)$ (resp. $\neg\varphi(f)$)" is equivalent to a Σ^0_1 formula over WKL^*_0. The assertion can be written as $\forall \tau \in \mathcal{S}^{\{0,1\}}_{\mathrm{II}}\varphi(\sigma\otimes\tau)$ (resp. $\forall \tau \in \mathcal{S}^{\{0,1\}}_{\mathrm{I}}\varphi(\tau \otimes \sigma)$), which is equivalent to a Σ^0_1 formula by Lemma 4.1.2.

2. For a Π^0_1 game $\varphi(f)$ in the Cantor space and the assertion "σ is a winning strategy for player I (or II) in $\varphi(f)$ (resp. in $\neg\varphi(f)$)" is equivalent to a Π^0_1 formula over RCA^*_0. The assertion can be written as $\forall \tau \in \mathcal{S}^{\{0,1\}}_{\mathrm{II}}\varphi(\sigma \otimes \tau)$ (resp. $\forall \tau \in \mathcal{S}^{\{0,1\}}_{\mathrm{I}}\varphi(\tau \otimes \sigma)$), which is equivalent to a Π^0_1 formula by Lemma 4.1.3.

3. For any Σ^0_1 game $\varphi(f)$ in the Cantor space, the assertion "player I (or II) has a winning strategy in $\varphi(f)$ (resp. $\neg\varphi(f)$)" is equivalent to a Σ^0_1 formula over WKL^*_0. The assertion "player I (or II) has no winning strategy in $\varphi(f)$ (resp. $\neg\varphi(f)$)" can be written as $\forall \sigma \in \mathcal{S}^{\{0,1\}}_{\mathrm{I}}\exists \tau \in \mathcal{S}^{\{0,1\}}_{\mathrm{II}}\neg\varphi(\sigma \otimes \tau)$ (resp. $\forall \sigma \in \mathcal{S}^{\{0,1\}}_{\mathrm{II}}\exists \tau \in \mathcal{S}^{\{0,1\}}_{\mathrm{I}}\neg\varphi(\tau \otimes \sigma)$). The part "$\exists \tau \in \mathcal{S}^{\{0,1\}}_{\mathrm{II}}\neg\varphi(\sigma \otimes \tau)$ (resp. $\exists \tau \in \mathcal{S}^{\{0,1\}}_{\mathrm{I}}\neg\varphi(\tau \otimes \sigma)$)" is equivalent to a Π^0_1 formula by Lemma 4.1.3, the whole statement is equivalent to a Π^0_1 formula by Lemma 4.1.2 and so the negation is equivalent to a Σ^0_1 formula.

4. For any Π^0_1 game $\varphi(f)$ in the Cantor space, the assertion "player I (or II) has a winning strategy in $\varphi(f)$ (resp. $\neg\varphi(f)$)" is equivalent to a Π^0_1 formula over WKL^*_0. The assertion can be written as "$\exists \sigma \in \mathcal{S}^{\{0,1\}}_{\mathrm{I}}\forall \tau \in \mathcal{S}^{\{0,1\}}_{\mathrm{II}}\varphi(\sigma \otimes \tau)$ (resp. $\exists \sigma \in \mathcal{S}^{\{0,1\}}_{\mathrm{II}}\forall \tau \in \mathcal{S}^{\{0,1\}}_{\mathrm{I}}\varphi(\tau \otimes \sigma)$)." By 2, the part "$\forall \tau \in \mathcal{S}^{\{0,1\}}_{\mathrm{II}}\varphi(\sigma \otimes \tau)$ (resp. $\forall \tau \in \mathcal{S}^{\{0,1\}}_{\mathrm{I}}\varphi(\tau \otimes \sigma)$)" is equivalent to a Π^0_1 formula, and so the assertion is equivalent to Π^0_1 by Lemma 4.1.2.

It is easy to see that the above implies the following.

Remark 4.1.5.

1. For a Σ^0_1 game $\varphi(f)$ in the Cantor space and $s \in 2^{<\mathbb{N}}$, the assertion "σ is a winning s-strategy for player I (or II) in $\varphi(f)$ (resp. $\neg\varphi(f)$)" is equivalent to a Σ^0_1 formula over WKL^*_0

2. For a Π^0_1 game $\varphi(f)$ in the Cantor space and $s \in 2^{<\mathbb{N}}$, the assertion "σ is a winning s-strategy for player I (or II) in $\varphi(f)$ (resp. $\neg\varphi(f)$)" is equivalent to a Π^0_1 formula over RCA^*_0

3. For a Σ^0_1 game $\varphi(f)$ in the Cantor space, the assertion "player I (or II) wins $\varphi(f)$ (resp. $\neg\varphi(f)$) at s" is equivalent to a Σ^0_1 formula over WKL^*_0.

4. For a Π^0_1 game $\varphi(f)$ in the Cantor space, the assertion "player I (or II) wins $\varphi(f)$ (resp. $\neg\varphi(f)$) at s" is equivalent to a Π^0_1 formula over WKL^*_0.

Now, we prove that WKL^*_0, Σ^0_1-Det^* and Δ^0_1-Det^* are pairwise equivalent.

Theorem 4.1.6. RCA^*_0 *proves that* Δ^0_1-Det^* *implies weak König's lemma.*

Proof. Assume Δ^0_1-Det^*. For contradiction, assume that weak König's lemma does not hold. Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite tree in which there is no infinite path, i. e., there is no f such that $\forall n f[n] \in T$. We consider a game $\varphi(f)$ in the Cantor space, which consists of the following four stages:

- Player I chooses $t \in 2^{<\mathbb{N}}$ by playing $t(i)$ at his $(2i+2)$-th turn for $i < |t|$, 0 at his $(2i+1)$-th turn for $i < |t|$, and 1 at his $(2|t|+1)$-th turn. If $t[i] \notin T$ for some $i < |t|$, then player I loses.

- If $t \in T$, then player II chooses $i < 2$.

- Player I produces a new sequence u of $2^{<\mathbb{N}}$ in a similar way as in the choice of t. If $t * \langle i \rangle * u \notin T$, then player I loses.

- If $t * \langle i \rangle * u \in T$, player II then produces a sequence v of $2^{<\mathbb{N}}$ in a similar way in the choice of t. If $t * \langle 1-i \rangle * v[i] \notin T$ for some $i < |v|$, player I wins. If $t * \langle 1-i \rangle * v \in T$, player I wins when $|u| > |v|$.

Now, let us see some more details of each stage.

$$
\begin{array}{lllllllllllll}
\mathrm{I} & 0 & t(0) & t(1) & t(1) & \cdots 0 & t(|t|-1) & \overbrace{1}^{\text{signal}} & & 0 & u(0) & 0 & u(1)\cdots \\
 & & & & & & & & & & & & \\
\mathrm{II} & * & * & * & * & * & & & \underbrace{i}_{\text{answer}} & * & * & * & \cdots
\end{array}
$$

$$\begin{array}{lllllllllll}
\text{(I continues)} \cdots 0 & u(|u|-1) & \overbrace{1}^{\text{signal}} & * & * & * \cdots & * & & * \\
\text{(II continues)} \cdots & * & & * & 0 & v(0) & v(1) & \cdots 0 & v(|v|-1) & \underbrace{1}_{\text{signal}}
\end{array}$$

Here, $*$'s denote 0 or 1, but moves marked $*$ are of no effect.

Formally, the game $\varphi(f)$ can be realized as follows:

$$\exists n \exists m \exists l \left[\begin{array}{l}
(\psi(f[4n]) \wedge f(4n) = 1 \wedge \overline{\overline{f[4n]}} \in T) \wedge \\
\psi(f[4n+4m+2] \ominus (4n+2)) \wedge \\
\overline{\overline{f[4n]}} * \langle f(4n+1) \rangle * \overline{\overline{f[4n+4m+2] \ominus (4m+2)}} \in T \wedge \\
\psi(f[4n+4m+4l+3] \ominus (4n+4m+3)) \wedge \\
(\overline{\overline{f[4n]}} * \langle 1 - f(4n+1) \rangle * \overline{\overline{f[4n+4m+4l+3] \ominus (4n+4m+3)}} \notin T \vee \\
\hfill (f(4n+4m+4l+3) = 1 \wedge l < m))
\end{array}\right]$$

where $\psi(s) \equiv \forall 4i < |s|(s(4i) = 0)$, where $\overline{\overline{s}} = \langle s(2), s(6), ..., s(4k_s+2) \rangle$, and where $k_s =$ the maximal k with $4k+2 < |s|$. Since T has no infinite path, the game is always determined after finitely many moves, which ensures that it is Δ^0_1. Actually $\neg\varphi(f)$ is equivalent to the following formula:

$$\exists n \exists m \exists l \left[\begin{array}{l}
\{\psi(f[4n]) \wedge \overline{\overline{f[4n]}} \notin T)\} \vee \\
\{\psi(f[4n]) \wedge f(4n) = 1 \wedge \\
\quad \psi(f[4n+4m+2] \ominus (4n+2)) \wedge \\
\quad \overline{\overline{f[4n]}} * \langle f(4n+1) \rangle * \overline{\overline{f[4n+4m+2] \ominus (4n+2)}} \notin T\} \vee \\
\{\psi(f[4n]) \wedge f(4n) = 1 \wedge \\
\quad \psi(f[4n+4m+2] \ominus (4n+2)) \wedge f(4n+4m+2) = 1 \wedge \\
\quad \psi(f[4n+4m+4l+3] \ominus (4n+4m+3)) \wedge \\
\quad f(4n+4m+4l+3) = 1 \wedge \\
\quad \overline{\overline{f[4n]}} * \langle 1 - f(4n+1) \rangle * \overline{\overline{f[4n+4m+4l+2] \ominus (4n+4m+2)}} \in T \wedge \\
\hfill m \leq l\}
\end{array}\right]$$

Thus $\varphi(f)$ is determinate by Δ^0_1-Det^*. Next, we will show that player I has no winning strategy. For contradiction, assume that player I has a winning strategy σ. Let w be a $(4k+1)$-length sequence such that $w(4l) = 0$ for all $l < k$, $w(2l+1) = 0$ for all $2l+1 < 4k$, $w(4k) = 1$ and $w(4k+2) = \sigma(w[4k+2])$. In other words, w is a finite play such that player I has just constructs $t = \overline{\overline{w}}$ in the first stage following σ. Note that such w exists, otherwise player I loses. We may assume that $t * \langle i \rangle \in T$ for each $i = 0, 1$. Otherwise, II always wins. Let $T_i = \{s : t * \langle i \rangle * s \in T\}$ for $i = 0, 1$. If T_i is bounded by n (i. e. any sequence of T_i has length $\leq n$) and T_{1-i} is not bounded by $n-1$ (i. e., T_{1-i} has a sequence of length n), then player II can easily win $\varphi(f)$ by choosing $1-i$ in the second stage. If both T_i and T_{1-i} are not bounded, then player II could choose either 0 or 1 in the second stage to win the game. Thus, player I can not have a winning strategy. Therefore, by Δ^0_1-Det^*, player II has a winning strategy τ. For each finite sequence $t \in 2^{<\mathbb{N}}$, t^τ be the finite play of the first and second stages in which player I constructs t at the first stage of the game and II follows τ. We then define $f : \mathbb{N} \to \{0, 1\}$ as follows:

$$f(n) = 1 - \tau(f[n]^\tau).$$

Namely, $f(n)$ is the opposite of player II's answer to $f[n]$, in which direction one can construct a longer sequence in T.

Claim. *$f[n] \in T$ for all n.*

Proof of the claim. We prove by Σ^0_0 induction. Clearly, $f[0] = \langle\rangle \in T$. Assume $f[n] \in T$. We show that if $f[n+1] \notin T$, T is bounded by $n+1$, i. e., for all $t \in T$, $|t| < n+1$. Assume $f[n+1] \notin T$. If $f[n] * \langle 1 - f(n) \rangle \in T$, player I can win by choosing $t = f[n]$ in the first stage of the game, and so $f[n] * \langle 1 - f(n) \rangle \notin T$. Set $\eta(m) \equiv (n < m) \vee \forall t \in 2^{n+1}(f[n-m] \subseteq t \to t \notin T)$. Now, $\eta(0)$ holds. Assume that $\eta(m)$ holds. If $n < m+1$, then clearly $\eta(m+1)$ holds. If $m < n$, there is no t_0 in T such that $f[n-(m+1)] * \langle 1 - f(n - (m+1)) \rangle \subseteq t_0$ and $|t_0| = n+1$, otherwise, player I can win by choosing $f[n-(m+1)]$ in the first stage of the game, and so $\eta(m+1)$ holds. By Σ^0_0 induction, for all m, $\eta(m)$ holds. Especially, $\eta(n)$ holds, which means there is no $t \in T$ with length $n+1$. This contradicts to the assumption that T is infinite.

Therefore $f[n] \in T$ implies $f[n+1] \in T$. By Σ^0_0 induction, $f[n] \in T$ for all n and T is a path of T. □

Therefore T has an infinite path, which contradicts to our assumption that T has no infinite path. Thus, Δ^0_1-Det^* implies weak König's lemma, which completes the proof of the theorem. □

Theorem 4.1.7. WKL^*_0 *proves* Σ^0_1-Det^*.

Proof. Let $\varphi(f)$ be a Σ^0_1 game in the Cantor space. By Lemma 3.6.1, we can find a Σ^0_0 formula $\theta(x)$ such that RCA^*_0 proves $\varphi(f) \leftrightarrow \exists k \theta(f[k])$ for all $f \in 2^{\mathbb{N}}$.

Precisely, for any $n \in \mathbb{N}$ and $k < n$, the induction on k yields $u_n^k : (2^{\leq n} - 2^{<n-k}) \to \{0,1\}$ such that

$$u_n^k(s) = \begin{cases} 1 & \text{if } |s| = n \text{ and } \exists t \subseteq s\theta(t), \\ 0 & \text{if } |s| = n \text{ and } \neg\exists t \subseteq s\theta(t), \\ \max\{u_n^k(s * \langle 0\rangle), u_n^k(s * \langle 1\rangle)\} & \text{if } |s| \leq n \text{ and } |s| \text{ is even, and} \\ \min\{u_n^k(s * \langle 0\rangle), u_n^k(s * \langle 1\rangle)\} & \text{if } |s| \leq n \text{ and } |s| \text{ is odd.} \end{cases}$$

Because u_n^k is bounded by some iterated power of n, here we only need Σ_0^0 induction. Set $u_n : 2^{\leq n} \to \{0,1\}$ by $u_n(s) = u_n^n(s)$.

We show that if there exists n such that $u_n(\langle\rangle) = 1$, then player I has a winning strategy. Fix such n and define a strategy σ for player I by:

$$\sigma(s) = \begin{cases} 0 & \text{if } |s| < n \text{ and } u_n(s * \langle 0\rangle) = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Let f be a play in which player I follows σ. We check below that $u_n(f[k]) = 1$ for all $k \leq n$ by Σ_0^0 induction on $k \leq n$. Clearly $u_n(f[0]) = u_n(\langle\rangle) = 1$. Assume $u_n(f[k]) = 1$ and $k + 1 \leq n$. We have the following two cases. (i) If k is even, player I chooses $f(k)$, following σ. By the definition of u_n, either $u_n(f[k] * \langle 0\rangle) = 1$ or $u_n(f[k] * \langle 1\rangle) = 1$ holds, and so by the definition of σ, $u_n(f[k] * \sigma(f[k])) = 1$. (ii) If k is odd, then both $u_n(f[k] * \langle 0\rangle) = 1$ and $u_n(f[k] * \langle 1\rangle) = 1$ hold by the definition of u_n, and so $u_n(f[k+1]) = 1$. Therefore, $u_n(f[n]) = 1$ and $\exists m \leq n\theta(f[m])$ hold, which means σ is a winning strategy for player I.

Now we show that if $u_n(\langle\rangle) = 0$ for all n then player II has a winning strategy. Since a strategy τ for player II can be regarded as a sequence in $2^{\mathbb{N}}$, each $s \in 2^{<\mathbb{N}}$ is a code of a fragment of a strategy for player II. We say s is partially winning for player II in $\varphi(f)$ if $\forall u \subseteq t\neg\theta(u)$ for any t such that t is a play following s, i. e., $t[2k+1] \in \mathrm{dom}(s)$ and $t(2k+1) = s(t[2k+1])$ for all $2k+1 < |t|$. Note that "s is partially winning for player II" can be written in a Π_0^0 formula. Δ_1^0 comprehension yields T defined by

$$T = \{s \in 2^{<\mathbb{N}} : s \text{ is partially winning for player II}\}.$$

Then T is a tree. Recall $u_n(\langle\rangle) = 0$ for all n. T is infinite, because for any n, $s : 2^{\leq n} \cap 2^{\mathrm{odd}} \to \{0,1\}$ defined by

$$s(t) = \begin{cases} 0 & \text{if } u_n(t) = 0, \\ 1 & \text{otherwise,} \end{cases}$$

is partially winning for player II. By weak König's lemma, T has an infinite path, which yields a winning strategy in $\varphi(f)$ for player II. $\square$

Corollary 4.1.8. *Weak König's lemma,* Δ^0_1-Det^* *and* Σ^0_1-Det^* *are equivalent over* RCA^*_0.

4.2 WKL_0 and determinacy

In this section, we consider the strength of Σ^0_1 induction+weak König's lemma over RCA^*_0. For this purpose, we introduce a set comprehension axiom which is equivalent to Σ^0_1 induction.

Lemma 4.2.1 ([30, Theorem II.3.8, Theorem X.4.4])**.** Σ^0_1 *induction is equivalent to bounded* Σ^0_1 *comprehension over* RCA^*_0.

The theorems below show Σ^0_1 induction+weak König's lemma is equivalent to $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)$-$\mathsf{Det}^*$ over RCA^*_0.

Theorem 4.2.2. RCA^*_0 *proves that* $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)$-$\mathsf{Det}^*$ *implies* Σ^0_1 *induction and weak König's lemma.*

Proof. Assume $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)$-$\mathsf{Det}^*$. Since $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)$-$\mathsf{Det}^*$ implies Σ^0_1-Det^*, which further implies weak König's lemma, it is sufficient to prove bounded Σ^0_1 comprehension, which is equivalent to Σ^0_1 induction by Lemma 4.2.1. Let $\varphi(x)$ be a Σ^0_1 formula. For fixed n, we consider the following game. Player I chooses $k < n$ and asks whether $\varphi(k)$ or not. Player II answers yes or no. She wins if and only if her answer is correct. Such a game is realized as follows:

- Player I chooses $k < n$ by playing 0 at his first k turns and 1 at his $(k+1)$-th turn.
- When player II chooses 1 (yes) at her $(k+1)$-th turn, she wins if $\varphi(k)$ holds.
- When player II chooses 0 (no) at her $(k+1)$-th turn, she wins if $\neg\varphi(k)$ holds.

The game goes as follows:

$$\begin{array}{lccc} \text{player I} & \overbrace{0, ..., 0,}^{k \text{ times}} & 1 & \cdots \\ \text{player II} & *, ..., * & \underbrace{i}_{\text{answer}} & \cdots \end{array}$$

Player I wins if and only if one of the following holds:

- Player I chooses $k < n$, player II answers "yes," and $\neg\varphi(k)$ holds.
- Player I chooses $k < n$, player II answers "no," and $\varphi(k)$ holds.

Formally, this winning condition can be written as below:

$$(\exists k < n(\psi(f, k, 1) \wedge \neg\varphi(k))) \vee (\exists k < n(\psi(f, k, 0) \wedge \varphi(k))), \qquad (\flat)$$

where $\psi(f, k, i)$ is the Π^0_0 formula $\forall i < k(f(2i) = 0) \wedge f(2k) = 1 \wedge f(2k+1) = i$. $(\flat)$ is equivalent to the following formula:

$$((\underbrace{\exists k < n\psi(f, k, 1))}_{(1)} \wedge \underbrace{\forall k < n(\psi(f, k, 1) \rightarrow \neg\varphi(k)))}_{(2)} \vee$$
$$((\underbrace{\exists k < n\,\psi(f, k, 0))}_{(3)} \wedge \underbrace{\exists k < n(\psi(f, k, 0) \wedge \varphi(k)))}_{(4)}).$$

Clearly (1) and (3) are Σ^0_1, and we can easily see that (1) and (3) are equivalent to some Π^0_1 formulae. There is no $f \in 2^{\mathbb{N}}$ which satisfies both (1) and (3). (2) is equivalent to some Π^0_1 formula and (4) is equivalent to some Σ^0_1 formula. Player I has no winning strategy in $(\flat)$, since for any $k < n$, player I cannot win if player II's answer is correct. By $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)\text{-}\mathsf{Det}^*$, player II has a winning strategy τ in $(\flat)$. We can also see that, $\tau(k^\tau * \langle 1\rangle) = 1$ if and only if $\varphi(k)$ holds for any $k < n$, where k^τ is defined in Notation 3.4.2. Δ^0_1 comprehension, provided by RCA^*_0, yields a set Y such that $k \in Y \leftrightarrow k < n \wedge \tau(k^\tau * \langle 1\rangle) = 1$. Intuitively, Y is the set of natural numbers $k < n$ such that the winning strategy τ tells player II to answer "yes" when she is asked about it. We can check that $\forall k((\varphi(k) \wedge k < n) \leftrightarrow k \in Y)$. □

For the converse, we need some fragments of axiom of choice.

Lemma 4.2.3.

1. RCA^*_0 *proves* Σ^0_1 *axiom of choice.*
2. WKL^*_0 *proves* Π^0_1 *axiom of choice.*

Proof. 1. Let $\varphi(n, Y)$ be a Σ^0_1 formula. Assume $\forall n \exists Y \varphi(n, Y)$. By Lemma 3.6.1, we can find a Π^0_0 formula $\theta(x, y)$ such that RCA^*_0 proves $\forall Y \forall n(\varphi(n, Y) \leftrightarrow \exists m \theta(n, Y[m]))$. Then, by Δ^0_1 comprehension, define $Z = \{\langle n, k\rangle : \theta(n, e(k))\}$, where $e : \mathbb{N} \to 2^{<\mathbb{N}}$ is the canonical enumeration of $2^{<\mathbb{N}}$. By the assumption $\forall n \exists Y \exists m \theta(n, Y[m])$, for each n, there is k with $\langle n, k\rangle \in Z$. Take $f : \mathbb{N} \to \mathbb{N}$ such that $f(n)$ is the least k such that $\langle n, k\rangle \in Z$ holds for all n. Clearly

$$Y = \{\langle m, n\rangle : m < |e(f(n))| \wedge e(f(n))(m) = 1\}$$

enjoys the desired property.
2. We work in WKL^*_0. Let $\varphi(n, Y)$ be a Π^0_1 formula. Assume $\forall n \exists Y \varphi(n, Y)$. As in the previous case, we can find a Π^0_0 formula $\theta(x, y)$ such that RCA^*_0 proves $\forall Y \forall n(\varphi(n, Y) \leftrightarrow \forall m \theta(n, Y[m]))$. For any $s \in 2^{<\mathbb{N}}$, let

$$s_{n,m} = \langle s(\langle n, 0\rangle), s(\langle n, 1\rangle), ..., s(\langle n, m-1\rangle)\rangle.$$

Define T by

$$\{s \in 2^{<\mathbb{N}} : \forall \langle n, m\rangle < |s| \theta_1(n, s_{n,m})\}.$$

Then T is infinite if and only if $\forall n \exists Y \forall m \theta_1(n, Y[m])$. Using weak König's lemma, take a path f of T. Then $Y = \{\langle m, n\rangle : f(\langle m, n\rangle) = 1\}$ enjoys the desired property. □

Theorem 4.2.4. WKL_0 *proves* $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)$-$\mathsf{Det}^*$.

Proof. By Remark 3.6.6, it suffices to prove $\mathsf{Sep}(\Delta^0_1, \Sigma^0_1)$-$\mathsf{Det}^*$ in WKL_0, instead of $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)$-$\mathsf{Det}^*$. Let $\varphi(f)$ be a game of the form $(\psi(f) \wedge \eta_0(f)) \vee (\neg\psi(f) \wedge \eta_1(f))$, where both $\psi(f)$ and $\eta_1(f)$ are Σ^0_1, where $\eta_0(f)$ is Π^0_1, and where $\forall f \in 2^{\mathbb{N}}(\psi(f) \leftrightarrow \psi'(f))$ for some Π^0_1 formula $\psi'(f)$. By Lemma 3.6.1, there are Π^0_0 formulae $\theta_0(x)$ and $\theta_1(x)$ such that $\psi(f) \leftrightarrow \exists n \theta_0(f[n])$ and $\neg\psi(f) \leftrightarrow \exists n \theta_1(f[n])$. By Δ^0_1 comprehension, set a tree $T = \{s \in 2^{<\mathbb{N}} : \forall t \subseteq s(\neg\theta_0(t) \wedge \neg\theta_1(t))\}$. T has no infinite path, otherwise there exists an infinite sequence $f \in 2^{\mathbb{N}}$ such that $\forall n \neg\theta_0(f[n])$ and $\forall n \neg\theta_1(f[n])$, which is impossible. By weak König's lemma, T is a finite tree. Let n_T be the least $n \in \mathbb{N}$ such that $|t| < n$ for all $t \in T$. Note that every $s \in 2^{<\mathbb{N}}$ of length n_T satisfies exactly one of $\exists t \subseteq s \theta_0(t)$ and $\exists t \subseteq s \theta_1(t)$. By Remarks 4.1.5.3 and 4, bounded Σ^0_1 comprehension, which is equivalent to Σ^0_1 induction, provides

$$W_i = \{s \in 2^{\mathbb{N}} : |s| = n_T \wedge \exists t \subseteq s \theta_i(t) \wedge (\text{player I wins } \eta_i(f) \text{ at } s)\} \quad (i < 2).$$

Then we can define a new game $\varphi^*(f) \equiv f[n_T] \in W_0 \cup W_1$. By Δ^0_1-Det^*, which is provided by WKL_0, one of the players has a winning strategy in $\varphi^*(f)$. We show the following claim, which completes the proof of this theorem.

Claim. *The player who wins $\varphi^*(f)$ also wins the original game $\varphi(f)$.*

Proof of the claim. First, assume that player I has a winning strategy σ^* in $\varphi^*(f)$. He can win $\varphi(f)$ by playing as follows. He follows σ till he reaches s of length n_T. By the assumption, s satisfies $s \in W_0 \cup W_1$. If he reaches $s \in W_i$ then he switches the strategy to a winning strategy σ_s in $\eta_i(s * f)$. More precisely, by Remarks 4.1.5.1 and 2 and Lemma 4.2.3, we have a sequence $\langle \sigma_s : s \in W_0 \cup W_1 \rangle$ of strategies for player I such that σ_s is a winning strategy in $\eta_i(s * f)$ for each $s \in W_i$. Now, a new strategy σ for player I in $\varphi(f)$ is defined as follows:

$$\sigma(t) = \begin{cases} \sigma_{t[n_T]}(t_{n_T}) & \text{if } |t| \geq n_T \text{ and } t[n_T] \in W_0 \cup W_1, \\ \sigma^*(t) & \text{otherwise.} \end{cases}$$

It can be checked that, for any strategy τ for player II, $(\sigma \otimes \tau)[n_T] \in W_0 \cup W_1$ and so $\sigma \otimes \tau = \sigma_{f[n_T]} \otimes \tau$. Therefore $\eta_i(\sigma \otimes \tau)$, if $(\sigma \otimes \tau)[n_T] \in W_i$. Thus player I wins $\varphi(f)$.

Next, assume that player II has a winning strategy τ^* in $\varphi^*(f)$. She can win $\varphi(f)$ by playing as follows. She follows τ^* till she reaches s of length n_T. By the assumption, s is not in $W_0 \cup W_1$. Note that s satisfies exactly one of $\exists t \subseteq s\theta_0(t)$ and $\exists t \subseteq s\theta_1(t)$. Also note that if $\exists t \subseteq s\theta_i(t)$ and $s \notin W_i$, then player II wins $\eta_i(f)$ at s by Σ^0_1-Det^*. Then she switches the strategy to a winning strategy τ_s in $\eta_i(s * f)$, if s satisfies $\exists t \subseteq s\theta_i(t)$. More precisely, we have a sequence $\langle \tau_s : |s| = n_T \wedge s \notin W_0 \cup W_1 \rangle$ of strategies for player II such that if $\exists t \subseteq s\theta_i(t)$, then τ_s is a winning s-strategy in $\eta_i(f)$ by Lemma 4.2.3 as in the previous case. Define a strategy τ for player II as follows:

$$\tau(t) = \begin{cases} \tau_{t[n_T]}(t) & \text{if } |t| \geq n_T \text{ and } t[n_T] \notin W_0 \cup W_1, \\ \tau^*(t) & \text{otherwise.} \end{cases}$$

Then we can prove that τ is a winning strategy for player II in $\varphi(f)$ in a similar way to the previous case. □

Finally, we have the following.

Corollary 4.2.5. $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)$ *is equivalent to weak König's lemma plus Σ^0_1 induction over* RCA^*_0.

4.3 ACA_0 and determinacy

Next, we show the equivalence between Σ^0_1 comprehension and the determinacy of Boolean combinations of Σ^0_1 formulae in the Cantor space.

Theorem 4.3.1. ACA_0 *proves* $(\Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}^*$.

Proof. Let φ be a $\Sigma^0_1 \wedge \Pi^0_1$ game. By Lemma 3.6.1, we can find a Π^0_0 formula $\theta(x)$ and a Π^0_1 formula $\psi(f)$ such that RCA^*_0 proves $\varphi(f) \leftrightarrow \exists n\theta(f[n]) \wedge \psi(f)$ for all $f \in 2^{\mathbb{N}}$. By Remark 4.1.5.4, ACA_0 provides the following set:

$$W = \{u \in 2^{<\mathbb{N}} : \exists v \subseteq u\theta(v) \wedge (\text{I wins } \psi(f) \text{ at } u)\}.$$

Define the game $\varphi^*(f)$ in the Cantor space by $\varphi^*(f) \equiv \exists n(f[n] \in W)$ with $f \in 2^{\mathbb{N}}$.

The formula $\varphi^*(f)$ is Σ^0_1, and hence it is determinate by Theorem 4.1.7.

We show the following claim, which completes the proof of this theorem.

Claim The player who wins $\varphi^*(f)$ also wins $\varphi(f)$.

Proof of the claim. First, suppose I has a winning strategy σ^*. By Remark 4.1.5.2 and Lemma 4.2.3.2, we have a sequence $\langle \sigma^*_s : s \in W\rangle$ of winning s-strategies for player I in $\psi(f)$. Define σ as follows:

$$\sigma(t) = \begin{cases} \sigma^*_s(t) & \text{if } s \text{ is the } \subseteq\text{-least initial segment of } t \text{ with } t \in W, \\ \sigma^*(t) & \text{if there is no such } s \subseteq t. \end{cases}$$

We show that σ is a winning strategy for player I in $\varphi(f)$. Since σ^* is a winning strategy in $\varphi^*(f)$, there is n with $f[n] \in W$. Take the least such n. Then for any $2m > n$, $f(2m) = \sigma^*_{f[n]}(f[2m])$ holds, and so $\psi(f)$ holds by the assumption that σ^*_s is a winning $f[n]$-strategy for player I in $\psi(f)$. Since $f[n] \in W$, $\exists n\theta(f[n])$ also holds, and so σ is a winning strategy for player I.

Now, assume that player II has a winning strategy τ^* in $\varphi^*(f)$. ACA_0 yields the following set:

$$W' = \{s \in 2^{<\mathbb{N}} : \exists t \subseteq s\theta(t) \wedge s \notin W\}.$$

Note that, player II wins $\psi(f)$ at each $s \in W'$ by Σ^0_1-Det^*, which is proved in ACA_0. By Remark 4.1.5.2, we have a sequence $\langle \tau^*_s : s \in W'\rangle$ of winning s-strategies for player II in $\psi(f)$. Define a strategy τ for player II by

$$\tau(t) = \begin{cases} \tau^*_s(t) & \text{if } s \text{ is the } \subseteq\text{-least initial segment of } t, \\ \tau^*(t) & \text{if there is no such } s \subseteq t. \end{cases}$$

We show that τ is a winning strategy for player II in $\varphi(f)$. Let f be a play in which player II follows τ. If there is no n with $\theta(f[n])$, then player II wins $\varphi(f)$. Assume that there is n with $\theta(f[n])$. Take the least such n. Since τ^* is a winning strategy in $\varphi^*(f)$, $f[n] \notin W$. Then f is a play in which player II follows $\tau^*_{f[n]}$, and so $\neg\psi(f)$ holds, which means player II wins $\varphi(f)$. Therefore τ is a winning strategy for player II in $\varphi(f)$. □

Theorem 4.3.2. RCA_0^* *proves that* $(\Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}^*$ *implies* Σ^0_1 *comprehension.*

Proof. Assume $(\Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}^*$. Let $\psi(n)$ be a Σ^0_1 formula. We need to construct a set Y such that $\psi(n) \leftrightarrow n \in Y$ for any $n \in \mathbb{N}$.

To construct Y, consider the following game in the Cantor space: Player I constructs n to ask whether $\psi(n)$ holds or not, and player II answers yes or no. Player II wins if one of the following cases holds. Such a game $\varphi(f)$ can be realized as follows:

- Player I chooses $n \in \mathbb{N}$ by playing 0 at his first n turns and playing 1 at his $(n+1)$-th turn. If he plays 0 for ever, he loses.
- Player II answers either 0 (no) or 1 (yes).
- Player I wins if either (1) player II chooses 0 and $\psi(n)$ holds or (2) player II chooses 1 and $\neg\psi(n)$ holds.

Formally, $\varphi(f)$ is defined as follows:

$$\exists n(\forall k < n(f(2k) = 0) \wedge f(2n) = 1 \wedge (f(2n+1) = 0 \rightarrow \psi(n)))\wedge$$
$$\forall n((\forall k < n(f(2k) = 0) \wedge f(2n) = 1 \wedge f(2n+1) = 1) \rightarrow \neg\psi(n))$$

Thus, $\varphi(f)$ is $\Sigma^0_1 \wedge \Pi^0_1$. We show that player I has no winning strategy. For contradiction, assume that player I has a winning strategy σ in $\varphi(f)$. Consider such a play f: Player I follows σ and $f(2n+1) = 0$ if $f(2n) = 0$. Note that there exists n with $f(2n) = 1$, otherwise player I loses despite of following σ. Take the least such n. By the assumption, player I wins both the cases $f(2n+1) = 0$ and $f(2n+1) = 1$, however, this means both $\psi(f)$ and $\neg\psi(f)$ holds, which is a contradiction.

By $(\Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}^*$, player II has a winning strategy τ in $\varphi(f)$. Recall that n^τ is a finite play of length $2n$ such that player I plays 0 at his each turn and II plays following τ.

We defined a set Y by:

$$n \in Y \leftrightarrow \tau(n^\tau * \langle 1 \rangle) = 1.$$

Namely, Y is a set of $n \in \mathbb{N}$ such that player II answers "yes" when she is asked about n, and Y exists by Π^0_0 comprehension. Moreover, we can verify that $\forall n(\varphi(n) \leftrightarrow n \in Y)$, which completes the proof. □

Corollary 4.3.3. $(\Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}^*$ *is equivalent to* Σ^0_1 *comprehension over* RCA^*_0.

4.4 ATR_0 and determinacy

In this section we show that Δ^0_2-Det^*, Σ^0_2-Det^* and arithmetical transfinite recursion are pairwise equivalent.

The following game theoretic characterization of arithmetical transfinite recursion was given by Steel [31] as one of the earliest results of reverse mathematics.

Lemma 4.4.1. *The following are pairwise equivalent over* RCA^*_0*:*

$$\Delta^0_1\text{-}\mathsf{Det},\ \Sigma^0_1\text{-}\mathsf{Det},\ \textit{and arithmetical transfinite recursion.}$$

Proof. See [30, section V.8] or [32]. Note that the original proof are done over RCA_0, they work also on RCA^*_0. □

To consider the Cantor space version of this theorem, we need some definitions and notations.

Definition 4.4.2. A finite sequence $s \in 2^{<\mathbb{N}}$ is *regular* if:

(r1) $s = \langle\rangle$ or $|s| > 1 \wedge s(|s|-1) \neq s(|s|-2)$,

(r2) for all $i < 2$, $2k+i < |s|-1$ and $s(2k+i) = i$ imply $s(2k+i+1) = i$.

If $s \in 2^{<\mathbb{N}}$ is a non-empty regular sequence, $\overline{s} = \langle n_0, ..., n_k \rangle \in \mathbb{N}^{<\mathbb{N}}$ is the sequence such that

$$s = \langle \underbrace{0, ..., 0}_{2n_0 \text{ times}}, 1, \underbrace{1, ..., 1}_{2n_1 \text{ times}}, 0, ... \underbrace{i_k, ..., i_k}_{2n_k \text{ times}}, 1 - i_k \rangle,$$

where i_k is 0 if k is even, $i_k = 1$ otherwise. We set $\overline{\langle\rangle} = \langle\rangle$, for convenience. The assertion "s is regular" is equivalent to a Π^0_0 formula over RCA^*_0.

An infinite sequence $f \in 2^{\mathbb{N}}$ is *totally regular* if it satisfies

(r2') for all $i < 2$ and k, $f(2k+i) = i$ implies $f(2k+i+1) = i$,

(r3) for all $i < 2$ and k, if $f(k) = i$, then there exists $m > k$ with $f(m) = 1 - i$.

For a totally regular sequence $g \in 2^{\mathbb{N}}$, $\overline{g} = \langle n_0, n_1, n_2, ...\rangle \in \mathbb{N}^{\mathbb{N}}$ is the sequence such that

$$g = \langle \underbrace{0, ..., 0}_{2n_0 \text{ times}}, 1, \underbrace{1, ..., 1}_{2n_1 \text{ times}}, 0, \underbrace{0, ..., 0}_{2n_2 \text{ times}}, 1, ...\rangle.$$

The assertion "f is totally regular" is equivalent to a Π^0_2 formula over RCA^*_0.

The following lemma shows that games in the Baire space can be translated to ones in the Cantor space.

Lemma 4.4.3. *The following is proved in* RCA^*_0*: For any* Σ^0_n *(or* Π^0_n*) game* $\varphi(f)$ *in the Baire space, there is a game* $\varphi^*(f)$ *in the Cantor space such that*

1. *Player I wins* $\varphi(f)$ *if and only if player I wins* $\varphi^*(f)$*,*

2. *Player II wins* $\varphi(f)$ *if and only if player II wins* $\varphi^*(f)$*.*

Moreover, $\varphi^*(f)$ *is of the form* $(\varphi_0(f) \wedge \varphi_1(f)) \vee \varphi_2(f)$*, where* $\varphi_0(f)$ *is* Π^0_2*, where* $\varphi_1(f)$ *is* Σ^0_n *(resp.* Π^0_n*) and where* $\varphi_2(f)$ *is* Σ^0_2*.*

Proof. We only prove the case $\varphi(f)$ is Σ^0_1, because other cases can be proved similarly. Let $\varphi(f)$ be a Σ^0_1 game in the Baire space. By 3.6.1, we can find a Σ^0_0 formula $\theta(x)$ such that $\varphi(f) \leftrightarrow \exists n \theta(f[n])$.

We consider a game in the Cantor space as follows: Instead of playing in the Baire space, players I and II construct a totally regular sequence $f \in 2^{\mathbb{N}}$. The first player who gives up making f totally regular loses. If they succeed to construct totally regular f, then player I wins if and only if $\varphi(\overline{f})$.

To give a rigorous formulation to this idea, first let

$$\begin{aligned}
\psi_0(k,i,f) &\equiv f(2k+i) = i \rightarrow f(2k+i+1) = i,\\
\psi_1(i,f) &\equiv \exists n(\forall k < n \forall j < 2\ \psi_0(k,j,f) \wedge \\
&\qquad\qquad \forall l < (1-i)\psi_0(n,l,f) \wedge \neg\psi_0(n,1-i,f)),\\
\psi_2(i,f) &\equiv \forall n \forall j < 2 \psi_0(n,j,f) \wedge \exists m \forall l > m(f(2l+i) = i),\\
\psi(i,f) &\equiv \psi_1(i,f) \vee \psi_2(i,f).
\end{aligned}$$

Intuitively, $\psi_1(i,f)$ means that "player I (or II) first breaks **(r2')** of Definition 4.4.2" if $i = 0$ (resp. $i = 1$), $\psi_2(i,f)$ means that "both players keep **(r2')** but player I (or II) breaks **(r3)**" if $i = 0$ (resp. $i = 1$), and $\psi(i,f)$ means that "player I (or II) first gives up making f totally regular" if $i = 0$ (resp. $i = 1$). Let $\xi_0(x)$ be a Π^0_0 formula which asserts that x is regular and $\xi_1(f)$ a Σ^0_2 formula which asserts that f is totally regular.

Define $\varphi^*(f)$ by $(\xi_1(f) \wedge \exists n(\xi_0(f[n]) \wedge \theta(\overline{f[n]})) \vee \psi(1,f))$. It is easy to check that $\varphi^*(f)$ enjoys the desired properties. □

Theorem 4.4.4. RCA^*_0 *proves that* Δ^0_2-Det^* *implies arithmetical transfinite recursion.*

Proof. By Lemma 4.4.1, it is enough to show that RCA^*_0 proves that Δ^0_2-Det^* implies Δ^0_1-Det. Let $\varphi(f)$ be a Σ^0_1 game in the Baire space. Assume that there is a Σ^0_1 formula $\psi(f)$ such that $\varphi(f) \leftrightarrow \neg\psi(f)$ for all $f \in \mathbb{N}^{\mathbb{N}}$. By Theorem 3.6.1, there exist Π^0_0 formulae $\theta_0(x)$ and $\theta_1(x)$ such that RCA^*_0 proves $\varphi(f) \leftrightarrow \exists n\theta_0(f[n])$ and $\neg\varphi(f) \leftrightarrow \exists n\theta_1(f[n])$ for all $f \in \mathbb{N}^{\mathbb{N}}$. We define a game $\varphi^*(f)$ such that the player who wins $\varphi^*(f)$ also wins $\varphi(f)$. Although the basic idea to define the following $\varphi^*(f)$ is the same as in the proof of Lemma 4.4.3, to make the complexity of $\varphi^*(f)$ as low as possible, we define $\varphi^*(f)$, using $\xi_0(x)$, $\xi_1(f)$ and $\psi(i,f)$ in the proof of Lemma 4.4.3, define as follows:

$$\varphi^*(f) \equiv \exists n\theta'_0(f[n]) \vee (\neg\exists n\theta'_0(f[n]) \wedge \neg\exists n\theta'_1(f[n]) \wedge \psi(1,f)),$$

where $\theta'_i(x) \equiv \xi_0(x) \wedge \theta_i(\overline{x})$. The intuitive idea for $\varphi^*(f)$ is as follows: If players construct a regular sequence s with $\theta_0(\overline{s})$, then player I wins. If players construct a regular sequence s with $\theta_1(\overline{s})$, then player II wins. Otherwise, the first player who gives up to make f totally regular loses. It is easy to check that $\varphi^*(f)$ is equivalent to a Σ^0_2 formula.

Since for each $g \in \mathbb{N}^{\mathbb{N}}$, one of $\exists n\theta_0(g[n])$ or $\exists n\theta_1(g[n])$ holds, for any $f \in 2^{\mathbb{N}}$, if both $\neg\exists n\theta'_0(f[n])$ and $\neg\exists n\theta_1(f[n])$ hold, then one of the players gives up to make f totally regular, namely, exactly one of $\psi(0,f)$ or $\psi(1,f)$ holds. Thus, for all $f \in 2^{\mathbb{N}}$, the following holds:

$$\neg\varphi(f) \leftrightarrow \exists n\theta'_1(f[n]) \vee (\neg\exists n\theta'_0(f[n]) \wedge \neg\exists n\theta'_1(f[n]) \wedge \psi(0,f)).$$

Therefore Δ^0_2-Det^* proves the determinacy of $\varphi^*(f)$. It is easy to check that player who wins $\varphi^*(f)$ also wins $\varphi(f)$. Hence $\varphi(f)$ is determinate. □

Remark 4.4.5.

1. For any Π^0_2 game $\varphi(f)$ in the Cantor space, the assertion "σ is a winning s-strategy for player I (or II) in $\varphi(f)$ (resp. $\neg\varphi(f)$)" is equivalent to a Π^0_2 formula over WKL^*_0. Let $\varphi(f)$ be a Σ^0_2 formula. By Lemma 3.6.1, we can find a Π^0_0 formula $\theta(x, y)$ such that $\varphi(f) \leftrightarrow \forall n\exists m\theta(n, f[m])$ for all $f \in 2^{\mathbb{N}}$. The assertion can be written as $\forall\tau \in \mathcal{S}^{\{0,1\}}_{\mathrm{II}}\forall n\exists m\theta(n, (\sigma \otimes \tau)[m])$, which is equivalent to $\forall n\forall\tau \in \mathcal{S}^{\{0,1\}}_{\mathrm{II}}\exists m\theta(n, (\sigma \otimes \tau)[m])$. The part "$\forall\tau \in \mathcal{S}^{\{0,1\}}_{\mathrm{II}}\exists m\theta(n, (\sigma \otimes \tau)[m])$" is equivalent to a Σ^0_1 formula by Lemma 4.1.2, and so the whole statement is Π^0_2.

2. For any Σ^0_2 game $\varphi(f)$ in the Cantor space, the assertion "σ is a winning s-strategy for player I (or II) in $\varphi(f)$ (resp. $\neg\varphi(f)$)" is equivalent to a Π^1_1 formula over RCA^*_0. This is straightforward.

3. For any Π^0_2 game $\varphi(f)$ in the Cantor space, the assertion "player I (or II) has a winning strategy in $\varphi(f)$ (resp. $\neg\varphi(f)$)" is equivalent to a Σ^1_1 formula over WKL^*_0. This follows from 1.

Now we turn to show that ATR_0 proves Σ^0_2-Det^*. Our proof uses a technique called the *method of pseudo-hierarchies.*

Lemma 4.4.6 (existence of pseudo-hierarchies)**.** *The following is provable in* ACA_0*. Let $\theta(n, Z)$ be an arithmetical formula and $\varphi(Y, Z)$ be a Σ^1_1 formula. Let $\mathrm{LO}(Y, <_Y)$ is an arithmetical formula which asserts the linear-orderedness of $(Y, <_Y)$. Recall that $\mathrm{WO}(Y, <_Y)$ is the Π^1_1 formula which asserts the well-orderedness of $(Y, <_Y)$ and that $(Z)^j = \{\langle m, i\rangle \in Z : i <_Y j\}$. If*

$$\forall Y(\mathrm{WO}(Y, <_Y) \to \exists Z(\forall j\forall k(\langle k, j\rangle \in Z \leftrightarrow \theta(k, (Z)^j)) \wedge \varphi(Y, Z))),$$

then

$$\exists Y\exists Z(\mathrm{LO}(Y, <_Y) \wedge \neg\mathrm{WO}(Y, <_Y) \wedge \forall j\forall k(\langle k, j\rangle \in Z \leftrightarrow \theta(k, (Z)^j)) \wedge \varphi(Y, Z)).$$

Proof. See Lemma V.4.12 of [30]. □

Theorem 4.4.7. ATR_0 *proves* Σ^0_2-Det^*.

Proof. We work in ATR_0. Let $\varphi(f)$ be a Σ^0_2 formula. By Lemma 3.6.1, we can find a Π^0_0 formula $\theta(x, y)$ such that RCA^*_0 proves $\forall f(\varphi(f) \leftrightarrow \exists n\forall m\theta(n, f[m]))$. For $f \in 2^{\mathbb{N}}$ and $s \in 2^{<\mathbb{N}}$, we define $\psi(n, X, f)$ and $\theta'(s, X)$ as follows:

$$\psi(n, X, f) \equiv \forall m(\theta(n, f[m]) \vee f[m] \in X),$$
$$\theta'(s, X) \equiv \exists n(\text{player I wins } \psi(n, X, f) \text{ at } s).$$

By Remark 4.1.5.4, $\theta'(s, X)$ is a Σ^0_2 formula. Thus, by arithmetical transfinite recursion, there exists a set $\langle W_x : x \in Y \rangle$ for each countable well-ordering $(Y, <_Y)$ such that

$$\forall j \in Y \forall s \in 2^{<\mathbb{N}} (s \in W_j \leftrightarrow \theta'(s, W_{<_Y j})), \tag{$*$}$$

where $W_{<_Y j} = \{s \in 2^{<\mathbb{N}} : \exists i <_Y j (s \in W_i)\}$.

Now, we consider the following two cases.

Case 1. There exists a countable well ordering $(Y, <_Y)$ and a sequence $\langle W_j : j \in Y \rangle$ such that $(*)$ holds and in addition $\langle \rangle \in \bigcup_{x \in Y} W_x$.

Fix such a countable well-ordering $(Y, <_Y)$. Then define $h_1 : \bigcup_{x \in W} W_x \to W$ and $h_2 : \bigcup_{x \in \mathrm{W}} W_x \to \mathbb{N}$ by

$$\begin{aligned} h_1(s) &= \text{the } <_Y\text{-least } l \text{ with } s \in W_l, \\ h_2(s) &= \text{the least } n \text{ such that player I wins } \psi(n, W_{<_Y h_1(s)}, f). \end{aligned}$$

By Remark 4.1.5.2 and Lemma 4.2.3.2, we have a sequence $\langle \sigma_s : s \in \bigcup_{x \in Y} W_x \rangle$ of winning s-strategies for player I in $\psi(h_2(s), W_{<_Y h_1(s)}, f)$. Define $h_3 : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ by

$$h_3(\langle \rangle) = \langle \rangle,$$

$$h_3(s * \langle i \rangle) = \begin{cases} s * \langle i \rangle & \text{if } h_1(s * \langle i \rangle) \text{ is defined and } h_1(s * \langle i \rangle) <_Y h_1(h_3(s)), \\ h_3(s) & \text{otherwise.} \end{cases}$$

It can be proved that $h_3(s) \in \bigcup_{x \in Y} W_x$ for all s by induction on $|s|$. Define a strategy σ for player I by

$$\sigma(s) = \sigma_{h_3(s)}(s).$$

The intuitive idea for σ is as follows: $h_3(s)$ tells player I which strategy to follow at s. First, player I follows $\sigma_{\langle \rangle}$. If he reaches s with $h_1(s) <_Y h_1(\langle \rangle)$, then he switches the strategy to σ_s. Similarly, when he reaches u with $h_1(u) <_Y h_1(t)$, following σ_t, then he switches the strategy to σ_u.

We show that σ is a winning strategy for player I. Let f be a play in which player I follows σ. Since $h_3(f[n]) \in \bigcup_{x \in Y} W_x$ for all n, $h_1(h_3(f[n]))$ is defined for all n. By the definitions of h_1 and h_3, $h_1(h_3(f[n+1])) \leq_Y h_1(h_3(f[n]))$ for all n. Since $(Y, <_Y)$ is a well-ordering, there exists n such that $h_1(h_3(f[n])) = h_1(h_3(f[n+k]))$ for all k. We can check that if $h_1(h_3(f[n])) = h_1(h_3(f[n+k]))$, then $h_3(f[n]) = h_3(f[n+k])$ for all n and k, which means that player I switches strategies only finite times. Let n be the least n such that $h_1(h_3(f[n])) = h_1(h_3(f[n+k]))$ for all k. Then $h_3(f[n]) = f[n]$ and $\sigma(f[2k]) = \sigma_{f[n]}(f[2k])$ holds for all $2k \geq n$, and so

$$\psi(h_2(f[n]), W_{<_Y h_1(f[n])}, f) \equiv \forall m(\theta(h_2(f[n]), f[m]) \vee f[m] \in W_{<h_1(f[n])})$$

holds. Note that $f[n+k] \notin W_{h_1(f[n])}$, since if $f[n+k] \in W_{h_1(f[n])}$ for some k, then $h_3(f[n+k]) <_Y h_3(f[n])$. Therefore $\forall m\theta(h_2(f[n]), f[m])$ holds, and so player I wins $\varphi(f)$.

Case 2. Assume that the hypothesis of case 1 does not holds. Since we are working in ATR_0, we have

$$\forall Y(\mathrm{WO}(Y, <_Y) \to (\ \exists\langle W_j : j \in Y\rangle \text{ such that } (*) \text{ holds and } \langle\rangle \notin \bigcup_{x\in Y} W_x)).$$

By Lemma 4.4.6, there exists a non-well-founded linear ordering $(Y, <_Y)$ and a sequence $\langle W_j : j \in Y\rangle$ such that $(*)$ holds and in addition $\langle\rangle \notin \bigcup_{x\in Y} W_x$. Let $g : \mathbb{N} \to Y$ be a fixed descending chain, i. e., $g(k+1) <_Y g(k)$ for all k. Define $h_4 : 2^{\mathbb{N}} \to \mathbb{N}$ by

$$h_4(\langle\rangle) = 0,$$
$$h_4(s * \langle i\rangle) = \begin{cases} h_4(s) + 1 & \text{if there is } t \subseteq s * \langle i\rangle \text{ with } \neg\theta(h_4(s), t), \\ h_4(s) & \text{otherwise.} \end{cases}$$

Note that if $h_4(s) = n$, then for each $m < n$, there exists $t \subseteq s$ with $\neg\theta(m, t)$.

By Σ^0_1 determinacy, for any s and $x \in Y$, $s \notin W_x$ implies that, for any $n \in \mathbb{N}$, player II wins $\psi(n, W_{<_Y x}, f)$ at s. In particular, if $s \notin W_{g(h_4(s))}$ and if $h_4(s) \geq 0$, then player II wins $\psi(h_4(s), W_{<_Y g_4(h_4(s))}, f)$ at s.

By Remark 4.1.5.1, and Lemma 4.2.3.1, we have a sequence $\langle \tau_s : s \notin W_{g(h_4(s))}\rangle$ of winning s-strategies for player II in $\psi(h_4(n), W_{<g(h_4(s))}, f)$. Define $h_5 : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ by

$$h_5(s) = \text{the } \subseteq\text{-least initial segment } t \text{ of } s \text{ with } h_4(t) = h_4(s).$$

It can be proved that $h_5(s) \notin \bigcup_{x\in Y} W_{g(h_4(h_5(s)))}$ by induction on $|s|$.

Define a new strategy τ for player II by

$$\tau(s) = \tau_{h_5(s)}(s).$$

The intuitive idea for τ is as follows: $h_5(s)$ tells player II which strategy to follow at s. By the assumption, $\langle\rangle \notin W_{g(0)}$, and so player II follows

$\tau_{\langle\rangle}$ till she reaches the $\subseteq$-least s_0 with $\neg\theta(0, s_0) \wedge s_0 \notin W_{<g(0)}$. Note that $h_4(s_0) = 1$. Since $s_0 \notin W_{<g(1)} \subseteq W_{g(0)}$, she switches the strategy to a winning s_0-strategy τ_{s_0} for $\psi(1, W_{<g(1)}, f)$. Similarly, if player II reaches with $\neg\theta(h_4(s) - 1, s) \wedge s \notin W_{<g(h_4(s)-1)}$, she switches the strategy to a winning s-strategy τ_s in $\psi(h_4(s), W_{g(h_4(s))}, f)$. Note that then $h_5(s) = s$.

We show that τ is a winning strategy for player II in $\varphi(f)$. Let f be a play in which player II follows τ. We see that, for any n, there exists m with $\exists l < m \neg\theta(n, f[l]) \wedge f[m] \notin W_{<g(n)}$, $h_4(f[m]) = n + 1$ and $h_5(f[m]) = f[m]$ by induction as follows. Recall that $\langle\rangle \notin \bigcup_{x \in Y} W_x$. For $n = 0$, if $\theta(0, \langle\rangle)$, then $f[1]$ enjoys the desired property. Otherwise, since $\langle\rangle \notin W_{g(0)}$ and $h_4(\langle\rangle) = 0$, player II first follows a winning $\langle\rangle$-strategy $\tau_{\langle\rangle}$ in $\psi(0, W_{<g(0)}, f)$. Since, for each t, if there is no $u \subseteq t$ with $\neg\theta(0, u)$, then $h_4(t) = 0$, she follows $\sigma_{\langle\rangle}$ till she reaches $f[m]$ with $\neg\theta(0, f[m]) \wedge f[m] \notin W_{<g(0)}$. Note that such m must be exists since $\sigma_{\langle\rangle}$ is a winning $\langle\rangle$-strategy for player II in $\psi(0, W_{g(0)}, f)$. Also note that the least such m satisfies $h_4(f[m]) = 1$ and $h_5(f[m]) = f[m]$. For $n = k$, assume that m is the least natural number with $\exists l < m \neg\theta(k, f[m]) \wedge f[m] \notin W_{<g(k)}$, $h_4(f[m]) = k + 1$ and $h_5(f[m]) = f[m]$. Then she follows a winning $f[m]$-strategy $\tau_{f[m]}$ for her in $\psi(n + 1, W_{g(k+1)})$. By a similar argument to the case $n = 0$, she follows $\tau_{f[m]}$ till she reaches $f[m']$ with $\exists l < m' \neg\theta(k + 1, f[l]) \wedge f[m'] \notin W_{<g(k+1)}$, $h_4(f[m']) = k + 2$, and $h_5(f[m']) = f[m']$. By Σ^0_0 induction on n, there exists m with $\exists l < m \neg\theta(n, f[l]) \wedge f[m] \notin W_{<g(n)}$, $h_4(f[m]) = n + 1$ and $h_5(f[m]) = f[m]$, which means, in particular, player II wins $\varphi(f)$. □

Remark 4.4.8. For any Σ^0_2 game $\varphi(f)$ in the Cantor space, the assertion "player I (or II) wins $\varphi(f)$ (resp. $\neg\varphi(f)$) at s" is equivalent to a Π^1_1 formula over ATR_0. By Remark 4.4.5.3, the assertion "player II wins $\varphi(f)$ at s" can be written as a Σ^1_1 formula ψ. By Σ^0_2-Det^*, which is provided by ATR_0, the aimed assertion is equivalent to $\neg\psi$ and is Π^1_1.

Finally, we give the following corollary.

Corollary 4.4.9. *The following are equivalent over* RCA_0.

- Δ^0_2-Det^*,
- Σ^0_2-Det^*,
- Δ^0_1-Det,

- Σ^0_1-Det,
- *arithmetical transfinite recursion.*

Proof. This follows from Lemma 4.4.1 and Theorems 4.4.4 and 4.4.7. □

4.5 $\mathsf{ATR}_0 + \Sigma^1_1$ induction and determinacy

In this section, we consider the system $\mathsf{ATR}_0 + \Sigma^1_1$ induction and determinacy in the Cantor space.

Remark 4.5.1.

1. [30, Corollary IX.4.7] shows that $\mathsf{ATR}_0 + \Sigma^1_1$ induction proves the consistency of ATR_0. This means that Σ^1_1 induction really strengthens ATR_0.

2. $\mathsf{ATR}_0 + \Pi^1_1$ transfinite induction proves the consistency of $\mathsf{ATR}_0 + \Sigma^1_1$ induction, since [30, Theorem VIII.3.15] shows that ATR_0 can be axiomatized by ACA_0 plus a single Π^1_2 formula and since [30, Theorem VIII.5.12] shows that $\mathsf{ATR}_0 + \Pi^1_1$ transfinite induction implies the existence of a countable coded ω-model of ATR_0 and hence the consistency of ATR_0+ full induction.

We have the following equivalence on Σ^1_1 induction.

Lemma 4.5.2 ([28, 2.1 Lemma]). *Σ^1_1 induction is equivalent to bounded Σ^1_1 comprehension over ACA_0.*

[28] shows that Σ^1_1 induction is equivalent to Σ^1_1 transfinite induction over ATR_0.

In the proof of Theorem 4.2.4, we used Σ^0_1 and Π^0_1 axioms of choice. By Remark 4.4.5, for a given Σ^0_2 game $\varphi(f)$, even if we know that player I wins at each $s \in W$, it seems that we need Π^1_1 axiom of choice to take a sequence $\langle \sigma_s : s \in W\rangle$ of winning s-strategies for player I, which is not proved in $\mathsf{ATR}^*_0 + \Sigma^1_1$ induction. The following lemma proves that actually we do not need it.

Lemma 4.5.3. *Let $1 < n < \omega$. Let $\varphi(x, f)$ be a Σ^0_n game in the Cantor space. Assume that player I (or player II) wins $\varphi(x, f)$ at s for all $\langle s, x\rangle \in W \subseteq 2^{<\mathbb{N}} \times \mathbb{N}$. Then, RCA^*_0 proves that Σ^0_n-Det^* yields a sequence $\langle \sigma_{s,x} : \langle s, x\rangle \in W\rangle$ of winning s-strategies for player I (resp. II) in $\varphi(x, f)$.*

Proof. Assume Σ^0_n-Det^*. Let $\varphi(x, f)$ be a Σ^0_n game in $2^{\mathbb{N}}$ and $W \subseteq 2^{<\mathbb{N}} \times \mathbb{N}$. Assume that player I wins $\varphi(f)$ at each $s \in W$. Fix an enumeration $e : \mathbb{N} \to 2^{<\mathbb{N}} \times \mathbb{N}$. For any m, let $\langle m_0, m_1\rangle = e(m)$. Consider the following game $\varphi'(f)$:

First, player II chooses $\langle s, x\rangle \in 2^{<\mathbb{N}}$. If $\langle s, x\rangle \in W$, players starts game $\varphi(x, g)$ from s and player I wins when $\varphi(x, g)$ holds. Otherwise, player II wins.

Let f be a play. Such a game is realized as follows:

- Player II choose $m \in \mathbb{N}$ at by playing 0 at her first m turns and playing 1 at her $(n + 1)$-th turn. If $e(m) \notin W$, player I wins.
- If $e(m) \in W$ and $|m_0|$ is even, then player I wins if $\varphi(m_1, m_0 * f')$, where f' is defined by $f'(k) = f(k + (2m + 2))$.
- If $e(m) \in W$ and $|m_0|$ is odd, then player I wins if $\varphi(m_1, m_0 * f'')$, where f'' is defined by $f''(k) = f(k + (2m + 3))$.

Formally, $\varphi'(f)$ is defined as follows:

$$\begin{aligned}\varphi'(f) \equiv\ & \forall m \neg (\forall i < m)(f(2i + 1) = 0 \land f(2m + 1) = 1 \land e(m) \in W) \lor \\ & \{\exists m \exists k((\forall i < m)(f(2i + 1) = 0 \land f(2m + 1) = 1) \land \\ & \quad ((|m_0| = 2k \land \varphi(m_1, f')) \lor (|m_0| = 2k + 1 \land \varphi(m_1, f''))))\}.\end{aligned}$$

Since $n > 1$, Σ^0_n-Det^* proves that one of the players has a winning strategy in $\varphi'(f)$. We see that player II has no winning strategy in $\varphi'(f)$ as follows. For contradiction, suppose that player II had a winning strategy τ. Consider such a play f:

- Player I plays 0 until player II plays 1.
- Player II follows τ.

Note that player II must play 1 at some turn, i. e., $f(2m + 1) = 1$ for some m, and $e(m) \in W$, otherwise player II loses. If τ were a winning strategy, then it would yield a winning m_0-strategy for player II in $\varphi(m_1, f)$. Since player I wins $\varphi(m_1, f)$ at m_0, this is impossible. Hence player I has a winning strategy σ in $\varphi'(f)$.

For $\langle s, x\rangle \in 2^{<\mathbb{N}} \times \mathbb{N}$, let $\overline{e}(s, x)$ be the least k with $e(k) = \langle s, x\rangle$ and s_x the $(2\overline{e}(s, x) + 2)$-length sequence t such that $t(2k + 1) = 0$ for each $2k + 1 <$

$2\overline{e}(s,x)$, $t(2\overline{e}(s,x)+1)=1$, and $t(2k)=\sigma(t[2k])$ for each $2k<2\overline{e}(s,x)+2$. In other words, s_x is the finite play in which player II has just chosen a sequence $\langle s,x\rangle \in 2^{\mathbb{N}}\times\mathbb{N}$ and player I followed σ.

Then, for $\langle s,x\rangle \in W$, define an s-strategy $\sigma_{s,x}$ in $\varphi(x,s)$ for player I by $\sigma_{s,x}(s*t)=\sigma(s_x*t)$ for even-length $s\in W$ and $\sigma_{s,x}(s*t)=\sigma(s_x*\langle\sigma(s_x)\rangle*t)$ for odd-length $s\in W$. Clearly $\sigma_{s,x}$ is a winning s-strategy for player I in $\varphi(x,f)$ for each $\langle s,x\rangle\in W$.

The assertion for player II can be proved similarly. □

The above lemma can be extended to Σ^0_1 game. Assume that Σ^0_1-Det^* and that player I (or II) wins a Σ^0_1 game $\varphi(x,f)$ at each $\langle s,x\rangle\in W$. Since Σ^0_1-Det^* implies weak König's lemma over RCA^*_0, it implies Σ^0_1 axiom of choice by Lemma 4.2.3. Then, by Remark 4.1.5.1, we have a sequence of $\langle\sigma_s:\langle s,x\rangle\in W\rangle$ of winning s-strategies for player I in $\varphi(x,s)$. The assertion for player II can be proved similarly.

Theorem 4.5.4. $\mathsf{ATR}_0+\Sigma^1_1$ *induction proves* $\mathsf{Bisep}(\Delta^0_1,\Sigma^0_2)$-$\mathsf{Det}^*$.

Proof. By Remark 3.6.6, it suffices to prove $\mathsf{Sep}(\Delta^0_1,\Sigma^0_2)$-$\mathsf{Det}^*$ in $\mathsf{ATR}_0+\Sigma^1_1$ induction. Let $\varphi(f)$ be a game of the form $(\exists n\theta_0(f[n])\wedge\eta_0(f))\vee(\exists n\theta_1(f[n])\wedge\eta_1(f))$, where $\theta_0(x)$ and $\theta_1(x)$ are Π^0_0 such that $\forall f(\exists n\theta_0(f[n])\leftrightarrow\neg\exists n\theta_1(f[n]))$, where $\eta_0(f)$ is Π^0_2, and where $\eta_1(f)$ is Σ^0_2. As in the proof of Theorem 4.2.4, define a tree $T=\{s\in 2^{<\mathbb{N}}:\forall t\subseteq s(\neg\theta_0(t)\wedge\neg\theta_1(t))\}$ and take the least n_T with $|s|<n_T$ for all $s\in T$. For each $i<2$, bounded Π^1_1 comprehension with help of Remarks 4.4.5.3 and 4.4.8 yields

$$W_i=\{s\in 2^{<\mathbb{N}}:|s|=n_T\wedge\exists t\subseteq s\theta_i(t)\wedge\text{player I wins }\eta_i(f)\text{ at }s\},$$
$$W'_i=\{s\in 2^{<\mathbb{N}}:|s|=n_T\wedge\exists t\subseteq s\theta_i(t)\wedge s\notin W_i\}.$$

Note that, by Σ^0_1-Det^*, player II wins $\eta_i(f)$ at $s\in W'_i$.

Define a new Δ^0_1 game $\varphi^*(f)\equiv f[n_T]\in W_0\cup W_1$. By Δ^0_1-Det^*, it is determinate.

Claim. *The player who wins $\varphi^*(f)$ also wins the original game $\varphi(f)$.*

Proof of the claim. First, assume that player I has a winning strategy σ^* in $\varphi^*(f)$. By Lemma 4.5.3, we have sequences $\langle\sigma_s:s\in W_0\rangle$ of winning s-strategies for player I in $\eta_0(f)$ and $\langle\sigma_s:s\in W_1\rangle$ of winning s-strategies for player I in $\eta_1(f)$.

Define a strategy σ for player I in $\varphi(f)$:

$$\sigma(s) = \begin{cases} \sigma_{s[n_T]}(s) & \text{if } |s| \geq n_T \text{ and } s[n_T] \in \bigcup_{i<2} W_i, \\ \sigma^*(s) & \text{otherwise.} \end{cases}$$

Then, for any strategy τ for player II in $\varphi(f)$, $(\sigma \otimes \tau)[n_T] \in W_0 \cup W_1$ holds and, for any $i < 2$, if $(\sigma \otimes \tau)[n_T] \in W_i$, then $\eta_i(\sigma \otimes \tau)$ holds, and so $\varphi(\sigma \otimes \tau)$.

Next, assume that player II has a winning strategy τ^* in $\varphi^*(f)$. Note that $\varphi^*(f) \leftrightarrow \neg(f[n_T] \in W_0' \cup W_1')$. As in the previous case, we have a sequences $\langle \sigma_s : s \in W_0' \rangle$ of winning s-strategies for player II in $\eta_0(f)$ and $\langle \sigma_s : s \in W_1' \rangle$ of winning s-strategies for player II in $\eta_1(f)$ by Lemma 4.5.3. Then, in a similar way to the previous case, it can be proved that the strategy τ for player II defined by

$$\tau(s) = \begin{cases} \tau_{s[n_T]}(s) & \text{if } |t| \geq n_t \text{ and } t[n_T] \in \bigcup_{i<2} W_i', \\ \tau^*(s) & \text{otherwise,} \end{cases}$$

is a winning strategy for player II in the original game $\varphi(f)$. □

Lemma 4.5.5 (normal form theorem 2, [30, Theorem V.1.4])**.** *For a Σ^1_1 formula $\varphi(X)$, we can find a Π^0_0 formula $\theta(X, f)$ such that ACA_0 proves $\forall Y(\varphi(Y) \leftrightarrow \exists g \in \mathbb{N}^{\mathbb{N}} \forall m \theta(Y[m], g[m]))$.*

Theorem 4.5.6. *RCA_0^* proves that $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_2)\text{-}\mathsf{Det}^*$ implies Σ^1_1 induction.*

Proof. Assume that $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_2)\text{-}\mathsf{Det}^*$. By Lemma 4.5.2, it suffices to prove bounded Σ^1_1 comprehension. Since RCA_0^* proves that $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_2)\text{-}\mathsf{Det}^*$ implies $(\Sigma^0_1 \wedge \Pi^0_1)\text{-}\mathsf{Det}^*$ and that $(\Sigma^0_1 \wedge \Pi^0_1)\text{-}\mathsf{Det}^*$ implies Σ^0_1 comprehension, we can work in ACA_0.

Let $\varphi(x)$ be a Σ^1_1 formula. By Lemma 4.5.5, we can find a Π^0_0 formula $\theta(x, s)$ such that ACA_0 proves $\varphi(x) \leftrightarrow \exists f \forall m \theta(x, f[m])$ for all $x \in \mathbb{N}$.

For any fixed n, consider the following game. First, player I chooses $k < n$ and asks whether $\varphi(k)$ or not. Then player II answers yes or no. If she answers "yes," she has to construct a witness g of $\varphi(k)$. If she answers "no," player I has to construct a witness of $\varphi(k)$. Such a game is realized as follows:

- Player I chooses $k < n$ by playing 0 at his first k turns and 1 at his $(k + 1)$-th turn.

- Player II chooses 0 (no) or 1 (yes) at her $(k+1)$-th turn.
- If player II answers 1 at her $(k+1)$-th turn, she wins by constructing g such that $\forall m \theta(k, \overline{g}[m]))$. (case 1)
- If player II chooses 0 at her $(k+1)$-th turn, player I wins by constructing g such that $\forall m \theta(k, \overline{g}[m])$. (case 2)

To illustrate, the game goes as follows:
Case 1

$$\begin{array}{lccccccc}
\text{player I} & \underbrace{0,...,0,}_{k \text{ times}} & 1, & *,...,*, & *, & *,...,*, & *, & ... \\
\text{player II} & *,...*, & \underbrace{1}_{\text{answer}}, & \underbrace{0,...,0,}_{2\overline{g}(0) \text{ times}} & 1, & \underbrace{1,...,1,}_{2\overline{g}(1) \text{ times}} & 0, & ...
\end{array}$$

Case 2

$$\begin{array}{lccccccc}
\text{player I} & \underbrace{0,...,0,}_{k \text{ times}} & 1, & \underbrace{0,...,0,}_{2\overline{g}(0) \text{ times}} & 1, & \underbrace{1,...,1,}_{2\overline{g}(1) \text{ times}} & 0, & ... \\
\text{player II} & *,...*, & \underbrace{0}_{\text{answer}}, & *,...,*, & *, & *,...,*, & *, & ...
\end{array}$$

Player I wins if and only if one of the following holds:

- Player I chooses $k < n$ and player II answers "yes" and player II fails to construct totally regular $g \in 2^{\mathbb{N}}$ with $\forall m \theta(k, \overline{g}[m])$.
- Player I chooses $k < n$ and player II answers "no" and player I successes to construct totally regular $g \in 2^{\mathbb{N}}$ with $\forall m \theta(k, \overline{g}[m])$.

Formally, this winning condition can be written as below:

$$\begin{aligned}
&[\exists k < l \psi(f, k, 1) \wedge \\
&\qquad \{(f_{k,1} \text{ is not totally regular}) \vee \exists l((f_{k,1}[l] \text{ is regular}) \wedge \neg\theta(k, \overline{f_{k,1}[l]}))\}] \vee \\
&[\exists k < l \psi(f, k, 0) \wedge \\
&\qquad \{(f_{k,0} \text{ is totally regular}) \wedge \forall l \exists m(|\overline{f_{k,0}[m]}| = l \wedge \theta(k, \overline{f_{k,0}[m]}))\}], \qquad (\sharp)
\end{aligned}$$

where $\psi(f, k, i)$ is the Π^0_0 formula $\forall i < k(f(2i) = 0) \wedge f(2k) = 1 \wedge f(2k+1) = i)$ and $f_{k,i}$ is $g \in 2^{\mathbb{N}}$ such that, for all l, $g(l) = f(2(k+1) + 2l + i)$.

Similarly to the proof of Theorem 4.2.4, $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_2)\text{-}\mathsf{Det}^*$ implies that $(\sharp)$ is determinate. Moreover, player I cannot have a winning strategy in

($\sharp$), since if $\varphi(k)$ holds, player I cannot win when player II answers "yes" and constructs a witness, otherwise, i. e., $\varphi(k)$ holds, player I cannot win when player II answers "no." By $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_2)\text{-}\mathsf{Det}^*$, player II has a winning strategy τ. Δ^0_1 comprehension yields $Y = \{k < n : \tau(k^\tau * \langle 1 \rangle) = 1\}$. Clearly, Y enjoys the desired property. □

Since $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_2)\text{-}\mathsf{Det}^*$ implies $\Sigma^0_2\text{-}\mathsf{Det}^*$, it implies arithmetical transfinite recursion by Theorem 4.4.4. Finally, we have the following.

Corollary 4.5.7. $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_2)\text{-}\mathsf{Det}^*$ *and arithmetical transfinite recursion* $+\Sigma^1_1$ *induction are equivalent over* RCA^*_0.

4.6 Π^1_1-CA_0 and determinacy

In this section, we find determinacy schemata which are equivalent to Π^1_1 comprehension.

By [30, Theorem V.5.1], Π^1_1-CA_0 includes ATR_0. Moreover, since Π^1_1-CA_0 proves both Π^1_1 and Σ^1_1 transfinite induction by Lemma 3.1.3, Π^1_1-CA_0 includes $\mathsf{ATR}_0+\Sigma^1_1$ induction and, by Remark 4.5.1.2, proves the consistency of ATR_0+ Σ^1_1 induction.

By determinacies in the Baire space, Π^1_1 comprehension is characterized as follows:

Proposition 4.6.1. *The following are pairwise equivalent over* RCA^*_0*:*

$$\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)\text{-}\mathsf{Det},\ (\Sigma^0_1 \wedge \Pi^0_1)\text{-}\mathsf{Det} \text{ and } \Pi^1_1 \text{ comprehension.}$$

Proof. See [30, Lemma VI.5.2, Lemma VI.5.3]. Although the proof of [30, Lemma VI.5.3] shows the implication from $(\Sigma^0_1)_2$-Det to Π^1_1 comprehension in RCA_0, $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)\text{-}\mathsf{Det}$ implies the determinacy of the game defined in the proof and so we need only $\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)\text{-}\mathsf{Det}$ for the proof. □

By the above proposition, it turns out that there is no determinacy schema in the Baire space corresponding to Wadge classes which are equivalent to $\mathsf{ATR}_0 + \Sigma^1_1$ induction.

In the following proof, we construct a game which is similar to that in the proof of Theorem 4.5.6. In the following game player I can ask about any natural number k, whereas he can ask only about $k < n$ in the game defined in the proof of Theorem 4.5.6. This point corresponds to the difference between (unbounded) Σ^1_1 comprehension and bounded Σ^1_1 comprehension.

Theorem 4.6.2. RCA_0^* *proves that* $\mathsf{Bisep}(\Sigma_1^0, \Sigma_2^0)\text{-}\mathsf{Det}^*$ *implies* Π_1^1 *comprehension.*

Proof. Assume $\mathsf{Bisep}(\Sigma_1^0, \Sigma_2^0)\text{-}\mathsf{Det}^*$. Since we can easily see $\mathsf{Bisep}(\Sigma_1^0, \Sigma_2^0)\text{-}\mathsf{Det}^*$ implies $(\Sigma_1^0 \wedge \Pi_1^0)\text{-}\mathsf{Det}^*$, by Theorem 4.3.2 we can work in ACA_0. We show Σ_1^1 comprehension, which is equivalent to Π_1^1 comprehension over RCA_0^*. Let $\varphi(n)$ be a Σ_1^1 formula. By 4.5.5, we can find a Π_0^0 formula $\theta(x, f)$ such that ACA_0 proves $\varphi(n) \leftrightarrow \exists f \forall m \theta(n, f[m])$ for all $n \in \mathbb{N}$. Now we consider the following game: Player I chooses $k \in \mathbb{N}$ and asks whether $\varphi(k)$ or not. Player II answers "yes" or "no." If she answers "yes," she has to construct a witness for $\varphi(k)$. If she answers "no," player I has to construct a witness for $\varphi(k)$.

Such a game is realized as follows:

- Player I chooses $k \in \mathbb{N}$ by playing 0 at his first k turns and 1 at his $(k+1)$-th turn.
- Player II chooses 0 (no) or 1 (yes) at her $(k+1)$-th turn.
- If player II chooses 1, she wins by constructing totally regular $g \in 2^{\mathbb{N}}$ with $\forall m \theta(k, \overline{g}[m])$.
- If player II chooses 0, player I wins by constructing totally regular $g \in 2^{\mathbb{N}}$ with $\forall m \theta(k, \overline{g}[m])$.

Player I wins if and only if one of the following conditions holds:

- Player I chooses k and player II answers "yes," and she fails to construct g with $\forall m \theta(k, \overline{g}[m])$.
- Player I chooses k and player II answers "no," and he successes to construct g with $\forall m \theta(k, \overline{g}[m])$.

Rigorously, the above winning condition can be written as below:

$$\begin{aligned}
&[\exists k \psi(f, k, 1) \wedge \\
&\qquad \{(f_{k,1} \text{ is not totally regular}) \vee \exists l((f_{k,1}[l] \text{ is regular}) \wedge \neg\theta(k, \overline{f_{k,1}[l]}))\}] \vee \\
&[\exists k \psi(f, k, 0) \wedge \\
&\qquad \{(f_{k,0} \text{ is totally regular}) \wedge \forall l \exists m(|\overline{f_{k,0}[m]}| = l \wedge \theta(k, \overline{f_{k,0}[m]}))\}], \qquad (\natural)
\end{aligned}$$

where $\psi(f, k, i)$ is the Π_0^0 formula $\forall i < k(f(2i) = 0) \wedge f(2k) = 1 \wedge f(2k+1) = i)$ and $f_{k,i}$ is $g \in 2^{\mathbb{N}}$ such that, for all l, $g(l) = f(2(k+1) + 2l + i)$. As in the proof of Theorem 4.5.6, $\mathsf{Bisep}(\Sigma^0_1, \Sigma^0_2)\text{-}\mathsf{Det}^*$ implies that player II has a winning strategy τ in (♮). Then $Y = \{k : \tau(k^\tau * \langle 1 \rangle) = 1\}$ enjoys the desired property. □

Next, we consider the converse. It suffices to show the implication from $\Pi^1_1\text{-}\mathsf{CA}_0$ to $\mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)\text{-}\mathsf{Det}^*$, since it is easy to prove that $\mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)\text{-}\mathsf{Det}^*$ implies $\mathsf{Bisep}(\Sigma^0_1, \Sigma^0_2)\text{-}\mathsf{Det}^*$. It can be proved in a similar way to Theorem 4.5.4.

Theorem 4.6.3. $\Pi^1_1\text{-}\mathsf{CA}_0$ *proves* $\mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)\text{-}\mathsf{Det}^*$.

Proof. Let $\varphi(f)$ be a $\mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)$ game, say $(\exists n\theta(f[n]) \wedge \psi(f)) \vee (\neg\exists n\theta(f[n]) \wedge \eta(f))$, where $\theta(x)$ is Π^0_0, $\psi(f)$ is Π^0_2 and $\eta(f)$ is Σ^0_2. By Remark 4.4.5.3, Π^1_1 comprehension implies the existence of W and W' such that

$$W = \{s : \exists t \subseteq s\theta(t) \text{ and player I wins } \psi(f) \text{ at } s\},$$
$$W' = \{s : \exists t \subseteq s\theta(t) \text{ and } s \notin W\}.$$

Define a new game $\varphi^*(f)$ as follows: $\varphi^*(f) \equiv \exists n(f[n] \in W) \vee (\neg\exists n\theta(f[n]) \wedge \eta(f))$. By $\Sigma^0_2\text{-}\mathsf{Det}^*$, which is proved in ATR_0 (cf. Theorem 4.4.7), $\varphi^*(f)$ is determinate. We show the following claim:

Claim. *The player who wins $\varphi^*(f)$ also wins the original game $\varphi(f)$.*

Proof of the claim. First, assume that player I has a winning strategy σ^* in $\varphi^*(f)$. By Lemma 4.5.3, we have a sequence $\langle \sigma_s : s \in W \rangle$ of winning s-strategies for player I in $\psi(f)$. Now we define a new strategy σ for player I in $\varphi(f)$ as follows:

$$\sigma(s) = \begin{cases} \sigma^*(s) & \text{if } t \notin W \text{ for all } t \subseteq s, \\ \sigma_t(s) & \text{if } t \text{ is the } \subseteq\text{-least initial segment of } s \text{ with } t \in W. \end{cases}$$

We prove that σ is a winning strategy for player I in $\varphi(f)$. Let f be a play in which player I follows σ. If there is no $k \in \mathbb{N}$ with $\theta(f[k])$, then f satisfies $\neg\exists n\theta(f[n]) \wedge \eta(f)$, and so player I wins. If there is k with $\theta(f[k])$, then f satisfies $\exists m f[m] \in W$. Take the least such m. Then $\sigma(f[2l]) = \sigma_{f[2m]}(f[2l])$ holds for each $2l \geq m$, and so $\varphi(f)$ holds. Thus player I wins $\varphi(f)$.

Next, assume that player II has a winning strategy τ^* in the game $\varphi^*(f)$. Since ATR_0 implies $\Sigma^0_2\text{-}\mathsf{Det}^*$, player II wins $\psi(f)$ at each $s \in W'$. By Lemma

4.5.3, we have a sequence $\langle \tau_s : s \in W' \rangle$ of winning s-strategies for player II in $\psi(f)$.

Now, define a new strategy τ for player II in $\varphi(f)$ as follows:

$$\tau(s) = \begin{cases} \tau^*(s) & \text{if } t \notin W' \text{ for all } t \subseteq s, \\ \tau_t(s) & \text{if } t \text{ is the } \subseteq\text{-least initial segment of } s \text{ with } t \in W'. \end{cases}$$

We prove that τ is a winning strategy for player II in $\varphi(f)$. Let f be a play in which player II follows τ. Then $f[m] \notin W$ holds for all m. If there is no m with $\theta(f[m])$, then $\neg\eta(f)$ holds, and so player II wins. Otherwise, the least m with $\theta(f[m])$ satisfies $f[m] \in W'$ by the assumption $\forall m f[m] \notin W$, and so $f(2k+1) = \tau(f[2k+1]) = \tau_{f[m]}(f[2k+1])$ holds for each $2k+1 > m$. Hence $\neg\psi(f)$ holds. Therefore player II wins $\varphi(f)$. □

Corollary 4.6.4. *The following are pairwise equivalent over* RCA_0^**:*

$$\mathsf{Bisep}(\Sigma^0_1, \Sigma^0_2)\text{-}\mathsf{Det}^*,\ \mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)\text{-}\mathsf{Det}^* \textit{ and } \Pi^1_1 \textit{ comprehension.}$$

Proof. The implication from $\mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)\text{-}\mathsf{Det}^*$ to $\mathsf{Bisep}(\Sigma^0_1, \Sigma^0_2)\text{-}\mathsf{Det}^*$ is trivial. With this fact, we have the above equivalence by Theorems 4.6.2 and 4.6.3. □

4.7 Π^1_1-TR_0 and determinacy

Now let us turn to considering the relation between Π^1_1 transfinite recursion and determinacy.

Π^1_1 transfinite recursion is characterized by determinacy as follows.

Proposition 4.7.1 ([32, Theorem 6.1])**.** Δ^0_2-Det *and* Π^1_1 *transfinite recursion are equivalent over* RCA_0^*.

In this section, we prove the equivalence between $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)\text{-}\mathsf{Det}^*$ and Π^1_1 transfinite recursion over RCA_0^*.

Theorem 4.7.2. RCA_0^* *proves that* $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)\text{-}\mathsf{Det}^*$ *implies* Δ^0_2-Det.

Proof. Let $\varphi(f)$ be a Σ^0_2 game in the Baire space. Assume that there exist Π^0_2 formula $\varphi'(f)$ with $\varphi(f) \leftrightarrow \varphi'(f)$ for all $f \in \mathbb{N}^{\mathbb{N}}$. Then, by Lemma 3.6.1, we can find Π^0_0 formulae $\theta_0(x, y)$ and $\theta_1(x, y)$ such that RCA_0^* proves

$$\forall f \in \mathbb{N}^{\mathbb{N}}(\varphi(f) \leftrightarrow \forall n \exists m \theta_0(n, f[m]))$$

and

$$\forall f \in \mathbb{N}^{\mathbb{N}}(\neg\varphi(f) \leftrightarrow \forall n \exists m \theta_1(n, f[m])).$$

As in the proof of Theorem 4.4.4, we shall define a game $\varphi^*(f)$ in the Cantor space such that the player who wins $\varphi^*(f)$ also wins $\varphi(f)$ and such that $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$-$\mathsf{Det}^*$ proves its determinacy.

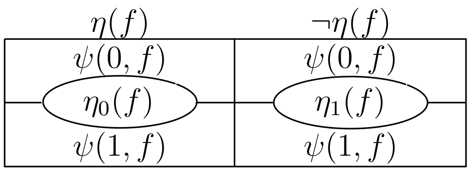
We use $\psi(i, f)$ in the proof of Lemma 4.4.3. Note that $\psi(i, f)$ is equivalent to a Σ^0_2 formula over RCA^*_0. Then, for $i < 2$, set $\eta_i(f)$ by

$$(f \text{ is totally regular}) \land \forall n \exists m (f[m] \text{ is regular} \land \theta_i(n, \overline{f[m]})).$$

Clearly η_i is equivalent to a Π^0_2 formula over RCA^*_0. It is also clear that there is no $f \in 2^{\mathbb{N}}$ with $\eta_0(f) \land \eta_1(f)$. By Lemma 3.6.7, RCA^*_0 yields a Σ^0_2 formula $\eta(f)$ and a Π^0_2 formula $\eta'(f)$ with

$$\forall f \in 2^{\mathbb{N}}((\eta(f) \leftrightarrow \eta'(f)) \land (\eta_0(f) \to \eta(f)) \land (\eta_1(f) \to \neg\eta(f))).$$

Furthermore, we can check that each f satisfies exactly one of $\psi(0, f)$, $\psi(1, f)$, $\eta_0(f)$ and $\eta_1(f)$.



Define a new game $\varphi^*(f)$ by $(\eta(f) \land \neg\psi(0, f)) \lor (\neg\eta(f) \land \psi(1, f))$. $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$-$\mathsf{Det}^*$ implies that $\varphi^*(f)$ is determinate. Since $\varphi^*(f)$ holds if and only if $\eta_0(f) \lor \psi_1(1, f)$, we can check that player who has a winning strategy in $\varphi^*(f)$ also has a winning strategy in $\varphi(f)$. □

For the converse, we need some preparations motivated by the reformulation of Δ^0_2 in [32].

Lemma 4.7.3. *For any Π^0_2 formula $\varphi(f)$ with a distinguished function variable $f \in 2^{\mathbb{N}}$, we can find, in RCA_0, a Π^0_0 formula $\theta'(x)$ in which n does not occur such that $\forall f \in 2^{\mathbb{N}}(\varphi(f) \leftrightarrow \forall n \exists m > n\theta'(f[m]))$.*

Proof. See [17, Lemma 3.1]. Although the original proof is done in ACA_0, it works also in RCA_0. □

In descriptive set theory, Hausdorff proved (cf. [14, §37. III. Theorem]) that a $\boldsymbol{\Delta}^0_{n+1}$ set can be represented as a boolean combination of transfinitely many $\boldsymbol{\Pi}^0_n$ sets, i. e., for any $\boldsymbol{\Delta}^0_{n+1}$ set A of Polish space $\mathcal{X}$, there exists an ordinal $\gamma < \omega_1$ and a decreasing sequence $\langle A_\alpha : \alpha < \gamma \rangle$ of $\boldsymbol{\Pi}^0_n$ sets such that

$$A = \{x \in A_0 : \min\{\alpha : x \notin A_\alpha\} \text{ is odd}\}.$$

The following lemma is a formalization of the case $n = 2$ in second order arithmetic.

Lemma 4.7.4. *For any pair of a Σ^0_2 formula $\psi_0(f)$ and a Π^0_2 formula $\psi_1(f)$, we can find a Π^0_0 formula $\theta(x, i, y)$ such that ACA_0 proves the following.*

$$\forall f \in 2^{\mathbb{N}}(\psi_0(f) \leftrightarrow \psi_1(f)) \rightarrow$$
$$\begin{bmatrix} \exists Y(\mathrm{WO}(Y, <_Y))\wedge \\ (\forall f \in 2^{\mathbb{N}})(((y,j) <^*_Y (x,i) \wedge \forall n\theta(x,i,f[n])) \rightarrow \forall n\theta(y,j,f[n]))\wedge \\ (\forall f \in 2^{\mathbb{N}})(\psi_0(f) \leftrightarrow \exists x \in Y(\forall n\theta(x,0,f[n]) \wedge \neg\forall n\theta(x,1,f[n])))\wedge \\ (\forall f \in 2^{\mathbb{N}})(\neg\psi_0(f) \leftrightarrow \exists x \in Y(\forall n\theta(x,1,f[n]) \wedge \neg\forall n\theta(x',0,f[n])))\end{bmatrix} \quad (\star)$$

where x' is the $<_Y$-successor of x, and where $(Y \times 2, <^*_Y)$ is a well ordering defined by

$$(x, i) <^*_Y (y, j) \leftrightarrow x <_Y y \vee (x = y \wedge i < j).$$

Proof. See Theorem 3.5 of [17]. □

Theorem 4.7.5. Π^1_1-TR_0 *proves* $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$-$\mathsf{Det}^*$.

Proof. Let $\varphi(f)$ be a game in the Cantor space such that

$$\forall f \in 2^{\mathbb{N}}(\varphi(f) \leftrightarrow ((\psi(f) \wedge \eta_0(f)) \vee (\neg\psi(f) \wedge \eta_1(f))))$$

for some Σ^0_2 formulae $\psi(f)$ and $\eta_1(f)$ and Π^0_2 formula $\eta_0(f)$. Assume that $\forall f \in 2^{\mathbb{N}}(\psi(f) \leftrightarrow \psi'(f))$ for some Π^0_2 formula $\psi'(f)$. Note that the following holds:

$$\forall f \in 2^{\mathbb{N}}(\neg\varphi(f) \leftrightarrow ((\psi(f) \wedge \neg\eta_0(f)) \vee (\neg\psi(f) \wedge \neg\eta_1(f)))).$$

By applying Lemma 4.7.4, taking $\psi(f)$ and $\psi'(f)$ as $\psi_0(f)$ and $\psi_1(f)$ respectively, we can find a Π^0_1 formula $\forall n\theta(x, i, f[n])$ and a well ordering $(Y, <_Y)$ such that $(\star)$ holds. In particular, the following holds:

$$\forall f \in 2^{\mathbb{N}}(\psi(f) \leftrightarrow \exists x(\forall n\theta(x,0,f[n]) \wedge \neg\forall n\theta(x,1,f[n])))$$

and

$$\forall f \in 2^{\mathbb{N}}(\neg\psi(f) \leftrightarrow \exists x(\forall n\theta(x,1,f[n]) \wedge \neg\forall n\theta(x',0,f[n]))),$$

where x' is the $<_Y$-successor of x. Then, by Π^1_1 transfinite recursion, define, for each $x \in Y$ and $i < 2$, $V_{x,i}$ and $W_{x,i}$ as follows:

$$\begin{aligned}
V_{x,0} =& \{s : \exists t \subseteq s \neg\theta(x,1,t) \text{ and player II wins } \eta'_0(x,f) \text{ at } s\},\\
W_{x,0} =& \{s : \exists t \subseteq s \neg\theta(x,1,t) \text{ and } s \notin V_{x,0}\},\\
W_{x,1} =& \{s : \exists t \subseteq s \neg\theta(x',0,t) \text{ and player I wins } \eta'_1(x,f) \text{ at } s\},\\
V_{x,1} =& \{s : \exists t \subseteq s \neg\theta(x',0,t) \text{ and } s \notin W_{x,1}\},
\end{aligned}$$

where $\eta'_0(x,f)$ and $\eta'_1(x,f)$ are defined by

$$\begin{aligned}
\eta'_0(x,f) &\equiv ((\forall n\theta(x,0,f[n]) \wedge \eta_0(f)) \vee \exists n f[n] \in W_{<^*_Y(x,i)}),\\
\eta'_1(x,f) &\equiv ((\forall n\theta(x,1,f[n]) \wedge \eta_1(f)) \vee \exists n f[n] \in W_{<^*_Y(x,i)}),
\end{aligned}$$

and where $W_{<^*_Y(x,i)} = \bigcup_{(y,j)<^*_Y(x,i)} W_{y,i}$.

Set a new game

$$\varphi^*(f) \equiv \exists n((\forall m < n)(f[m] \notin \bigcup_{x\in Y, i<2} V_{x,i}) \wedge f[n] \in \bigcup_{x\in Y,i<2} W_{x,i}).$$

We show that

$$\neg\varphi^*(f) \leftrightarrow \exists n((\forall m < n)(f[m] \notin \bigcup_{x\in Y,i<2} W_{x,i}) \wedge f[n] \in \bigcup_{x\in Y,i<2} V_{x,i}),$$

as follows. First, we see $\bigcup_{x\in Y,i<2} W_{x,i} \cap \bigcup_{x\in Y,i<2} V_{x,i} = \emptyset$. Assume $s \in W_{x,i} \cup V_{x,i}$. We may assume that (x,i) is the $<^*_Y$-least such one. Note that $W_{x,i} \cap V_{x,i} = \emptyset$. Consider the case $s \in W_{x,i}$. Then $s \notin \bigcup_{(y,j)<^*_Y(x,i)} W_{y,j} \cup \bigcup_{(y,j)<^*_Y(x,i)} V_{y,j}$. If $\exists t \subseteq s\neg\theta(y,1,t)$ for some $(y,1)$ with $(x,i) <^*_Y (y,1)$, then player I wins $\eta'_0(y,f)$ at s and so $s \notin V_{y,0}$. If $\exists t \subseteq s\neg\theta(y',0,t)$ for some $(y',0)$ with $(x,i) <^*_Y (y',0)$, then player I wins $\eta'_1(y,f)$ at s and so $s \notin V_{y,1}$. In a similar way, we can prove that if $s \in V_{x,i}$, then $s \notin \bigcup_{x\in Y,i<2} W_{x,i}$. Thus $\bigcup_{x\in Y,i<2} W_{x,i} \cap \bigcup_{x\in Y,i<2} V_{x,i} = \emptyset$.

Next we see that, for any $f \in 2^{\mathbb{N}}$, exactly one of $\varphi^*(f)$ or $\exists n((\forall m < n)(f[m] \notin \bigcup_{x\in Y,i<2} W_{x,i}) \wedge f[n] \in \bigcup_{x\in Y,i<2} V_{x,i})$ holds.

Since, for all $f \in 2^{\mathbb{N}}$, either $\exists x \in Y(\forall n\theta(x,0,f[n]) \wedge \neg\forall n\theta(x,1,f[n]))$ or $\exists x \in Y(\forall n\theta(x,1,f[n]) \wedge \neg\forall n\theta(x',0,f[n]))$ holds, and since $W_{x,0} \cup V_{x,0} = \{s \in$

$2^{<\mathbb{N}} : \exists t \subseteq s \neg\theta(x,1,t)\}$ and $W_{x,1} \cup V_{x,1} = \{s \in 2^{<\mathbb{N}} : \exists t \subseteq s \neg\theta(x',0,t)\}$ hold, for all $f \in 2^{\mathbb{N}}$, there exists n such that $f[n] \in \bigcup_{x \in Y, i<2} W_{x,i} \cup \bigcup_{x\in Y,i<2} V_{x,i}$. Since $\bigcup_{x\in Y,i<2} W_{x,i} \cap \bigcup_{x\in Y,i<2} V_{x,i} = \emptyset$, the $\subseteq$-least such $f[n]$ is in exactly one of $\bigcup_{x\in Y,i<2} W_{x,i}$ and $\bigcup_{x\in Y,i<2} V_{x,i}$.

Thus, for any $f \in 2^{\mathbb{N}}$, exactly one of the following holds:

- $\varphi^*(f) \equiv \exists n((\forall m < n)(f[m] \notin \bigcup_{x\in Y,i<2} V_{x,i}) \wedge f[n] \in \bigcup_{x\in Y,i<2} W_{x,i})$,
- $\exists n((\forall m < n)(f[m] \notin \bigcup_{x\in Y,i<2} W_{x,i}) \wedge f[n] \in \bigcup_{x\in Y,i<2} V_{x,i})$.

The next claim completes the proof.

Claim. *The player who wins $\varphi^*(f)$ also wins $\varphi(f)$.*

Proof of the claim. First, assume that player I has a winning strategy σ^* in $\varphi^*(f)$. For $s \in \bigcup_{x\in Y,i<2} W_{x,i}$, let (x_s, i_s) be the $<^*_Y$-least (x,i) with $s \in W_{x,i}$. By Lemma 4.5.3, take a sequence $\langle \sigma_s : s \in \bigcup_{x\in Y,i<2} W_{x,i}\rangle$ such that σ_s is a winning s-strategies for player I in $\eta'_{i_s}(x_s, f)$. Then, for any $s \in \bigcup_{x\in Y,i<2} W_{x,i}$, define an s-strategy σ^*_s for player I by arithmetical transfinite recursion along $(Y \times 2, <^*_Y)$ as follows:

$$\sigma^*_s(t) = \begin{cases} \sigma^*_u(t) & \text{if } u \text{ is the } \subseteq\text{-least initial segment of } t \\ & \text{with } s \subsetneq u \text{ and } u \in W_{<^*_Y(x_s,i_s)}, \\ \sigma_s(t) & \text{if there is no such } u. \end{cases}$$

Now we prove, by Π^1_1 transfinite induction on $(Y \times 2, <^*_Y)$, for any $s \in \bigcup_{x\in Y,i<2} W_{x,i}$, σ^*_s is a winning s-strategy for player I in $\varphi(f)$. Assume that σ^*_t is a winning t-strategy for player I in $\varphi(f)$ for all $(y,j) <^*_Y (x,i)$ and for all $t \in W_{<^*_Y(x,i)}$. Take $s \in W_{x,i}$ with $(x,i) = (x_s,i_s)$ and an s-strategy ρ for player II. If there is k such that $t = (\sigma^*_s \otimes \rho)[k] \in W_{<^*_Y(x,i)}$, take the least such k. Then $\sigma^*_s \otimes \rho = \sigma^*_t \otimes \rho'$, where $\rho' = \rho \upharpoonright (t)_2$, and so $\varphi(\sigma^*_s \otimes \rho)$ holds by induction hypothesis. If there is no k with $(\sigma \otimes \rho)[k] \in W <^*_Y (x,i)$ and if $i = 0$, then $\forall n \theta(x, 0, (s * f)[n]) \wedge \eta_0(s * f)$ hold. Since $s \in W_{x,i}$, there is $t \subseteq s$ with $\neg\theta(x,1,t)$, and so $\varphi(f)$. If there is no k with $(\sigma \otimes \rho)[k] \in \bigcup_{<^*_Y(x,i)} W_{x,i}$ if $i = 1$, we can similarly prove that $\varphi(f)$ holds.

It is now easy to check that σ defined by

$$\sigma(t) = \begin{cases} \sigma^*_u(t) & \text{if } u \text{ is the } \subseteq\text{-least initial segment of } t \\ & \text{with } u \in \bigcup_{x\in Y,i<2} W_{x,i}, \\ \sigma^*(t) & \text{if there is no such } u. \end{cases}$$

is a winning strategy for player I in $\varphi(f)$.

Let us turn to the case in which player II has a winning strategy τ^* in $\varphi^*(f)$. For each $s \in \bigcup_{x\in Y,i<2} V_{x,i}$, let (y_s, i_s) be the $<^*_Y$-least (y,j) with $s \in V_{y,j}$. As in the previous case, take a sequence $\langle \tau_s : s \in \bigcup_{x\in Y,i<2} V_{x,i}\rangle$ of winning s-strategies for player II in $\eta'_{j_s}(y_s, f)$ and define a sequence of strategies $\langle \tau^*_s : s \in \bigcup_{x\in Y,i<2} V_{x,i}\rangle$ by

$$\tau^*_s(t) = \begin{cases} \tau^*_u(u) & \text{if } u \text{ is the } \subseteq\text{-least initial segment of } t \\ & \text{with } s \subsetneq u \text{ and } u \in V_{<^*_Y(x,i)} = \bigcup_{(y,j)<^*_Y(x,i)} V_{y,j}, \\ \tau_s(t) & \text{otherwise.} \end{cases}$$

Then, we can prove that, for any $s \in \bigcup_{x\in Y,i<2} V_{x,i}$, τ^*_s is a winning s-strategy for player II in $\varphi(f)$ by Π^1_1 transfinite induction as follows. Assume that τ^*_t is a winning t-strategy for player II in $\varphi(f)$ for any $(y,j) <^*_Y (x,i)$ and for any $t \in V_{y,j}$. Take $s \in V_{x,i}$ with $(x,i) = (y_s, j_s)$ and an s-strategy ν for player I. If there is k such that $t = (\nu \otimes \tau^*_s)[k] \in V_{<^*_Y(x,i)}$, then $\nu \otimes \tau^*_s = \nu' \otimes \tau^*_t$, where $\nu' = \nu \restriction (t)_2$, and so $\varphi(\nu \otimes \tau^*_s)$ holds by induction hypothesis. Next we consider the case in which there is no k with $(\nu \otimes \tau)[k] \in V_{<^*_Y(x,i)}$. We may assume $i = 0$, because the case $i = 1$ can be proved similarly. If $\exists n \neg\theta(x, 0, (\nu \otimes \tau^*_s)[n])$ holds, then, there exists m with $(\nu \otimes \tau^*_s)[m] \in \bigcup_{(y,j)<^*_Y(x,i)} (W_{y,j} \cup V_{y,j})$, since, for any $f \in 2^{\mathbb{N}}$, either $\psi(f)$ or $\neg\psi(f)$ holds, and since for all (z,l) and (z',l') in $Y \times \{0,1\}$, $(z,l) <^*_Y (z',l')$ and $\forall n\theta(z', l', f[n])$ implies $\forall n\theta(z, l, f[n])$. By the fact that τ^*_t is a winning strategy for player II in $\eta'_0(x,f)$, $(\nu \otimes \tau^*_s)[n]$ is not in $W_{<^*_Y(x,i)}$, and, by the assumption, $(\nu \otimes \tau^*_s)[n]$ is not in $V_{<^*_Y(x,i)}$, which is a contradiction. Therefore $\forall n\theta(x, 0, (\nu \otimes \tau^*_s)[n])$ holds, and so $\nu \otimes \tau^*_s$ satisfies both $\forall n\theta(x, 0, (\nu \otimes \tau^*_s)[n]) \wedge \exists n \neg\theta(x, 1, (\nu \otimes \tau^*_s)[n])$ and $\neg\eta_0(\nu \otimes \tau^*_s)$, which means that player II wins $\varphi(f)$.

Now it is easy to check that τ defined by

$$\tau(t) = \begin{cases} \tau^*_u(t) & \text{if } u \text{ is the } \subseteq\text{-least initial segment of } t \\ & \text{with } u \in \bigcup_{x\in Y,i<2} V_{x,i}, \\ \tau^*(t) & \text{if there is no such } u. \end{cases}$$

□

Corollary 4.7.6. $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$-$\mathsf{Det}^*$, Δ^0_2-Det *and* Π^1_1 *transfinite recursion are pairwise equivalent over* RCA^*_0.

Chapter 5

Hierarchy of complete determinacy

In this chapter, we consider a variation of determinacy.

5.1 Complete determinacy

Complete determinacy is defined as follows.

Definition 5.1.1 (complete determinacy)**.** Let $X \subseteq \mathbb{N}$. $W \subseteq X^{<\mathbb{N}}$ is the *winning set for player I in* $\varphi(f)$ if, for any $s \in X^{<\mathbb{N}}$, $s \in W$ implies that player I wins $\varphi(f)$ at s and $s \notin W$ implies that player II wins $\varphi(f)$ at s.

For a class Γ of formulae, Γ *complete determinacy* in $X^{\mathbb{N}}$ consists of all axioms of the form

$$\exists W(W \subseteq X^{<\mathbb{N}} \text{ is the winning set for player I in } \varphi(f)),$$

where $\varphi(f)$ is a game in $X^{\mathbb{N}}$ which belongs to Γ and W is not free in $\varphi(f)$.

Clearly Γ complete determinacy in $X^{\mathbb{N}}$ implies Γ determinacy in $X^{\mathbb{N}}$. Γ and $\neg\Gamma$ complete determinacies in $X^{\mathbb{N}}$ are equivalent over RCA_0^*.

Γ-$\mathsf{comp.Det}$ (or Γ-$\mathsf{comp.Det}^*$) stands for Γ complete determinacy in the Baire space (resp. in the Cantor space).

As well as determinacy schemata, we define complete determinacy schemata.

Definition 5.1.2. For a game $\varphi(f)$ in $X^{\mathbb{N}}$, $\mathsf{comp.Det}^X[\varphi]$ is an abbreviation for $\exists W(W \subseteq X^{<\mathbb{N}}$ is the winning set for player I).

Let $1 \leq n, m < \omega$. We define the following schemata.

Δ^0_n complete determinacy in $X^{\mathbb{N}}$:
$\forall f \in X^{\mathbb{N}}(\varphi(f) \leftrightarrow \psi(f)) \to \mathsf{comp.Det}^X[\varphi]$, where $\varphi(f)$ is Σ^0_n and where $\psi(f)$ is Π^0_n.

$(\Sigma^0_n)_2$ complete determinacy in $X^{\mathbb{N}}$:
$\mathsf{comp.Det}^X[\varphi \wedge \psi]$, where $\varphi(f)$ is Σ^0_n and where $\psi(f)$ is Π^0_n.

$\mathsf{Sep}(\Sigma^0_m, \Sigma^0_n)$ complete determinacy in $X^{\mathbb{N}}$:
$\mathsf{comp.Det}^X[(\psi \wedge \eta_0) \vee (\neg\psi \wedge \eta_1)]$,
where $\psi(f)$ is Σ^0_m, where $\eta_0(f)$ is Π^0_n and where $\eta_1(f)$ is Σ^0_n.

$\mathsf{Bisep}(\Sigma^0_m, \Sigma^0_n)$ complete determinacy in $X^{\mathbb{N}}$:
$\forall f \in X^{\mathbb{N}}\neg(\psi_0(f) \wedge \psi_1(f)) \to \mathsf{comp.Det}^X[(\psi_0 \wedge \eta_0) \vee (\psi_1 \wedge \eta_1)]$,
where $\psi_i(f)$'s are Σ^0_m, where $\eta_0(f)$ is Π^0_n and where $\eta_1(f)$ is Σ^0_n.

$\mathsf{Bisep}(\Delta^0_m, \Sigma^0_n)$ complete determinacy in $X^{\mathbb{N}}$:
$$\forall f \in X^{\mathbb{N}}((\neg(\psi_0(f) \wedge \psi_1(f))) \wedge (\psi_0(f) \leftrightarrow \xi_0(f)) \wedge (\psi_1(f) \leftrightarrow \xi_1(f))) \to \mathsf{comp.Det}^X[(\psi_0 \wedge \eta_0) \vee (\psi_1 \wedge \eta_1)],$$
where $\psi_i(f)$'s are Σ^0_m, where $\xi_i(f)$'s are Π^0_m, where $\eta_0(f)$ is Π^0_n and where $\eta_1(f)$ is Σ^0_n.

Remark 5.1.3. We have the same implications as Remark 3.6.5 between complete determinacies.

5.2 Weak König's lemma and complete determinacy

In this section, we prove that Δ^0_1-$\mathsf{comp.Det}^*$, Δ^0_1-Det^* is equivalent to weak König's lemma over RCA^*_0.

Recall theorems proved in the previous chapter, which characterize weak König's lemma by determinacy schemata.

Fact 5.2.1 (Corollary 4.1.8)**.** *Δ^0_1-Det^*, Σ^0_1-Det^* and Weak König's lemma are equivalent over RCA^*_0.*

Remark 5.2.2. Since Δ^0_1-$\mathsf{comp.Det}^*$ implies Δ^0_1-Det^* over RCA^*_0, Δ^0_1-$\mathsf{comp.Det}$ implies weak König's lemma.

The following Theorem is the converse of the above remark.

Theorem 5.2.3. WKL_0^* *proves* Δ^0_1-$\mathsf{comp.Det}^*$.

Proof. Let $\varphi(f)$ be a Σ^0_1 game in the Cantor space. Assume that there exists a Π^0_1 formula $\psi(f)$ such that, for $f \in 2^{\mathbb{N}}$, $\varphi(f) \leftrightarrow \psi(f)$. By Lemma 3.6.1, we can find Π^0_0 formulae $\theta(x)$ and $\theta'(x)$, in both of which f does not occur freely, with $\forall f \in 2^{\mathbb{N}}(\varphi(f) \leftrightarrow \exists n\theta(f[n]))$ and $\forall f \in 2^{\mathbb{N}}(\neg\varphi(f) \leftrightarrow \exists n\theta'(f[n]))$. Δ^0_1 comprehension yields $T = \{s \in 2^{<\mathbb{N}} : \forall t \subseteq s \neg(\theta(t) \vee \theta'(t))\}$. Intuitively, T is the set of those positions at which none of the players has yet won. Since there is no $f \in 2^{\mathbb{N}}$ with $\varphi(f) \wedge \neg\varphi(f)$, T has no infinite path.

By weak König's lemma, T is finite. Let n be the maximal length of sequences in T. Note that any $s \in 2^{<\mathbb{N}}$ of length $n+1$ satisfies exactly one of $\exists t \subseteq s\theta(t)$ and $\exists t \subseteq s\theta'(t)$. For $s \in 2^{<\mathbb{N}}$, we can determine whether player I wins $\varphi(f)$ at s as follows:

- If $|s| > n$, player I wins $\varphi(f)$ at s if and only if there exists $t \subseteq s$ with $\theta(t)$.
- If $|s| \leq n$ and $|s|$ is even (or odd), player I wins $\varphi(f)$ at s if and only if I wins $\varphi(f)$ either at $s * \langle 0\rangle$ or at $s * \langle 1\rangle$ (resp. both at $s * \langle 0\rangle$ and at $s * \langle 1\rangle$).

Precisely, the induction on k yields $u_k : (2^{\leq n+1} - 2^{<n+1-k}) \to \{0,1\}$ such that

$$u_k(s) = \begin{cases} 1 & \text{if } |s| = n+1 \text{ and } \exists t \subseteq s\theta(t), \\ 0 & \text{if } |s| = n+1 \text{ and } \exists t \subseteq s\theta'(t), \\ \max\{u_k(s * \langle 0\rangle), u_k(s * \langle 1\rangle)\} & \text{if } |s| \leq n \text{ and } |s| \text{ is even, and} \\ \min\{u_k(s * \langle 0\rangle), u_k(s * \langle 1\rangle)\} & \text{if } |s| \leq n \text{ and } |s| \text{ is odd.} \end{cases}$$

Because u_k is bounded by some iterated power of n, here we only need Σ^0_0 induction. Δ^0_1 comprehension yields $u : 2^{<\mathbb{N}} \to \{0,1\}$ with $\forall s \in 2^{<\mathbb{N}}(u(s) = u_{n+1}(s[n+1]))$ and $W = \{s : u(s) = 1\}$.

We show that if $s \in W$ then player I wins $\varphi(f)$ at s and otherwise II wins $\varphi(f)$ at s. Assume $s \in W$. Then define an s-strategy σ_s by

$$\sigma_s(t) = \begin{cases} 0 & \text{if } u(t * \langle 0\rangle) = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then, for any s-strategy τ for player II, $\sigma_s \otimes \tau$ satisfies $u((\sigma_s \otimes \tau)[m]) = 1$ for all $m \geq n$, and so there exists k such that $\theta((\sigma_s \otimes \tau)[k])$. Therefore player I wins. Similarly, player II wins $\varphi(f)$ at s if $s \notin W$. □

By Theorem 5.2.3, we have the following.

Corollary 5.2.4. *We can add* Δ^0_1-$\mathsf{comp.Det}^*$ *to the list of Fact 5.2.1.*

5.3 ACA_0 and complete determinacy

In this section we prove that Σ^0_1-$\mathsf{comp.Det}^*$ is equivalent to Σ^0_1 comprehension over RCA^*_0.

In the previous chapter, Σ^0_1 comprehension can be characterized game-theoretically as follows.

Fact 5.3.1 (Corollary 4.3.3)**.** $(\Sigma^0_1)_2$-Det^* *and* Σ^0_1 *comprehension are pairwise equivalent over* RCA^*_0.

Remark 5.3.2. By Remark 4.1.5.3, ACA_0 yields

$$X = \{s \in 2^{\mathbb{N}} : \text{player I wins } \varphi(f) \text{ at } s\}.$$

By Σ^0_1-Det^*, which is proved in ACA_0, $s \notin X$ implies that II wins $\varphi(f)$ at s, and so ACA_0 proves Σ^0_1-$\mathsf{comp.Det}^*$.

The next theorem is the converse of the above remark.

Theorem 5.3.3. RCA^*_0 *proves that* Σ^0_1-$\mathsf{comp.Det}^*$ *implies* Σ^0_1 *comprehension.*

Proof. Assume Σ^0_1-$\mathsf{comp.Det}$. Let $\psi(n)$ be a Σ^0_1 formula of the form $\exists m\theta(n, m)$, where $\theta(x, y)$ is a Π^0_0 formula. Consider the following game $\varphi(f)$ in the Cantor space:

- player I chooses $k \in \mathbb{N}$ by playing his first 1 at his $(k + 1)$-th turn.
- I wins if and only if there is m such that $\theta(k, m)$ holds.

To illustrate, the game goes as follows:

$$\begin{array}{lll} \text{player I} & \overbrace{0, ..., 0,}^{k \text{ times}} & 1 \quad \cdots \\ \text{player II} & *, ..., * & * \quad \cdots \end{array}$$

Formally, $\varphi(f)$ is defined as $\exists k\exists m(\forall l < k(f(2l) = 0) \wedge f(2k) = 1 \wedge \theta(k, m))$. Clearly $\varphi(f)$ is Σ^0_1. Σ^0_1-$\mathsf{comp.Det}^*$ yields the winning set W for player I in $\varphi(f)$. We can check that, for any n, player I wins at $0^{2n} * \langle 1\rangle$ if and only if $\psi(n)$. Δ^0_1 comprehension yields $X = \{n : 0^{2n} * \langle 1\rangle \in W\}$. It is easy to see $X = \{n : \psi(n)\}$. □

Since, by Lemma 3.1.6, Σ^0_1 comprehension implies Σ^0_1 induction over RCA^*_0, Theorem 5.3.3 shows that we do not need to care about whether the base theory is RCA^*_0 or RCA_0 when we discuss about complete determinacy schemata stronger than Σ^0_1-$\mathsf{comp.Det}^*$.

Γ determinacy does not always imply Γ complete determinacy, as this theorem with Fact 5.3.1 separates Σ^0_1-$\mathsf{comp.Det}^*$ and Σ^0_1-Det^*. Moreover the equivalence between Γ and Γ' determinacies does not necessarily imply that between Γ and Γ' complete determinacies. Indeed Δ^0_1-$\mathsf{comp.Det}^*$ and Σ^0_1-$\mathsf{comp.Det}^*$ are not equivalent, while Δ^0_1-Det^* and Σ^0_1-Det^* are.

However, the completion of $(\Sigma^0_1)_2$-Det^* does not lift up the comprehension axiom:

Theorem 5.3.4. ACA_0 *proves* $(\Sigma^0_1)_2$-$\mathsf{comp.Det}^*$.

Proof. Let $\varphi(f)$ be a $(\Sigma^0_1)_2$ game $\exists n\theta(f[n]) \wedge \eta(f)$, where $\theta(x)$ is Π^0_0 in which f does not occur freely, and where $\eta(f)$ is Π^0_1. By Remark 4.1.5.4, $V = \{s : \exists t \subseteq s\theta(t) \wedge$ (I wins $\eta(f)$ at s)$\}$ is provided by Π^0_1-$\mathsf{comp.Det}^*$, proved in ACA_0 and Δ^0_1 comprehension. Define a new Σ^0_1 game $\varphi'(f)$ by $\exists n(f[n] \in V)$.

We see that $W = \{s :$ player I wins $\varphi'(f)$ at $s\}$, yielded by Σ^0_1-$\mathsf{comp.Det}^*$, is the winning set for player I in $\varphi(f)$.

Assume that $s \in W$ and that σ' is a winning s-strategy for player I in $\varphi'(f)$. Since the assertion "σ is a winning s-strategy for player I" is equivalent to a Π^0_1 formula by Remark 4.1.5.2 and Lemma 4.2.3.2, [24, Lemma 3.6]), we have a sequence $\langle \sigma'_t : t \in V\rangle$ of winning t-strategies for player I in $\eta(f)$. Define a new s-strategy σ by

$$\sigma(u) = \begin{cases} \sigma'_t(u) & \text{if } t \text{ is the } \subseteq\text{-least initial segment of } u \text{ with } s \subseteq t \wedge t \in V, \\ \sigma'(u) & \text{if there is no such } t. \end{cases}$$

We see that σ is a winning s-strategy for player I in $\varphi(f)$. Take an s-strategy ν for player II. Since σ' is a winning s-strategy for player I in $\varphi'(f)$, there is m with $t = (\sigma \otimes \nu)[m] \in V$. Take the least such m. Then $\sigma \otimes \nu = \sigma'_t \otimes (\nu \upharpoonright (t)_2)$, and so $\sigma \otimes \nu$ satisfies both $\exists n\theta((\sigma \otimes \nu)[n])$ and $\eta(\sigma \otimes \nu)$, which means that σ is a winning s-strategy for player I in $\varphi(f)$.

Now assume $s \notin W$. By Σ^0_1-Det^*, player II has a winning s-strategy τ' in $\varphi'(f)$. Let $\overline{V} = \{t \in 2^{<\mathbb{N}} : (\exists t' \subseteq t)\theta(t') \wedge t \notin V\}$. Note that player II wins $\eta(f)$ at each $t \in \overline{V}$ by Σ^0_1-Det^*. By Remark 4.1.5.1 and Lemma 4.2.3.1, ACA_0 yields a sequence $\langle \tau'_t : t \in \overline{V}\rangle$ of winning t-strategies for player II in

$\eta(f)$. Define a new s-strategy τ for player II by

$$\tau(u) = \begin{cases} \tau'_t(u) & \text{if } t \text{ is the } \subseteq\text{-least initial segment of } u \text{ with } s \subseteq t \wedge t \in \overline{V}, \\ \tau'(u) & \text{if there is no such } t. \end{cases}$$

We see that τ is a winning s-strategy for player II. Take an s-strategy υ for player I. If there is m such that $(\upsilon \otimes \tau)[m]$ satisfies $\theta((\upsilon \otimes \tau)[m])$, take the least such m. Then $t = (\upsilon \otimes \tau)[m] \notin V$ since τ' is an s-winning strategy for player II in $\varphi'(f)$ and so $\upsilon \otimes \tau = (\upsilon \upharpoonright (t)_2) \otimes \tau'_t$ holds. Since τ'_t is a t-winning strategy for player II in $\eta(f)$, $\neg\eta(\upsilon \otimes \tau)$ holds and player II wins. If there is no such m with $\theta((\upsilon \otimes \tau)[m])$, then $\neg\exists n \theta((\upsilon \otimes \tau)[n])$ holds and player II wins. □

By Remark 5.3.2, Theorems 5.3.3 and 5.3.4, we have the following.

Corollary 5.3.5. *We can add* $(\Sigma^0_1)_2$-comp.Det*, Bisep(Δ^0_1, Σ^0_1)-comp.Det* *and* Σ^0_1-comp.Det* *to the list of Fact 5.3.1.*

5.4 Intermezzo—complete determinacy and universal strategies

In the proof of Theorem 5.3.4, we used some kinds of axiom of choice. When we investigate complete determinacy schemata further, we often use similar arguments. However, we do not always have sufficient fragment of axiom of choice. Fortunately, we can avoid using the axiom of choice by the following lemma. Lemma 4.5.3 is a variation of these lemma.

Lemma 5.4.1. *Let* $1 \leq n < \omega$. *In* RCA^*_0, *the following is provable: If player I (or II) wins* Σ^0_n *game* $\varphi(f)$ *in* $\mathbb{N}^\mathbb{N}$ *at each* $s \in A \subseteq \mathbb{N}^{<\mathbb{N}}$, *then* Σ^0_n-Det *yields a sequence* $\langle \sigma_s : s \in A \rangle$ *of winning* s*-strategies for player I (resp. II) in* $\varphi(f)$.

Proof. We work in RCA^*_0. Let $\varphi(f)$ be a Σ^0_n game in $\mathbb{N}^\mathbb{N}$. Assume that player I wins $\varphi(f)$ at each $s \in A \subseteq \mathbb{N}^{<\mathbb{N}}$. Although strictly speaking, on our notation all finite sequences are denoted by their codes, when a number is regarded as a code of a finite sequence, we clarify it by applying the fixed enumeration $e : \mathbb{N} \to \mathbb{N}^{<\mathbb{N}}$ of $\mathbb{N}^{<\mathbb{N}}$ to the number. Consider the following Σ^0_n game $\varphi'(f)$:

- Player II chooses $m \in \mathbb{N}$ at her first turn. If $e(m) \notin A$, player I wins.

- If $e(m) \in A$ and $|e(m)|$ is even, then player I wins if $\varphi(e(m) * (f \ominus 2))$.
- If $e(m) \in A$ and $|e(m)|$ is odd, then player I wins if $\varphi(e(m) * (f \ominus 3))$.

Σ^0_n-Det implies that one of the player has a winning strategy in $\varphi'(f)$. We can check that player II has no winning strategy in $\varphi'(f)$. For contradiction, suppose that player II has a winning strategy τ. Consider such a play f:

- Player I first play 0, i. e., $f(0) = 0$.
- Player II play m, following τ.

Note that $e(m) \in A$, otherwise player II loses. Then τ yields a winning $e(m)$-strategy for player II in $\varphi(f)$, which contradicts the assumption that player I wins $\varphi(f)$ at $e(m)$. Hence player I has a winning strategy σ in $\varphi'(f)$.

For $s \in \mathbb{N}^{<\mathbb{N}}$, let $\overline{e}(s)$ be the least k with $e(k) = s$. Let $\widehat{s}$ be the sequence $\langle \sigma(\langle\rangle), \overline{e}(s)\rangle$ if $|s|$ is even and $\langle \sigma(\langle\rangle), \overline{e}(s), \sigma(\langle \sigma(\langle\rangle), \overline{e}(s)\rangle)\rangle$ if $|s|$ is odd. Then define σ_s by $\sigma_s(t) = \sigma(\widehat{s} * (t \ominus |s|))$ for each $s \in A$. Clearly σ_s is a winning s-strategy for player I in $\varphi(f)$ for each $s \in A$.

The statement for player II can be proved similarly. □

Together with complete determinacy, we have the following:

Lemma 5.4.2. *Let $1 \leq n < \omega$.* RCA^*_0 *proves that, for any Σ^0_n game $\varphi(f)$ in $\mathbb{N}^\mathbb{N}$ (or $2^\mathbb{N}$),* Σ^0_n-comp.Det *(resp.* Σ^0_n-comp.Det**) yields a universal winning strategy in $\varphi(f)$, i. e., a sequence $\langle \sigma_s : s \in \mathbb{N}^{<\mathbb{N}}$(resp. $2^{<\mathbb{N}}$)$\rangle$, such that σ_s is a winning s-strategy for player I for all $s \in W$, and such that σ_s is a winning s-strategy for player II for any $s \notin W$, where W is the winning set for player I in $\varphi(f)$.*

Proof. Take $A = W$ (or the complement of W) and apply Lemma 5.4.1 or Lemma 4.5.3 to it. Note that Σ^0_1-comp.Det implies $(\Sigma^0_1 \wedge \Pi^0_1)$-Det*. □

5.5 ATR_0 and complete determinacy

We briefly mention complete determinacy for ATR_0.

Fact 5.5.1 (Corollary 4.4.9)**.** *The arithmetical transfinite recursion is equivalent to* Σ^0_2-Det*, Δ^0_2-Det*, Σ^0_1-Det, Δ^0_1-Det *and* Δ^0_1-comp.Det *over* RCA^*_0*:*

Theorem 5.5.2. ATR_0 *proves* Δ^0_2-comp.Det*.

Proof. We can prove this in an almost same way as the proof of [32, Theorem 6.1], by replacing Π^1_1 transfinite recursion with arithmetical transfinite recursion. Note that, although the original theorem states the equivalence between Π^1_1 transfinite recursion and Δ^0_2-Det, the proof actually shows that between Π^1_1 transfinite recursion and Δ^0_2-comp.Det. □

By Theorem 5.5.2, we have the following.

Corollary 5.5.3. *We can add* Δ^0_2-comp.Det* *to the list of Fact 5.5.1.*

5.6 Π^1_1-CA_0 and complete determinacy

We turn to Π^1_1-CA_0. We have the following game-theoretical characterization.

Fact 5.6.1 (Corollary 4.6.4)**.** Sep(Σ^0_1, Σ^0_2)-Det*, Bisep(Σ^0_1, Σ^0_2)-Det*, $(\Sigma^0_1 \wedge \Pi^0_1)$-Det, Bisep$(\Delta^0_1, \Sigma^0_1)$-Det *and* Π^1_1 *comprehension are pairwise equivalent over* RCA^*_0.

The strengths of Σ^0_1-comp.Det and of Δ^0_1-comp.Det are separated by the next theorem, similarly to the case of the Cantor space.

Theorem 5.6.2. RCA^*_0 *proves that* Σ^0_1-comp.Det *implies* Π^1_1 *comprehension.*

Proof. Assume Σ^0_1-comp.Det. Since Σ^0_1-comp.Det implies Σ^0_1 comprehension, we can work in ACA_0. We prove Σ^1_1 comprehension, which is equivalent to Π^1_1 comprehension. Let $\psi(n)$ be a Σ^1_1 formula. By Lemma 4.5.5, we can find a Π^0_0 formula $\theta(x, y)$ such that ACA_0 proves $\psi(n) \leftrightarrow \exists f \forall m \theta(n, f[m])$ for all n. We prove the existence of $X = \{n : \psi(n)\}$. Set a Π^0_1 game $\varphi(f)$ in the Baire space by $\forall m \theta(f(0), \widetilde{f}[m])$, where $\widetilde{f}$ is a sequence in $\mathbb{N}^{\mathbb{N}}$ defined by $\widetilde{f}(k) = f(2k + 2)$ for all k. To illustrate, the game goes as follows:

player I	n		$\widetilde{f}(0)$		$\widetilde{f}(1)$		$\widetilde{f}(2)$	$\cdots$
player II		$*$		$*$		$*$		$\cdots$

Intuitively, when player I chooses n first, he wins if and only if he constructs an infinite sequence witnessing $\psi(n)$. Σ^0_1-comp.Det yields the winning set W for player I in $\varphi(f)$. Then Δ^0_1 comprehension yields $X = \{n : \langle n \rangle \in W\}$. For n with $\psi(n)$, player II cannot win at $\langle n \rangle$ if player I constructs $\widetilde{f}$ with $\forall m \theta(n, \widetilde{f}[m])$. By Σ^0_1-Det, player I have a winning strategy in $\varphi(f)$. Hence $n \in X$ if $\psi(n)$. Conversely, if $n \in X$, then $\langle n \rangle$-winning strategy for player I yields a witness for $\psi(n)$. Thus $n \in X$ if and only if $\psi(n)$. □

Now we modify the last theorem to get the Cantor space version. Recall the definition of total regularity of $f \in 2^{\mathbb{N}}$ in Definition 4.4.2.

Theorem 5.6.3. RCA_0^* *proves that* Σ^0_2-$\mathsf{comp.Det}^*$ *implies* Π^1_1 *comprehension.*

Proof. This can be proved similarly to the last theorem. Assume Σ^0_2-$\mathsf{comp.Det}^*$. Since Σ^0_2-$\mathsf{comp.Det}^*$ implies Σ^0_1 comprehension by Theorem 4.3.1, we can work in ACA_0. Let $\psi(n)$ be a Σ^1_1 formula. By Lemma 4.5.5, we can find Π^0_0 formula $\theta(x, y)$, in which f does not occur freely, such that $\forall n(\psi(n) \leftrightarrow \exists f \forall m \theta(n, f[m]))$.

Define a game $\varphi(f)$ by

$$(f \text{ is totally regular}) \wedge \forall m \theta(\overline{f_0}(0), \langle \overline{f_0}(1), ..., \overline{f_0}(m+1)\rangle),$$

where f_0 is defined by $f_0(m) = f(2m)$. By Σ^0_2-Det^*, $\varphi(f)$ is determinate. Intuitively, player I chooses n by playing his first 1 at his $(n+1)$-th turn, and tries to construct the sequence $\langle \underbrace{0, ..., 0}_{2g(0)\ times}, 1, \underbrace{0, ..., 0}_{2g(1)\ times}, 1... \rangle$, instead of a sequence $g \in \mathbb{N}^{\mathbb{N}}$ with $\forall m \theta(n, g[m])$.

Σ^0_2 complete determinacy yields the winning set $W \subseteq 2^{<\mathbb{N}}$ for player I in $\varphi(f)$. Δ^0_1 comprehension yields $X = \{n : 0^{2n} * \langle 1 \rangle \in W\}$. As in the proof of the last theorem, we can also check $X = \{n : \psi(n)\}$. □

Remark 5.6.4. By Remark 4.4.5.3, for a Σ^0_2 game $\varphi(f)$, Π^1_1-CA_0 yields

$$W = \{s \in \mathbb{N}^{\mathbb{N}} : \text{player I wins } \varphi(f) \text{ at } s\}.$$

By Σ^0_2-Det^*, proved in Π^1_1-CA_0, W is a winning set for player I in $\varphi(f)$. Hence Σ^0_2-$\mathsf{comp.Det}^*$ is implied by Π^1_1-CA_0.

Theorem 5.3.4 shows that the completion of $(\Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}^*$ does not strengthen the system. Similarly, neither does that of $(\Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}$ nor of $\mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)$-$\mathsf{Det}^*$.

Theorem 5.6.5.

1. Π^1_1-CA_0 *proves* $(\Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{comp.Det}$
2. Π^1_1-CA_0 *proves* $\mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)$-$\mathsf{comp.Det}^*$.

Proof. 1. It can be proved in an almost same way to Theorem 5.3.4. Note that, over ACA_0, for any Π^0_1 formula $\varphi(f)$, $\forall f \in \mathbb{N}^\mathbb{N}\varphi(f)$ is equivalent to a Π^0_1 formula, because, by Lemma 4.5.5, we can find a Π^0_1 formula $\theta(x, y)$ such that $\varphi(f) \leftrightarrow \forall m\theta(f[m])$ for all $f \in \mathbb{N}^\mathbb{N}$, which implies that $\forall f \in \mathbb{N}^\mathbb{N}\varphi(f)$ is equivalent to a Π^0_1 formula $\forall s \in \mathbb{N}^{<\mathbb{N}}\theta(s)$. Therefore, for a Π^0_1 game $\varphi(f)$ in the Baire space, the assertion "σ is a winning s-strategy for player I in $\varphi(f)$" is Π^0_1 over ATR_0 and ATR_0 proves Σ^1_1 choice. (see [30, Lemma VI.5.2]).

2. Let $\varphi(f)$ be a game. Assume that $\varphi(f) \leftrightarrow (\zeta(f) \wedge \eta_0(f)) \vee (\neg\zeta(f) \wedge \eta_1(f))$ for all $f \in 2^\mathbb{N}$, where $\zeta(f)$ is Σ^0_1, where $\eta_0(f)$ is Π^0_2 and where $\eta_1(f)$ is Σ^0_2. By Lemma 3.6.1, we can find a Π^0_0 formula $\theta(x)$ such that RCA^*_0 proves $\zeta(f) \leftrightarrow \exists n\theta(f[n])$. By Remark 4.4.5.3, Π^1_1 comprehension yields the following V and W:

$$V = \{s \in 2^{<\mathbb{N}} : \exists t \subseteq s\theta(t) \wedge \text{ (player I wins } \eta_0(f) \text{ at } s)\},$$
$$W = \{s \in 2^{<\mathbb{N}} : \text{player I wins } \varphi'(f) \text{ at } s\},$$

where $\varphi'(f) \equiv \exists n f[n] \in V \vee (\neg\exists n\theta(f[n]) \wedge \eta_1(f))$.

We prove that W is the winning set for player I in $\varphi(f)$.

Assume that $s \in W$ and σ' is a winning s-strategy for player I in $\varphi'(f)$. By Lemma 4.5.3, we have a sequence $\langle \sigma'_t : t \in V\rangle$ of winning t-strategy for player I in $\eta_0(f)$. Then define an s-strategy σ by

$$\sigma(u) = \begin{cases} \sigma'_t(u) & \text{if } t \text{ is the } \subseteq\text{-least initial segment of } u \text{ with } s \subseteq t \wedge t \in V, \\ \sigma'(u) & \text{if there is no such } t. \end{cases}$$

We see that σ is a winning s-strategy for player I in $\varphi(f)$. Take an s-strategy ν for player II. If there is m with $(\sigma\otimes\nu)[m] \in V$, take the least such m. Then $t = (\sigma\otimes\nu)[m]$ satisfies $\sigma\otimes\nu = \sigma'_t\otimes(\nu \upharpoonright (t)_2)$ and so $\exists n\theta((\sigma\otimes\nu)[n])\wedge\eta_0(\sigma\otimes\nu)$. If there is no such m, then $\neg\exists n\theta((\sigma \otimes \nu)[n]) \wedge \eta_1(\sigma \otimes \nu)$ holds. Therefore σ is a winning s-strategy for player I in $\varphi(f)$.

Now, assume $s \notin W$. Note that $\neg\varphi(f) \leftrightarrow ((\exists n\theta(f[n]) \wedge \neg\eta_0(f)) \vee (\neg\exists n\theta(f[n]) \wedge \neg\eta_1(f)))$. By Σ^0_2-Det^*, which is proved in ATR_0, player II has a winning s-strategy τ in $\varphi'(f)$. Let $\overline{V} = \{s : \exists t \subseteq s\theta(t) \wedge s \notin V\}$. Note that, by Σ^0_2-Det^*, player II wins $\eta_0(f)$ at each $s \in \overline{V}$. Again by Lemma 4.5.3 we have a sequence $\langle \tau'_s : s \in \overline{V}\rangle$ of winning s-strategy for player II in $\eta_0(f)$. Then define an s-strategy τ by

$$\tau(u) = \begin{cases} \tau'_t(u) & \text{if } t \text{ is the } \subseteq\text{-least initial segment of } u \text{ with } s \subseteq t \wedge s \in \overline{V}, \\ \tau'(u) & \text{if there is no such } t. \end{cases}$$

Then τ is a winning s-strategy for player II in $\varphi(f)$. Take an s-strategy υ for player I. If there is m with $\theta((\upsilon \otimes \tau)[m])$, take the least such m. Then $t = (\upsilon \otimes \tau)[m] \in \overline{V}$ and $\upsilon \otimes \tau = (\upsilon \upharpoonright (t)_2) \otimes \tau'_t$ holds, and so $\exists n \theta((\upsilon \otimes \tau)[n]) \wedge \neg\eta_0(\upsilon \otimes \tau)$. If there is no such m, then $\neg\exists n\theta(\upsilon \otimes \tau) \wedge \neg\eta_1(\upsilon \otimes \tau)$ holds. Therefore τ is a winning s-strategy for player II in $\varphi(f)$. □

By Remark 5.6.4 and Theorems 5.6.3 and 5.6.5, we have the following.

Corollary 5.6.6. *We can add the following to the list of Fact 4:* $\mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)$-$\mathsf{comp.Det}^*$*;* $\mathsf{Bisep}(\Sigma^0_1, \Sigma^0_2)$-$\mathsf{Det}^*$ Σ^0_2-$\mathsf{comp.Det}^*$*,* $(\Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{comp.Det}$*,* Bisep-(Δ^0_1, Σ^0_1)-$\mathsf{comp.Det}$ *and* Σ^0_1-$\mathsf{comp.Det}$*.*

5.7 Π^1_1-TR_0 and complete determinacy

In the previous works, stronger determinacy statements also have been investigated. Here we make brief comments on some stronger complete determinacy statements.

As mentioned in the proof of Theorem 5.5.2, [32, Theorem 6.1] essentially shows that Π^1_1-TR_0 proves Δ^0_2-$\mathsf{comp.Det}$.

On complete determinacy in the Cantor space, we have the following.

Theorem 5.7.1. Π^1_1-TR_0 *proves* $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$-$\mathsf{comp.Det}^*$*.*

Proof. It can be proved in an almost similar way to the Theorem 4.7.5. Let $\varphi(f)$ be a game such that $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$-$\mathsf{Det}^*_0$ implies its determinacy. Recall $\varphi^*(f)$ defined in the proof of the Theorem 4.7.5. We proved that the player who wins $\varphi^*(f)$ also wins the original game $\varphi(f)$. In a similar way, we can prove that $W = \{s : \text{player I wins } \varphi^*(f) \text{ at } s\}$ is the winning set for player I. □

Finally, we have the following corollary.

Corollary 5.7.2. $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$-$\mathsf{Det}^*$*,* $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$-$\mathsf{comp.Det}^*$*,* Δ^0_2-Det*,* Δ^0_2-$\mathsf{comp.Det}$ *and* Π^1_1 *transfinite recursion are pairwise equivalent over* RCA^*_0*.*

Chapter 6

Related works to Part 1

In this chapter, we overview related works on this part.

6.1 Determinacy of more complex classes

An operator $\Phi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is in a class Γ of formulae if $x \in \Phi(X)$ is equivalent to a formula in Γ. For an operator $\Phi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$, consider the following sequence $\langle \Phi_\alpha : \alpha \in \mathrm{On} \rangle$ of sets:

- $\Phi_0 = \emptyset$
- $\Phi_\alpha = \Phi(\bigcup_{\beta<\alpha} \Phi_\beta) \cup \bigcup_{\beta<\alpha} \Phi_\beta$

Since $\langle \Phi_\alpha : \alpha \in \mathrm{On} \rangle$ is an increasing sequence, there exists, in ZFC, $\alpha < \omega_1$ such that $\Phi_\alpha = \Phi_\beta$ for all $\beta > \alpha$ and the least such Φ_α is called the *fixed point* of Φ.

However, in a weak set theory, such as KP, the existence of the fixed point is not always guaranteed. Γ *inductive definition* asserts that, for any operator in Γ, there exists a fixed point of Φ. Now we formalize this notion in second order arithmetic.

Definition 6.1.1 (pre-wellordering)**.** A binary relation $W \subseteq \mathbb{N} \times \mathbb{N}$ is *pre-ordering* on its *field* $\mathrm{field}(W) = \{x : \exists y((x, y) \in W \vee (y, x) \in W)\}$, if it satisfies the following properties:

reflexivity $\forall a \in \mathrm{field}(W)((a, a) \in W)$

connectivity $\forall a \in \mathrm{field}(W)\forall b \in \mathrm{field}(W)((a, b) \in W \vee (b, a) \in W)$

transitivity $\forall a \in \mathrm{field}(W)\forall b \in \mathrm{field}(W)\forall c \in \mathrm{field}(W)((a,b) \in W \wedge (b,c) \in W \to (a,c) \in W)$

For a binary relation W, $W_x = \{y : (y,x) \in W\}$ and $W_{<_W x} = \{y : (y,x) \in W \wedge (x,y) \notin W\}$. A pre-ordering W is a *pre-wellordering* if it is well-founded, i.e., there is no $f : \mathbb{N} \to \mathrm{field}(W)$ such that, for all n, $(f(n+1), f(n)) \in W$ and $(f(n), f(n+1)) \notin W$.

Definition 6.1.2 (Γ-ID)**.** Let Γ be a class of L_2 formulae. Γ *inductive definition* (Γ-ID) asserts that, for any Γ operator Φ, there exists a set $W \subseteq \mathbb{N} \times \mathbb{N}$ such that

1. W is a pre-wellordering on $\mathrm{field}(W)$
2. $\forall x \in F\ W_x = \Phi(W_{<_W x}) \cup (W_{<_W x})$.
3. $\Phi(\mathrm{field}(W)) = \mathrm{field}(W)$.

For an operator Φ, $\mathrm{field}(W)$ of the set W with above three properties is called a *fixed point* of Φ.

An operator Φ is *monotone* if, for any $X \subset \mathbb{N}$ and $Y \subset \mathbb{N}$, $X \subseteq Y$ implies $\Phi(X) \subseteq \Phi(Y)$. *Monotone* Γ *inductive definition* asserts that, for any monotone Γ operator Φ, there exists a fixed point.

$(\Sigma^0_2 \wedge \Pi^0_2)$-$\mathsf{Det}^*$ and Σ^0_2-Det are characterized by inductive definition as follows.

Proposition 6.1.3 ([20] and [33])**.** *$(\Sigma^0_2 \wedge \Pi^0_2)$-$\mathsf{Det}^*$, Σ^0_2-Det, Σ^1_1 inductive definition, monotone Σ^0_1 inductive definition are pairwise equivalent over RCA^*_0.*

Lemma 4.7.4 can be extended as follows.

Proposition 6.1.4 ([17])**.** *Let $1 \leq n < \omega$. For any Σ^0_{n+1} formula $\psi_0(f)$ and Π^0_{n+1} formula $\psi_1(f)$, we can find a Π^0_n formula $\psi(x,i,f)$ such that ACA_0 prove the following.*

$$\forall f \in 2^{\mathbb{N}}(\psi_0(f) \leftrightarrow \psi_1(f)) \to \left[\begin{array}{l} \exists Y(\mathrm{WO}(Y, <_Y))\wedge \\ (\forall f \in 2^{\mathbb{N}})(((y,j) <^*_Y (x,i) \wedge \forall n\psi(x,i,f)) \to \forall n\psi(y,j,f))\wedge \\ (\forall f \in 2^{\mathbb{N}})(\psi_0(f) \leftrightarrow \exists x \in Y(\psi(x,0,f) \wedge \neg\psi(x,1,f)))\wedge \\ (\forall f \in 2^{\mathbb{N}})(\neg\psi_0(f) \leftrightarrow \exists x \in Y(\psi(x,1,f) \wedge \neg\psi(x',0,f))) \end{array}\right] \qquad (\star)$$

where x' is the $<_Y$-successor of x, and where $(Y \times 2, <^*_Y)$ is a well ordering defined by

$$(x, i) <^*_Y (y, j) \leftrightarrow x <_Y y \vee (x = y \wedge i < j).$$

Definition 6.1.5. Let $0 < n < \omega$ and $0 < k < \omega$. The class $(\Sigma^0_n)_k$ of formulae are defined as follows.

- $(\Sigma^0_n)_1 = \Sigma^0_n$
- $(\Sigma^0_n)_{2k} = (\Sigma^0_n)_{2k-1} \wedge \Sigma^0_n$
- $(\Sigma^0_n)_{2k+1} = (\Sigma^0_n)_{2k} \vee \Sigma^0_n$

In [18], iterations of inductive definition along countable well-ordering are considered. $[\Sigma^1_n]^\alpha$ inductive definition asserts the existence of fixed point of α Σ^1_1 operators. For detail, see [18].

Proposition 6.1.6 ([18] and [20])**.** *Let $0 < k < \omega$. $(\Sigma^0_n)_{k+1}$-Det^*, $(\Sigma^0_n)_k$-Det, monotone $[\Sigma^1_1]^k$ inductive definition and $[\Sigma^1_1]^k$ inductive definition are pairwise equivalent over RCA^*_0.*

In particular, we have the following.

Proposition 6.1.7. Δ^0_3-Det^*, Δ^0_3-Det *are equivalent over* RCA^*_0

Remark 6.1.8. The above proposition can be proved from Lemma 4.4.3. In [18], an inductive definition schema which is equivalent to Δ^0_3-Det^* is investigated.

About Σ^0_3 determinacy, it is easy to check Π^1_3 comprehension implies Σ^0_3 by formalizing the proof of [6]. In [37], it is proved that Δ^1_3 comprehension does not implies Σ^0_3 determinacy over RCA^*_0.

For more stronger determinacy, in [7], it is proved that Π^1_∞ comprehension does not implies Σ^0_5-Det^*, which means any subsystem of second order arithmetic does not imply Σ^0_5 determinacy. [9] comment that any subsystem of second order arithmetic does not implies Σ^0_4-Det.

In [17], the relationship among determinacy and various comprehension and induction schemata are investigated.

6.2 Labeling and complete determinacy

In this section, we consider the relationship between complete determinacy and labeling of positions defined in [2].

The notion of labeling of positions generalizes the method of proving Σ^0_1 determinacy by [8].

Definition 6.2.1 (Labeling and soundness)**.** We say that $\mathbf{L} = \langle L_I, <_I, L_{II}, <_{II} \rangle$ is a *labeling system* if L_I and L_{II} are disjoint sets, $<_I$ is a well-ordering on L_I, and $<_{II}$ is a well-ordering on L_{II}. We call any partial function $l : X^{<\mathbb{N}} \to L_I \cup L_{II}$ a *labeling*, where X is either $\mathbb{N}$ or $\{0, 1\}$.

For a labeling l and a position $s \in X^{<\mathbb{N}}$, an s-strategy σ for player I is *l-good* if it satisfies the following property: if $t \in \mathrm{dom}(\sigma)$ and there is j with $l(t * \langle j \rangle) \in L_I$, then $l(t * \sigma(t))$ is the $<_I$-least element of the set $\{l(t * \langle j \rangle) : j \in X\} \cap L_I$. The player II case is handled analogously.

For a game $\varphi(f)$ in $X^{\mathbb{N}}$, we say that l is *φ-sound at s* if either $l(s) \in L_I$ and every l-good s-strategy for player I is winning in $\varphi(f)$, or $l(s) \in L_{II}$ and every l-good s-strategy for player II is winning in $\varphi(f)$. If l is total and φ-sound at every $s \in X^{<\mathbb{N}}$, then we say that l is *globally φ-sound.*

Note that the definition of labeling system and labeling are given on the setting of set theory in [2], this definition works also on the setting of second order arithmetic.

Using labeling system, Σ^0_1-Det is proved as follows. Note that this proof is done in the context of descriptive set theory, not in second order arithmetic. Given Σ^0_1 game $\exists n \theta(f[n])$, where $\theta(x)$ is Π^0_0, consider $L_I = \omega_1$ with canonical order $<_I$ on ω_1 and $L_{II} = \{\infty\}$. Then define partial function l_α from $\mathbb{N}^{<\mathbb{N}}$ to ω_1 by $l_0(s) = 0$ if $\theta(s)$ and, for $\alpha \geq 1$,

$$l_\alpha(s) = \begin{cases} l_\beta(s) & \beta \text{ is the minimum } \beta \text{ with } s \in \mathrm{dom}(l_\beta), \text{ if such } \beta \text{ exists,} \\ \alpha & s \notin \bigcup_{\beta<\alpha} \mathrm{dom}(l_\beta) \text{ and } \forall i \in \mathbb{N}(s * \langle i \rangle \in \bigcup_{\beta<\alpha} \mathrm{dom}(l_\beta)). \end{cases}$$

Let $l' = \bigcup_{\alpha<\omega_1} l$ and set $l : \mathbb{N}^{<\mathbb{N}} \to \omega_1 \cup \{\infty\}$ by $l(s) = l'(s)$ if $s \in \bigcup_{\alpha\in\omega_1} \mathrm{dom}(l_\alpha)$ and $l(s) = \infty$ otherwise. Then we can check that player I wins at s if $l(s) \in L_I$ and player II wins otherwise.

In [2], the equivalence between determinacy and the existence of both a labeling system and a globally sound labeling is proved under a kind of weak axiom of choice. The following theorem asserts that we can remove the choice axiom by replacing determinacy with complete determinacy.

Theorem 6.2.2. *Let $1 \leq n < \omega$. Assume X is either $\mathbb{N}$ or $2 = \{0,1\}$. For any game $\varphi(f)$ in $X^{\mathbb{N}}$, following are equivalent in ACA_0:*

1. *$\varphi(f)$ is completely determinate;*
2. *The existence of a labeling system and a globally $\varphi(f)$-sound labeling.*

Proof. For (1)→(2), assume that a game $\varphi(f)$ in $\mathbb{N}^{\mathbb{N}}$ is completely determinate and W the winning set for player I in $\varphi(f)$. By Lemma 5.4.2, we have a universal winning strategy $\langle \sigma_s : s \in \mathbb{N}^{<\mathbb{N}} \rangle$. As before, we use the fixed enumeration $e : \mathbb{N} \to \mathbb{N}^{<\mathbb{N}}$. For a strategy ν in $\varphi(f)$, Δ^0_1 comprehension yields $T_\nu = \{s : \forall t \subsetneq s(t \in \mathrm{dom}(\nu) \to t * \nu(t) \subseteq s)\}$. In other words, T_ν is the set of all finite plays following ν.

Then $\mathbf{L} = \langle L_I, <_I, L_{II}, <_{II} \rangle$ and l defined by

$$
\begin{aligned}
&l(s) = \text{the minimum } m \text{ with } s \in T_{\sigma_{e(m)}}, \\
&L_I = \{m : e(m) \in W\} \qquad\qquad <_I = < \cap\, (L_I \times L_I), \\
&L_{II} = \{m : e(m) \notin W\} \qquad\qquad <_{II} = < \cap\, (L_{II} \times L_{II}),
\end{aligned}
$$

are a labeling system and a globally φ-sound labeling. Note that l is totally defined since $s \in T_{\sigma_s}$.

For (2)→(1), assume that $\langle L_I, <_I, L_{II}, <_{II} \rangle$ is a labeling system and that l is a globally φ-sound labeling for a Σ^0_n game $\varphi(f)$. In ACA_0, we can define an s-strategy σ_s for $l(s) \in L_I$ (or L_{II}) by

$$
\sigma_s(t) = \begin{cases} n & \text{if } n \text{ is the least } m \text{ such that } l(t * \langle m \rangle) \text{ is the least element of} \\ & \{l(t * \langle i \rangle) : l(t * \langle i \rangle) \in L_I (\text{resp. } L_{II})\}, \\ 0 & \text{if there is no such } n. \end{cases}
$$

Then, for each s, σ_s is a l-good strategy for player I (resp. II), and so it is a winning s-strategy. Therefore $W = \{s : l(s) \in L_I\}$ is the set of all winning positions for player I. □

In the above proof, we cannot replace the base theory with RCA^*_0 nor RCA_0, because we cannot define a function in a way " $v(t) =$ the $<_I$-least t such that..." by Δ^0_1 comprehension in general. Since [2] considered stronger settings, it is not unnatural that the definition given there does not work well in RCA^*_0 or RCA_0. However, if we change the definition of "l is globally φ-sound labeling" as follows, we can prove the equivalence over RCA^*_0:

l is total and *there exists l-good winning s-strategy for each $s \in X^{<\mathbb{N}}$.*

Part II

Determinacy in intuitionistic mathematics

Chapter 7

Introduction to Part 2

This chapter treats determinacy in Brouwerian intuitionistic mathematics.

This mathematics is the mathematics over *intuitionistic* logic with some special axioms, which are based on the philosophy of the founder, L. E. J. Brouwer.

Intuitionistic logic allows all the inference rules in classical logic but the *law of excluded middle* (*LEM*), which asserts that, for any statement A, $A \vee \neg A$ holds. (for detail, see Chapter 8),

The lack of LEM causes many *strange* theorems from a viewpoint of classical mathematician. First, we cannot compare real numbers in general, i. e., there exist real numbers x and y such that we cannot say that $x < y \vee x = y \vee y < x$, because we cannot have De Morgan's law without LEM.

Furthermore, special axioms of intuitionistic mathematics cause more strange theorems. The *continuity principle*, which asserts that every function from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$ is continuous, implies that every functions on $\mathbb{R}$ is continuous. For more about interesting (or strange) world of intuitionistic mathematics, see [10].

Determinacy in intuitionistic mathematics is previously investigated by [34]. Note that determinacy statement "player I has a winning strategy, *or* player II has a winning strategy" is disjunctive statement. Because we do not have De Morgan's law, various formalizations of determinacy, which are all equivalent over classical logic, are considerable. Actually, in [34], three formalizations of determinacy have proposed. Among them, most important one is *predeterminacy*. We say that a game G is *predeterminate* if, whenever there exists a (continuous) function η such that, for all (continuous) strategy τ for player II, $\eta(\tau)$ is a play in which player II follows τ and player II lose

G, player I wins G.

[34] treated ordinary ω-length game in $\mathbb{N}^{\mathbb{N}}$, i. e., the game such that, for a given set $A \subseteq \mathbb{N}$, players I and II alternately choose natural number to form an infinite sequence f and such that player I wins if and only if $f \in A$. [34] proved that any game $A \subseteq \mathbb{N}^{\mathbb{N}}$ such that player II has only finitary many options at each her turn is predeterminate. In such games, strategies for players are partial functions from $\mathbb{N}^{<\mathbb{N}}$ to $\mathbb{N}$, which can be regarded as functions from $\mathbb{N}$ to $\mathbb{N}$. Such functions are always continuous, and so they are not restricted by continuity principle.

In this part, we consider variations of games such that strategies for players are functions from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$. By continuity principle, the strategies are much restricted than classical mathematics.

We show, by giving some examples, that the continuity principle and the lack of LEM make the behavior of determinacy drastically different from that in classical mathematics. To explicate the role of classical principles in determinacy, we also treat predeterminacy in classical mathematics.

Chapter 8

Preliminaries for Part 2

8.1 Logic and axioms of intuitionistic mathematics

In this section, we clarify the mathematical setting of Chapters 8 and 9.

The logical constants have their constructive meanings and the rules of intuitionistic logic are employed. In particular, a disjunctive statement $A \vee B$ means there exists a proof of A or one of B, and an existential statement $(\exists x \in V)A(x)$ means there exist an element a of V and an proof of $A(a)$. $\neg A$ means that a contradiction can be implied from the proof of A.

In this logic, *the law of excluded middle (LEM)*, which asserts that $A \vee \neg A$ holds for every statement A, is not allowed in general. For example, consider the following statement P, called *Goldbach's conjecture*:

> Every even integer greater than 2 can be written as the sum of two primes.

At present, we do not have a proof of either P or $\neg P$. Therefore $P \vee \neg P$ does not holds.

De Morgan's law, which asserts $\neg(A \wedge B) \leftrightarrow \neg A \vee \neg B$ for all statements A and B, also does not allowed in general. Let A and B be the following statements:

A: If there exists 99-length uninterrupted sequence of 9 in the decimal expansion of π, the least such one starts from an even digit.

B: If there exists 99-length uninterrupted sequence of 9 in the decimal expansion of π, the least such one starts from an odd digit.

While $A \wedge B$ implies contradiction clearly, we do not have a proof either $\neg A$ or $\neg B$. Therefore we have $\neg(A \wedge B)$ but $\neg A \vee \neg B$.

A statement A is *decidable* if $A \vee \neg A$ holds. A set $X \subseteq V$ is decidable if the statement $a \in X$ is decidable for each $a \in V$.

Notations without mentions follows ones in Part 1.

An infinite sequence α of natural numbers $\alpha(0), \alpha(1), \alpha(2), ...$ may be determined by some finitely described algorithm, i. e., the n-th element $\alpha(n)$ of α is the result of the algorithm for input n. Sometimes, however, such an infinite sequence may be constructed step by step by choosing its elements one by one. In this case, the construction of the sequence is never finished: At any point in time, only finitely many elements have been chosen, and so we can only know a finite part of the sequence.

The latter construction is not permitted in the constructive mathematics, and so this point divides intuitionistic mathematics from the constructive mathematics.

Note that every infinite sequence, even if it is given by an algorithm, can be regarded as a result of step-by-step-construction. This is the reason we do not distinguish infinite sequences of natural numbers by their manners of construction.

The following axiom is employed in the mathematical setting of this and the next Chapter, which is not accepted in the classical setting.

The continuity principle

If $R \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ is a binary relation such that, for every α, there is m with $\alpha R m$, then, for every α, there exist m and n such that, for every β, $\alpha[n] = \beta[n]$ implies $\beta R m$.

This axiom is adopted for the following reason. If, for every α, we can find a suitable m, then m must be determined by a finite part of α, because α must be step-by-step-constructed and so we can only know a finite part of α at each moment of time.

Definition 8.1.1. $T \subseteq \mathbb{N}$ is a *spread-law* if such that $\langle\rangle \in T$ and, for each s, $s \in T$ if and only if there is n with $s * \langle n \rangle \in T$. For a spread-law T, $[T]$ denotes the set of all α such that $\alpha[m] \in T$ for all m. $S \subseteq \mathbb{N}^{\mathbb{N}}$ is a *spread* if $S = [T]$ for some spread-law T. (In classical mathematics, a spread-law is often called an non-empty tree without leaves and spread $[T]$ is the set of paths of T.) For a spread $S \subseteq \mathbb{N}^{\mathbb{N}}$, η is a *code of a continuous function* if, for any $\alpha \in S$ and

m, there is n with $\eta(\langle m\rangle * \alpha[n]) \neq 0$. For a code η of a continuous function, $\eta|\alpha$ is $\beta \in \mathbb{N}^{\mathbb{N}}$ such that, for all m, $\beta(m) = \eta(\langle m\rangle * \alpha[p]) - 1$, where p is the least n with $\eta(\langle m\rangle * \alpha[n]) \neq 0$.

Since, in intuitionistic logic, $(\forall x \in V)A(x)$ means that there is a function $\mathcal{F}$ such that, for all $a \in V$, $\mathcal{F}(a)$ is a proof of $A(a)$, the continuity principle leads the following.

The second axiom of continuous choice

Let S be a spread on $\mathbb{N}$ and $R \subseteq S \times \mathbb{N}^{\mathbb{N}}$ a relation. If, for all α in S, there is β with $\alpha R \beta$, then there is a code η of a continuous function such that $\alpha R(\eta|\alpha)$ for all α.

Definition 8.1.2. A spread-law T is a *fan-law* if, for each s in T, there are only finitely many n with $s * \langle n\rangle \in T$. (Classically a fan-law is often called a finitary branching tree.) A spread S is a *fan* if $S = [T]$ for some fan-law T. $B \subseteq \mathbb{N}$ is a *bar* in a spread S if, for every sequence α in S, there is n with $\alpha[n] \in B$. A bar B is *bounded* if there is n such that $|b| < n$ for each b in B.

The following is intuitionistic counterpart of König's lemma. Because Brouwer implied it from *Bar induction principle*, it is called a theorem. Here we treat it as an axiom here.

The strict fan theorem

For a fan S and a decidable bar B in S, there is a bounded sub-bar $B' \subseteq B$ in S.

While König's lemma and the strict fan theorem are equivalent in classical mathematics, they are not in intuitionistic mathematics. Actually we can construct a "so-called" intuitionistic counterexample, i. e., a fan T which has sequences of any finite length such that we cannot prove that T has an infinite path, i. e., $\alpha : \mathbb{N} \to \mathbb{N}$ such that $\alpha[n] \in T$ for all n. Let $i^n \in \{0,1\}^n$ be such that $i^n(k) = i$ for all $k < n$ and let $i^{\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ be such that $i^{\mathbb{N}}(n) = i$ for all n. Define $T \subseteq \{i^n : i < 2, n \in \mathbb{N}\}$ by

$0^n \in T \leftrightarrow$ there is no $k < n$ such that $p_{k+i} = 9$ for all $i < 99$, or if the least such k is even,

$1^n \in T \leftrightarrow$ there is no $k < n$ such that $p_{k+i} = 9$ for all $i < 99$, then the least such k is odd, where p_k denotes the k-th digit of the decimal expansion of π. We can easily see that T is a fan which has sequences of any finite length and that if T has an infinite path α, then $\alpha = 0^{\mathbb{N}}$ or $\alpha = 1^{\mathbb{N}}$. Assume that T has an infinite path α. If $\alpha(0) = 0$ (or 1), then we must have a proof of the statement "if there is uninterrupted occurrences of 9 of length 99 in the decimal expansion of π, the least such one starts at an even (resp. odd) digit." Up to now, we do not have any proof of such statements, and so "there is an infinite path in T" does not hold. (If we have a proof in future, we can find another so-called counterexample using another unsolved problem in a similar way.)

8.2 Determinacy in intuitionistic mathematics

In this section, we introduce the notion of determinacy and variants.

As in the previous part, for $A \subseteq \mathbb{N}^{\mathbb{N}}$, the *game* $G(A)$ in $\mathbb{N}^{\mathbb{N}}$ is defined as follows. Two players, called players I and II, starting with player I, alternately choose a natural number to construct $\alpha \in \mathbb{N}^{\mathbb{N}}$. Player I wins if and only if the resulting play α is in A. Player II wins if and only if player I does not win. A strategy for player I (or II) is a function which assigns a natural number to each even-(resp. odd-)length sequence in $\mathbb{N}^{<\mathbb{N}}$.

For a strategy σ for player I (or II), $\alpha \in_I \sigma$ (resp. $\alpha \in_{II} \sigma$) denotes that α is a play in which player I (or II) follows σ at all his turns. Note that if $\alpha \in_I \sigma$ and $\alpha \in_{II} \tau$, then α is uniquely determined. Since finite sequences of natural number can be coded by natural number, We can regard a strategy α for a player as a function from $\mathbb{N}$ to $\mathbb{N}$. A winning strategy for player I (or II) in $G(A)$ is a strategy for player I (resp. II) such that player I (resp. II) wins if he/she follows it.

In classical mathematics, we say that $G(A)$ is determinate if one of the players has a winning strategy in $G(A)$. Note that the classical determinacy is a disjunctive statement.

There are many variations of game.

Game $G_{X^{\mathbb{N}}}$ in $X^{\mathbb{N}}$: For $A \subseteq X^{\mathbb{N}}$, players alternately choose an element of X to construct $\alpha \in X^{\mathbb{N}}$. Player I wins if $\alpha \in A$ and player II wins if player I does not win.

γ-length game G_γ: For a given ordinal γ and $A \subseteq \mathbb{N}^\gamma$, players alternately choose a natural number to construct $\alpha \in \mathbb{N}^\gamma$. Player I wins if $\alpha \in A$ and player II wins if player I does not win.

Game $G_{[S]}$ in a spread $[S]$: For a given spread-law S and $A \subseteq [S]$, players alternately choose a natural number to construct α so that $\alpha[n] \in S$ for every n. Player I wins if $\alpha \in A$ and player II wins if player I does not win.

[34] introduced three formalization of determinacy in intuitionistic mathematics.

The following is the simplest formalization.

Definition 8.2.1. $G(A)$ is *strongly determinate* if, in $G(A)$, *either* player I *or* player II has a winning strategy.

Unfortunately, almost no game is strongly determinate. Since, in our setting, it might happen that we can not decide $f \in A$ or $f \notin A$. Therefore we consider other formalizations.

Definition 8.2.2. $G(A)$ is *determinate from the view point of player I* if, *if* for every strategy τ of player II, there is $\alpha \in_{II} \tau$ with $\alpha \in A$, *then* player I has a winning strategy in $G(A)$.

This statement corresponds to the classical statement "if player II has no winning strategy, then player I has one in $G(A)$," which is classically equivalent to "$G(A)$ is determinate."

Definition 8.2.3. An *anti-strategy for player I in* $G(A)$ is a function η which assigns $\alpha \in_{II} \tau$ to each strategy τ for player II in $G(A)$. An anti-strategy η for player I *secures* A if, for any strategy τ for player II, $\eta(\tau) \in A$. $G(A)$ is *predeterminate from the viewpoint of player I* if, *if* he has an anti-strategy securing A, *then* he has a winning strategy in $G(A)$.

Note that $G(A)$ is predeterminate from the viewpoint of player I, if $G(A)$ is determinate from his viewpoint.

Moreover, in a game $\mathcal{G}(X)$ in $\mathbb{N}^\mathbb{N}$ (or spread $[S]$), the second axiom of continuous choice yields the converse, i. e., predeterminacy implies determinacy, since a strategy for a player can be regarded as a function from $\mathbb{N}$ to $\mathbb{N}$ and since if there is $\alpha \in_{II} \tau$ with $\alpha \in X$ for all strategy τ for player II, then by the second axiom of continuous choice an anti-strategy for player I securing X is given by a code η of a continuous function.

Proposition 8.2.4 (The intuitionistic determinacy theorem [34, Theorem 3.5])**.** *Let $[S]$ be a II-finitary branching spread, i. e., S is a spread-law such that, for every odd-length $s \in S$, there are at most finitely many n with $s * \langle n \rangle \in T$. If there exists an anti-strategy η for player I securing A, then there exists a winning strategy σ for player I in $G_{[S]}(A)$ such that, for any $\alpha \in_I \sigma$, there exists a strategy δ for player II with $\eta | \delta = \alpha$. In particular, $G_{[S]}(A)$ is predeterminate from the viewpoint of player I for every $A \subseteq [S]$.*

In particular, if $A \subseteq \{0,1\}^{\mathbb{N}}$, $G_{\{0,1\}^{\mathbb{N}}}(A)$ is predeterminate from the viewpoint of player I. [34] also gave $A \subseteq \mathbb{N}^{\mathbb{N}}$ such that $G(A)$ is not predeterminate from the viewpoint of player I.

Remark 8.2.5*.* The notion of predeterminacy can be formalized from the viewpoint of player II and we can obtain similar results to the last proposition.

Chapter 9

Variations of games and predeterminacy

In this chapter, we consider other variations of games in the intuitionistic mathematics. For these games, we can define the three formalizations of determinacy in the same way.

9.1 2-length games in $\{0,1\}^{\mathbb{N}} \times \{0,1\}$

This section treats one of the simplest cases in which less strategies are allowed than in the classical context. $\{0,1\}^{\mathbb{N}} \times \{0,1\}$ denotes the product topological space of the Cantor space and discrete space $\{0,1\}$.

For given $A \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}$, the game $\mathcal{G}_1(A)$ is defined as follows:

- Player I chooses $\alpha \in \{0,1\}^{\mathbb{N}}$.
- Player II chooses $i \in \{0,1\}$.
- Player I wins if $(\alpha, i) \in A$ and player II wins if player I does not win.

Although $\{0,1\}^{\mathbb{N}} \times \{0,1\}$ is homeomorphic to the Cantor space topologically, we must be sensitive to the ordertype of the indexing set for the sequences.

In this game, a strategy for player I is his initial move α, and a strategy for player II is a function from $\{0,1\}^{\mathbb{N}}$ to $\{0,1\}$. The continuity principle forces all the strategies for player II to be continuous, and so we may regard a strategy τ for player II as a code of a continuous function such that $\tau(\alpha) = (\tau|\alpha)(0) \in \{0,1\}$ for all $\alpha \in \{0,1\}^{\mathbb{N}}$. $B = \{s \in \{0,1\}^{<\mathbb{N}} | \tau(s) > 0\}$ is a decidable bar in the fan $\{0,1\}^{\mathbb{N}}$. Then, by the strict fan theorem, there is a bounded sub-bar $B' \subseteq B$. Take n such that $|s| < n$ for every $s \in B'$. Then, $\{0,1\}^n$ is also a bar in $\{0,1\}^{\mathbb{N}}$, and, for every $\alpha, \beta \in \{0,1\}^{\mathbb{N}}$, $\alpha[n] = \beta[n]$ implies $\tau|\alpha(0) = \tau|\beta(0)$. Thus we can regard τ as a function from $\{0,1\}^n$ to $\{0,1\}$, which can be coded by a natural number. Because an anti-strategy η for player I is a function from the set of all strategies for player II to the set of plays in this game, it can be regarded as a function from $\mathbb{N}$ with the discrete topology to $\{0,1\}^{\mathbb{N}} \times \{0,1\}$.

The following examples show that even simpler sets, such as open or closed sets, are not predeterminate from the viewpoint of player I.

Example 9.1.1. An open game $\mathcal{G}_1(A)$ which is not predeterminate from the viewpoint of player I: Define $A_i = \{0^n * \langle 1, i\rangle : n \in \mathbb{N}\}$ and $A = \{(\alpha, i) : \exists n(\alpha[n] \in A_i)\}$. Then A is open. Let η be the anti-strategy for player I which assigns $(0^n_\tau * \langle 1, \tau(0^n_\tau)\rangle * 0^{\mathbb{N}}, \tau(0^n_\tau))$ to each strategy $\tau : \{0,1\}^n \to \{0,1\}$ for player II. Then $\eta(\tau) \in A$ for each strategy τ for player II, and so η is an anti-strategy for player I securing A. On the other hand, it is clear that player I has no winning strategy in $\mathcal{G}_1(A)$.

Example 9.1.2. A closed game $\mathcal{G}_1(B)$ which is not predeterminate from the viewpoint of player I: Let T be an intuitionistic counterexample to König's lemma, i. e., an unbounded binary tree such that we cannot prove that T has an infinite path. Let $T_i = \{t * i^n : t \in T \wedge n \in \mathbb{N}\}$. Then $B = \{(\alpha, i) : \forall n(\alpha[n] \in T_i)\}$ is a closed set. If player I had a winning strategy α in $\mathcal{G}_1(B)$, α would be an infinite path of T. Thus player I cannot have a winning strategy in $\mathcal{G}_1(B)$. On the other hand, player I has an anti-strategy securing B. Fix an enumeration of T and let t_n be the minimum $s \in T$ such that $|s| = n$ with respect to this enumeration. Let η be the anti-strategy for player I which assigns $(t_n * (\tau(t_n))^{\mathbb{N}}, \tau(t_n))$ to each strategy $\tau : \{0,1\}^n \to \{0,1\}$ for player II. Clearly η secures B.

9.2 $\omega + 1$ length games in $\{0,1\}^{\mathbb{N}} \times \{0,1\}$

In this section, we consider another kind of games in $\{0,1\}^{\mathbb{N}} \times \{0,1\}$.

For given $A \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}$, the game $\mathcal{G}_2(A)$ is defined as follows.

- Player I and player II alternately choose $i \in \{0,1\}$ to form $\alpha \in \{0,1\}^{\mathbb{N}}$.
- After α is formed, player I chooses $i \in \{0,1\}$.

- Player I wins $\mathcal{G}_2(A)$ if and only if $(\alpha, i) \in A$.

In this game, a strategy σ for player I is a pair (σ_0, σ_1) of functions $\sigma_0 : \bigcup_{n\in\mathbb{N}}\{0,1\}^{2n} \to \{0,1\}$ and $\sigma_1 : \{0,1\}^{\mathbb{N}} \to \{0,1\}$. By the strict fan theorem, we can regard, as well as in the last section, σ_1 as a function from $\{0,1\}^n$ to $\{0,1\}$ for some $n \in \mathbb{N}$.

A strategy for player II is a function $\tau : \bigcup_{n\in\mathbb{N}}\{0,1\}^{2n+1} \to \{0,1\}$, which can be regarded as an element of $\{0,1\}^{\mathbb{N}}$. Then an anti-strategy η for player I is a function from $\{0,1\}^{\mathbb{N}}$ to $\{0,1\}^{\mathbb{N}} \times \{0,1\}$, which can be regarded a pair (η_0, η_1) of codes of continuous functions such that, for any strategy τ for player II, $(\eta_0|\tau, (\eta_1|\tau)(0)) \in_{II} \tau$. By the strict fan theorem, there is n such that for any strategies τ and τ', $\tau[n] = \tau'[n]$ implies $(\eta_1|\tau)(0) = (\eta_1|\tau')(0)$, and so we can regard η_1 as a function from $\{0,1\}^n$ to $\{0,1\}$.

Theorem 9.2.1. *For any $C \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}$, $\mathcal{G}_2(C)$ is predeterminate from the viewpoint of player I.*

Proof. For $i < \{0,1\}$, set $C_i = \{\alpha : (\alpha, i) \in C\}$. Assume that $\eta = (\eta_0, \eta_1)$ is an anti-strategy for player I securing C and that η_1 can be regarded as a function from $\{0,1\}^n$ to $\{0,1\}$ for some n. Note that, in $G_{\{0,1\}^{\mathbb{N}}}(C_0 \cup C_1)$, η_0 is an anti-strategy for player I securing $C_0 \cup C_1$. Let σ_0 be a winning strategy for player I such that for any $\alpha \in_I \sigma_0$, there exists a strategy δ for player II with $\alpha = \eta_0|\delta$. By the second axiom of continuous choice, there exists a code of continuous function ζ such that, for any strategy $\alpha \in P_{\sigma_0}$, $\zeta|\alpha$ is a strategy for player II with $\eta_0|(\zeta|\alpha) = \alpha$. By the strict fan theorem, there exists a natural number m such that, for any α and β in P_{σ_0}, $\alpha[m] = \beta[m]$ implies $(\zeta|\alpha)[n] = (\zeta|\beta)[n]$. Then we can define $\sigma_1 : P_{\sigma_0} \to \{0,1\}$ by $\sigma_1(\alpha) = \eta_1((\zeta|\alpha)[n])$, since $\sigma_1(\alpha)$ is determined by $\alpha[m]$. Define a new strategy $\sigma = (\sigma_0, \sigma_1)$ for player I in $\mathcal{G}_2(C)$. Then, for any $(\alpha, i) \in_I \sigma$, the strategy $\delta = \zeta|\alpha$ for player II satisfies $(\alpha, i) = (\eta_0|\delta, (\eta_1|\delta)(0))$, and so σ is a winning strategy for player I in $\mathcal{G}_2(C)$. □

Comparing this theorem with the examples in the last section, we can conclude that predeterminacy depends *how* players construct the sequence rather than *what* sequence they do.

9.3 $\omega + 2$-length game in $\{0,1\}^{\mathbb{N}} \times \{0,1\}^2$

Next we consider slightly longer games.

For a given set $A \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}^2$, consider the following game $\mathcal{G}_3(A)$.

- First, player I and player II alternately choose $n \in \{0,1\}$ to form $\alpha \in \{0,1\}^{\mathbb{N}}$.
- After α is formed, player I chooses $i \in \{0,1\}$ and player II chooses $j \in \{0,1\}$.
- Player I wins if $(\alpha, \langle i,j \rangle) \in A$ and player II wins if player I does not win.

Similarly to the previous section, a strategy σ for player I is a pair (σ_0, σ_1), where σ_0 is a function from $\bigcup_{n\in\mathbb{N}}\{0,1\}^{2n}$ to $\{0,1\}$ and where σ_1 is a function from $\{0,1\}^{\mathbb{N}}$ to $\{0,1\}$. We can regard σ_1 as a function from $\{0,1\}^n$ to $\{0,1\}$ for some $n \in \mathbb{N}$.

A strategy τ for player II is a pair (τ_0, τ_1), where τ_0 is a function from $\bigcup_{n\in\mathbb{N}}\{0,1\}^{2n+1}$ to $\{0,1\}$ and where τ_1 is a function from $\{0,1\}^{\mathbb{N}} \times \{0,1\}$ to $\{0,1\}$. Note that since τ_1 is continuous, its restriction $\tau_{1,i}$ to $\{0,1\}^{\mathbb{N}} \times \{i\}$ is also continuous and so we can regard τ_1 as a pair (τ_{10}, τ_{11}) of functions from $\{0,1\}^{n_i}$ to $\{0,1\}$ for some n_i's.

Hence, the set of strategies for player II can be regarded as $\{0,1\}^{\mathbb{N}} \times \mathbb{N}$, and so an anti-strategy for player I can be regarded as a function η from $\{0,1\}^{\mathbb{N}} \times \mathbb{N}$ to $\{0,1\}^{\mathbb{N}} \times \{0,1\}^2$ such that $\eta(\tau) \in_{II} \tau$ for each strategy τ for player II.

As in the case of $\mathcal{G}_1(X)$, we have the following examples. In the following two examples, for any $s \in \{0,1\}^{<\mathbb{N}}$, let s' be the sequence $\langle s(0), s(2), ..., s(2n) \rangle$, where n is the maximal m with $2m < |s|$.

Example 9.3.1. Recall A_i defined in Example 9.1.1. Then the open game $\mathcal{G}_3(A')$ defined by $A' = \{(\alpha, \langle i,j \rangle) : \exists n((\alpha[n])' \in A_j)\}$ is not predeterminate from the viewpoint of player I.

Example 9.3.2. Recall T_i defined in Example 9.1.2. Then the closed game $\mathcal{G}_3(B')$ defined by $B' = \{(\alpha, \langle i,j \rangle) : \forall n((\alpha[n])' \in T_j)\}$ is not predeterminate from the viewpoint of player I.

Chapter 10

Predeterminacy in classical mathematics

In this chapter, we consider predeterminacy in classical mathematics in order to investigate the role of classical principles in predeterminacy. Note that all the definitions and statements in this chapter are made in classical mathematics which includes the countable axiom of choice.

10.1 Definition of predeterminacy in classical mathematics

Recall that, in intuitionistic mathematics, an anti-strategy is a function η such that $\eta(\tau) \in_{II} \tau$ for each strategy τ for player II. We translate this definition into classical mathematics, noticing that every function on $\mathbb{N}^{\mathbb{N}}$ is continuous in intuitionistic mathematics:

Definition 10.1.1. Let $\mathcal{G}(X)$ be any of games treated in the previous chapters. An *anti-strategy for player I in* $\mathcal{G}(X)$ is a continuous function which assigns $\alpha \in_{II} \tau$ to every *continuous* strategy τ for player II in $\mathcal{G}(X)$. An anti-strategy η for player I in $\mathcal{G}(X)$ *secures* X if $\eta(\tau) \in X$ for all continuous strategies τ for player II. $\mathcal{G}(X)$ is *predeterminate* from the viewpoint of player I if,

> if player I has an anti-strategy η securing X then player I has a winning strategy in $\mathcal{G}(X)$.

Note that the ordinary definition of the determinacy statement can be seen as "if there is a function η such that $\eta(\tau) \in_{II} \tau$ and $\eta(\tau) \in X$ for *all* strategies τ for player II, then player I has a winning strategy in $\mathcal{G}(X)$."

For $X \subseteq \mathbb{N}^{\mathbb{N}}$, strategies for players in the game $G(X)$ can be regarded as functions from $\mathbb{N}$ to $\mathbb{N}$, and so all the strategies are continuous. Therefore the condition "continuous" for strategies has no effect in games $G(X)$, but it does in the games $\mathcal{G}_1(X)$, $\mathcal{G}_2(X)$ and $\mathcal{G}_3(X)$. Moreover the continuity in the definition of anti-strategy is essential in the following discussion.

As mentioned in [34, 1.1], The intuitionistic determinacy theorem holds also in classical mathematics. In particular, for all $A \subseteq \{0,1\}^{\mathbb{N}}$, $G_{\{0,1\}^{\mathbb{N}}}(A)$ is predeterminate from the viewpoint of player I in the classical mathematics.

In the following section, we consider the predeterminacy of the games $\mathcal{G}_1(X)$, $\mathcal{G}_2(X)$ and $\mathcal{G}_3(X)$ which are defined in the last chapter, in classical mathematics. Due to König's lemma, the classical counterpart of the strict fan theorem, also in classical mathematics, a continuous function from $\{0,1\}^{\mathbb{N}} \to \{0,1\}$ or $\{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is given by its code η defined in Chapter 8. In particular, a strategy for player II in $\mathcal{G}_1(A)$ can be seen as a function $\tau : \{0,1\}^n \to \{0,1\}$ for some n and an anti-strategy for player I in $\mathcal{G}_2(A)$ can be seen as a pair (η_0, η_1) of a code η_0 of continuous function and $\eta_1 : \{0,1\}^m \to \{0,1\}$ for some m.

10.2 Predeterminacy of $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ in classical mathematics

The game $\mathcal{G}_1(A)$ is not predeterminate from the viewpoint of player I, where A is defined in the proof of Example 9.1.1. For closed games, the situation differs: Whereas Example 9.1.2 is a closed game which is not predeterminate from the viewpoint of player I in intuitionistic mathematics, we will show that there is no such closed game in classical mathematics.

For $X \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}$ and $s \in \{0,1\}^{<\mathbb{N}}$, η is an *anti-strategy for player I securing X above s* if η is an anti-strategy for player I such that, for each strategy τ for player II, $\eta(\tau) = (\alpha, i)$ satisfies $(s * \alpha, i) \in X$.

Note that, by the countable axiom of choice, player I has an anti-strategy securing X above s, if and only if, for any n and any strategy $\tau : \{0,1\}^n \to \{0,1\}$ for player II, there exists $\alpha \in \{0,1\}^{\mathbb{N}}$ such that $\langle s * \alpha, \tau(\alpha[n]) \rangle \in X$.

Lemma 10.2.1. *If player I has an anti-strategy securing X above s, then player I has an anti-strategy securing X above either $s * \langle 0 \rangle$ or $s * \langle 1 \rangle$.*

Proof. Assume that, for contradiction, player I has an anti-strategy η securing X above s, but neither above $s*\langle 0\rangle$ nor above $s*\langle 1\rangle$. Then, for each $i < 2$, there exist strategies $\tau^i : \{0,1\}^{n_i} \to \{0,1\}$ such that, for every $\alpha \in \{0,1\}^{\mathbb{N}}$, $(s*\langle i\rangle*\alpha, \tau_i(\alpha[n_i])) \notin X$. Fix such τ^0 and τ^1. Take $n = \max\{n_0+1, n_1+1\}$. Then define $\tau : \{0,1\}^n \to \{0,1\}$ by

$$\tau(t) = \begin{cases} \tau^0(\langle t(1), ..., t(n_0)\rangle) & \text{if } t(0) = 0, \\ \tau^1(\langle t(1), ..., t(n_1)\rangle) & \text{otherwise.} \end{cases}$$

It is easy to see that $(s*\alpha, \tau(\alpha[n])) = (s*\langle k\rangle*\beta, \tau^k(\beta[n_k]))$, where $k = \alpha(0)$ and where $\beta(m) = \alpha(m+1)$ for all m. By the assumption that η is an anti-strategy for player I securing X, $\eta(\tau) = (\alpha, \tau(\alpha[n]))$ satisfies $(s*\alpha, \tau(\alpha[n])) \in X$. However, letting $\alpha = \langle i\rangle * \beta$, we can say $(s*\langle i\rangle*\beta, \tau^i(\beta[n_i])) = (s*\alpha, \tau(\alpha[n])) \in X$ which contradicts the choice of τ^i. □

Theorem 10.2.2. *For closed $X \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}$, $\mathcal{G}_1(X)$ is predeterminate from the viewpoint of player I.*

Proof. Let $X \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}$ be a closed set and $X_i = \{\alpha \in \{0,1\}^{\mathbb{N}} : (\alpha, i) \in X\}$. Then, for each $i < 2$, there exists $X'_i \subseteq \{0,1\}^{<\mathbb{N}}$ such that $\alpha \in X_i$ if and only if $\alpha[n] \in X'_i$ for all n. Assume that player I has an anti-strategy securing X. Note that player I has an anti-strategy securing X above $\langle\rangle$. Define α by recursion as follows:

$$\alpha(n) = \begin{cases} 0 & \text{if player I has an anti-strategy securing } X \text{ above } \alpha[n], \\ 1 & \text{otherwise.} \end{cases}$$

By Lemma 10.2.1 and by induction, we can prove that player I has an anti-strategy securing X above $\alpha[n]$ for all n. In particular, for all n, $\alpha[n] \in X'_0 \cap X'_1$, and so α is a winning strategy for player I. □

For the $\omega+1$-length game $\mathcal{G}_2(X)$, we have the following theorem.

Theorem 10.2.3. *For any $X \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}$, $\mathcal{G}_2(X)$ is predeterminate from the viewpoint of player I.*

Proof. This can be proved in a similar way to Theorem 9.2.1. □

Let us turn to the game $\mathcal{G}_3(X)$. There is an open game $\mathcal{G}_3(X)$ which is not predeterminate from the viewpoint of player I ($\mathcal{G}_3(A')$ defined in Example 9.3.1 enjoys this property). How about closed game? The author has not yet solved it.

Chapter 11

Further problems

Many interesting problems concerning this thesis are left to solve. Besides others, the problems particularly interesting for the author are the followings:

1. Analysis of determinacy of Σ^0_3 sets in classical reverse mathematics: Since a sufficient set existence axiom to prove the determinacy of Σ^0_3 sets is already known, a natural next problem is what set existence axiom is necessary and sufficient. From the solution of this problem, the whole hierarchy of determinacy in second order arithmetic will be clear. Because the determinacy of Σ^0_2 sets is essentially equivalent to the existence of an ordinal with a certain property, it seems that the determinacy of Σ^0_3 sets will be also equivalent to a similar set theoretic statement.

2. Perfect set property in classical reverse mathematics: The set existence axiom that is necessary and sufficient to prove the perfect set property of Σ^1_1 sets, is known. Since perfect set property has a close connection to determinacy, there might be a class whose perfect set property cannot be implied by any set existence axiom formalized in the language of second order arithmetic. For this problem, a reverse mathematical analysis on the relationship between determinacy and the perfect set property must be needed.

3. Analysis of Δ^1_2 sets in classical reverse mathematics: The previous researches have unveiled that the class of Δ^1_2 sets, i. e. both themselves and their complements are definable by Σ^1_2 formulae, has interesting properties as the following two examples (cf. [17]). First, for any arithmetical game, if it is determinate, then there must be a winning strategy of Δ^1_2 complexity. Second, the fixed points of Σ^1_1 operators are also Δ^1_2 sets, although the existence of the fixed points of Σ^1_1 operators cannot be implied by the existence of all Δ^1_2 sets. The analysis on the subclasses consisting of winning strategies for determinate Borel games and of fixed points of Σ^1_1 operators must be useful to clarify the property of sets of reals.

4. Wadge hierarchy in intuitionistic mathematics: One of the most interesting characters of intuitionistic mathematics is that every function is continuous, and so an investigation on classes closed under continuous preimages is hopeful in intuitionistic descriptive set theory. Such classes form Wadge hierarchy, which is known as a refinement of Borel hierarchy in classical descriptive set theory. Since, in Chapter 4, the formalization of Wadge hierarchy played an important role to analyze the hierarchy of determinacy in classical reverse mathematics, it could yield profound results also in constructive reverse mathematics. The first question is whether we can define Wadge hierarchy also in intuitionistic mathematics. Her formalization of the hierarchy in classical reverse mathematics will help this investigation.

5. Determinacy in constructive reverse mathematics: In Chapter 9, it is shown that determinacy of several kinds of games holds under intuitionistic axioms. The next question is whether such axioms are crucial or not, especially, whether determinacy implies some intuitionistic axioms or not. Techniques used in classical reverse mathematics might be applied.

6. Baire property and perfect set property in Bishop's constructive mathematics and intuitionistic mathematics: The first problem on Baire property and perfect set property in Bishop's constructive mathematics and intuitionistic mathematics is how to formalize these properties on intuitionistic logic. The next problem is what kinds of sets have these properties. As mentioned before, many properties of sets are implied by determinacy in classical mathematics. Especially, the implications of Baire property and perfect set property from determinacy are proved in direct ways. Investigating on these implication in Bishop's constructive mathematics will be a good key to analyzing the properties on each setting.

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