Determinacy and Π^1_1 transfinite recursion along ω

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Abstract

It is known that almost all important set existence axioms in second order arithmetic can be characterized by determinacy schemes of infinite games. In this article, we sketch the proof of the equivalence between Π^1_1 transfinite recursion along ω and determinacy scheme.

1 Introduction

It is shown in [4], [6], [7], [10], [11], [12] and [13] that almost all important set existence axioms in reverse mathematics program can be characterized by determinacy schemes of infinite games, as summarized in the table, where the characterizing set existence axioms in the left row are equivalent to determinacy schemes in the same line over suitable base theory, e. g., RCA_0, RCA_0^*.

Subsystem of second order arithmetic	Determinacy in $2^{\mathbb{N}}$ (Det*)	Determinacy in $\mathbb{N}^{\mathbb{N}}$ (Det)
WKL_0^*	Δ^0_1 Σ^0_1	-
WKL_0	$\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)$	-
ACA_0	$(\Sigma^0_1)_2$	-
ATR_0	Δ^0_2 Σ^0_2	Δ^0_1 Σ^0_1
$\mathsf{ATR}_0 + \Sigma^1_1$ induction	$\mathsf{Bisep}(\Delta^0_1, \Sigma^0_2)$	
Π^1_1-CA_0	$\mathsf{Bisep}(\Sigma^0_1, \Sigma^0_2)$ $\vdots$ $\mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)$	$\mathsf{Bisep}(\Delta^0_1, \Sigma^0_1)$ $(\Sigma^0_1)_2$
Π^1_1-TR_0	$\mathsf{Bisep}(\Delta^0_2, \Sigma^0_2)$	Δ^0_2
Σ^1_1-ID_0	$(\Sigma^0_2)_2$	Σ^0_2
$\vdots$	$\vdots$	$\vdots$
$[\Sigma^1_1]^k$-ID_0	$(\Sigma^0_2)_{k+1}$	$(\Sigma^0_2)_k$
$\vdots$	$\vdots$	$\vdots$
$[\Sigma^1_1]^{\mathrm{TR}}$-$\mathsf{ID}_0$	Δ^0_3	Δ^0_3

Despite these "accordance" in weak systems of reverse mathematics, recent studies are unveiling some "discordance" between stronger set existence axioms and determinacy schemes in second order arithmetic. For example, in [5] [1], it is proved Π^1_{n+2}-$\mathsf{CA}_0 \vdash (\Sigma^0_3)_n$-Det and Δ^1_{n+2}-$\mathsf{CA}_0 \nvdash (\Sigma^0_3)_n$-Det, whereas, in [3], it is proved Δ^1_1-Det $\nvdash$ Δ^1_2-comprehension. In [5], it is also proved $\mathsf{Z}_2 \nvdash (\Sigma^0_3)_\omega$-Det. In other words, the series of determinacy schemes and that of comprehension schemes diverge beyond some point. Since Π^1_1-TR_0, Σ^1_1-ID_0 etc. have somehow different flavor from comprehension schemes, the diverging point seems to be located around but above Π^1_1-CA_0.

Therefore, in order to see the relationship between the two series, it seems quite meaningful to investigate this region, i. e., the relationship between determinacy scheme and restrictions of Π^1_1 transfinite recursion. In this paper, we focus on the first important instance of the restrictions, namely, Π^1_1 transfinite recursion along ω.

Main Theorem. Π^1_1-CA_0^+ and $(\Sigma^0_1)_\omega$-Det are equivalent over RCA_0.

This is a refinement of the result in [12] and the Π^1_1 analogue of the result in [8]:

Fact. ACA_0^+ and $(\Sigma^0_1)_\omega$-Det* are equivalent over RCA_0.

In this article, we sketch the proof of the main theorem. We also mention some prospective generalizations of this theorem as a conjecture at the end.

[1] $(\Sigma^0_3)_n$-Det and $(\Sigma^0_3)_\omega$-Det in our notation are denoted by n-Π^0_3-DET and n-Π^0_3-DET respectively, in [5].

2 Preliminaries

2.1 Subsystems of second order arithmetic

We work in an usual language L_2 of second order arithmetic. Most notations and definitions in this paper are fairly standard. The readers can refer to [10].

Definition 2.1. For any formula θ, let $H_\theta((Y, \prec_Y), Z)$ be the following formula:

$$\forall j \forall k(\langle k, j\rangle \in Z \leftrightarrow \theta(k, (Z)^j)),$$

where $(Z)^j = \{\langle m, i\rangle \in Z : i \prec_Y j\}$. For any class Γ of formulae, Γ *transfinite recursion*, denoted by Γ-TR, is the following scheme:

$$\forall (Y, \prec_Y)(\mathrm{WO}(Y, \prec_Y) \to \exists Z H_\theta((Y, \prec_Y), Z)),$$

where $\mathrm{WO}(Y, \prec_Y)$ is a Π^1_1 formula stating the wellorderedness of $(Y, \prec_Y)$, and where $\theta(x, Y)$ is a Γ formula in which Z does not occur freely. The system Π^1_1-TR_0 consists of RCA_0 plus the schema of Π^1_1 transfinite recursion.

For a wellordering $(W, \prec)$, the schema of Γ transfinite recursion along $(W, \prec)$, denoted by Γ-TR$(W, \prec)$ is as follows:

$$\exists Z H_\theta((W, \prec), Z),$$

where θ is any Γ formula in which Z does not occur freely. In particular, Γ-TR$(\mathbb{N}, <)$ is denoted by Γ-TRω. The system Π^1_1-CA^+_0 consists of RCA_0 plus Π^1_1-TRω.

This is Π^1_1 analogue of ACA^+_0 (cf. [10, X.3] and [9]). Π^1_1-CA^+_0 is between Π^1_1-CA_0 and Π^1_1-TR_0.

Remark 2.2. Over RCA_0, Π^1_1-TR is equivalent to Σ^1_1-TR and Π^1_1-TRω is equivalent to Σ^1_1-TRω.

2.2 Games and determinacy

Let X be either $\mathbb{N}$ or $\{0, 1\}$. A *game in* $X^\mathbb{N}$ is given by a formula $\varphi(f)$ with a distinguished function variable $f \in X^\mathbb{N}$. It is played as follows: Two players, say player I and player II, alternately choose $x \in X$ to form $f \in X^\mathbb{N}$. Player I wins if and only if $\varphi(f)$ holds and Player II wins otherwise. In this article, we assume that player I is male and that player II is female.

A *strategy* for player I (or II) is a function which assigns an element of X to each even-length (resp. odd-length) finite sequence from X. A strategy for a player is *winning* in a game $\psi(f)$ if the player wins $\psi(f)$ as long as he or she plays following it. A game is *determinate* if one of the players has a winning strategy. We regard a class Γ of formulae with a distinguished function variable as a class of games. Γ *determinacy* asserts that every game in Γ is determinate. In this article, Γ determinacy in $\mathbb{N}^\mathbb{N}$ is denoted by Γ-Det and Γ determinacy in $2^\mathbb{N}$ is denoted by Γ-Det*.

For a game $\psi(f)$ and $s \in X^{<\mathbb{N}}$, a strategy σ for a player is *s-winning* in $\psi(f)$ if the player always wins $\psi(f)$ in any play f with $s \subset f$ in which he or she follows σ at all steps after $|s|$. A player *wins* $\psi(f)$ *at* s if he or she has a winning s-strategy in $\psi(f)$.

The following is a determinacy scheme to each class in the hierarchy between Σ^0_1 and Δ^0_2, which is motivated by Hausdorff's difference hierarchy (cf. [1]).

Definition 2.3 **($(\Sigma^0_n)_{(Y,\prec)}$ determinacy).** A sequence $\langle \varphi(n, f) : n \in \mathbb{N}\rangle$ with variable $f \in X^\mathbb{N}$ of formulae is called $(Y, \prec)$-increasing if $\varphi(n, f) \to \varphi(m, f)$ holds for all $n \prec m$ and $f \in X^\mathbb{N}$. A $(\Sigma^0_n)_{(Y,\prec)}$ game is defined by $(\exists\, \text{even}\, x)(\varphi(x', f) \wedge \neg\varphi(x, f))$ with a $(Y, \prec)$-increasing sequence of Σ^0_n formulae, where x' is $\prec$-successor of x.

The scheme of $(\Sigma^0_n)_{(Y,\prec)}$ determinacy in $X^\mathbb{N}$ asserts that every $(\Sigma^0_n)_{(Y,\prec)}$ game in $X^\mathbb{N}$ is determinate. In particular, we call $(\Sigma^0_n)_{(\mathbb{N},<)}$ game as $(\Sigma^0_n)_\omega$.

2.3 Π^1_1-TR_0 and Δ^0_2 determinacy

In [12], the following theorem is proved. Actually, it is easy to see that the base theory ACA_0 can be replaced to RCA_0.

Theorem 2.4. *Over* ACA_0*,* Π^1_1-TR *and* Δ^0_2-Det *are equivalent.*

The following lemma is equivalent to [12, Theorem 5.2], which plays an important role in the proof of this theorem.

Lemma 2.5. *For any* Δ^0_2 *game* $\psi(f)$*, we can find a* Π^0_0 *formula* $\theta(x,i,y)$ *and* $(Y,\prec)$ *such that* ACA_0 *proves the following.*

- $\mathrm{WO}(Y,\prec)$*;*
- $(\forall f \in \mathbb{N}^{\mathbb{N}})(((x,i) \prec^* (y,j) \wedge \exists n\theta(x,i,f[n])) \to \exists n\theta(y,j,f[n]))$ *(increasing along* $(Y,\prec)$*);*
- $(\forall f \in \mathbb{N}^{\mathbb{N}})(\psi(f) \leftrightarrow \exists x \in Y(\exists n\theta(x,1,f[n]) \wedge \neg\exists n\theta(x,0,f[n])))$,

where $(Y \times 2, \prec)$ *is a wellordering defined by*

$$(x,i) \prec^* (y,j) \leftrightarrow x \prec y \vee (x = y \wedge i < j).$$

From this lemma, each Δ^0_2 game can be expressed by an increasing sequence of Σ^0_1 formulae (coded by a single Σ^0_1 formula) along some wellordering. This is a formalization of [1, §30.VI. Theorem 1], which states that every Δ^0_2 set belongs to some level of difference hierarchy starting from Σ^0_1. Then, it is natural to expect that Π^1_1 transfinite recursion along a certain wellordering $(Y,\prec)$ corresponds to $(\Sigma^0_1)_{(Y,\prec)}$-Det. In the following section, we show that this is the case for $(Y,\prec) = (\mathbb{N},<)$.

3 Π^1_1-CA^+_0 and $(\Sigma^0_1)_\omega$-Det

In this section, we sketch a proof of the equivalence between Π^1_1-TRω and $(\Sigma^0_1)_\omega$-Det. All proofs are refinements of those in [12].

The following lemma is easy to prove.

Lemma 3.1. *For any* Π^0_1 *(or* Σ^0_1*) game* $\psi(f)$ *in* $\mathbb{N}^{\mathbb{N}}$*, there exists a* Σ^1_1 *(resp.* Π^1_1*) formula* $\eta(x,f)$ *such that* RCA_0 *proves that player I (resp. II) wins* $\psi(f)$ *at* $s \in \mathbb{N}^{\mathbb{N}}$ *iff* $\eta(s,f)$ *holds.*

First, we see the implication from Π^1_1-$\mathsf{TR}\omega$ to determinacy.

Theorem 3.2. Π^1_1-CA^+_0 *proves* $(\Sigma^0_1)_\omega$-Det.

Sketch of proof: Let $\langle\varphi(n,f) : n \in \mathbb{N}\rangle$ be an $(\mathbb{N},<)$-increasing sequence of Σ^0_1 formulae. We show that a game $\eta(f) \equiv \exists n(\varphi(2n+1,f) \wedge \neg\varphi(2n,f))$ is determinate. By Normal form theorem [10, Lemma II.2.7] for Σ^0_1, there exists a Π^0_0 formula θ such that $\exists m\theta(n,f[m]) \leftrightarrow \varphi(n,f)$ for all m and $f \in \mathbb{N}^{\mathbb{N}}$. Then, Σ^1_1-$\mathsf{TR}\omega$, which is equivalent to Π^1_1-$\mathsf{TR}\omega$ by Remark 2.2, yields the following set W:

$$\langle s, 2n+1\rangle \in W \leftrightarrow \theta(2n+1,s) \wedge \text{player I wins } \varphi_1(2n+1,f),$$
$$\langle s, 2n+2\rangle \in W \leftrightarrow \text{player II wins } \varphi_2(2n,f), \text{ and}$$

where $\varphi_1(2n+1,f) \equiv \forall m(\neg\theta(2n,f[m]) \vee \forall k < n(\langle 2k+2, f[m]\rangle \notin (W)^{2n+1}))$ and where $\varphi_2(2n,f) \equiv \exists m\exists k < n(\langle 2k+1, f[m]\rangle \in (W)^{2n})$. Note that this construction can be done by Σ^1_1 by Lemma 3.1. Define new Σ^0_1 game $\psi(f) \equiv \exists n\exists m(\langle f[m], 2n+1)\rangle \in W$. Then, we can prove that

- If player I wins $\psi(f)$, player I also wins $\eta(f)$.
- If player II wins $\psi(f)$, player II also wins $\eta(f)$.

By Σ^0_1-Det, which is proved in Π^1_1-CA_0, the determinacy of $\eta(f)$ is implied. □

Next, we see the converse.

Theorem 3.3. *Over* RCA_0, $(\Sigma^0_1)_\omega$-Det *implies* Π^1_1-TRω.

Sc-ketch of proof: Instead of Π^1_1-TRω, we prove that $(\Sigma^1_1)_\omega$-Det implies Σ^1_1-TRω. Since Δ^0_1-Det implies arithmetical comprehension by [10, Lemma V.8.5], we can work in ACA_0. Let $\phi(x, X)$ be a Σ^1_1 formula. By [10, Lemma V.1.4], there exists a Π^0_0 formula $\theta(x, f, X)$ such that $\forall X \forall x(\psi(x, X) \leftrightarrow \exists f \theta(x, f[m], X[m]))$. We consider a game with the following property:

- Player II wins it;
- Player II's winning strategy in it yields Z such that $H_\theta((\mathbb{N}, <), Z)$.

Such a game can be defined as follows:

1. At the first step, player I asks whether $\langle n, i\rangle \in Z$ or not by playing $\langle n, i, 0\rangle$.
2. Set $\langle m, j\rangle := \langle n, i\rangle$. Then, player II answers "yes" by playing $\langle m, j, 2\rangle$, or "no" by playing $\langle m, j, 1\rangle$.
   If her answer is yes, go to 3, and otherwise go to 4.
3. Player II constructs the characteristic function $\chi_{(Z)^j}$ of $(Z)^j$ and a witness f such that $\forall k \theta(m, f[m], (Z)^j[m])$.

player I	player II
$*$	$\langle \chi_{(Z)^j}(0), f(0)\rangle$
$*$	$\langle \chi_{(Z)^j}(1), f(1)\rangle$
$*$	$\langle \chi_{(Z)^j}(2), f(2)\rangle$
$\vdots$	$\vdots$

   If player I attacks player II's decision on the value of $\chi_{(Z)^j}(\langle m_0, j_0\rangle)$ with $j_0 < j$ by playing $\langle m_0, j_0, 0\rangle$,

   - If $\chi_{(Z)^j}(\langle m_0, j_0\rangle) = 0$, set $\langle m, j\rangle := \langle m_0, j_0\rangle$ and go to 4.
   - If $\chi_{(Z)^j}(\langle m_0, j_0\rangle) = 1$, set $\langle m, j\rangle := \langle m_0, j_0\rangle$ and go to 3.

   If there is no such $\langle m_0, j_0\rangle$, then

   - Player I wins if f and $\chi_{(Z)^j}$ satisfy $\exists k \neg\theta(m, f[k], (Z)^j[m])$.
   - Player II wins if f and $\chi_{(Z)^j}$ satisfy $\forall k \theta(m, f[k], (Z)^j[m])$.
4. The same as 3, but the roles of player I and player II in 3 are exchanged.

We can see that the winning condition of this game can be expressed by $(\Sigma^0_1)_\omega$, by describing the argument between players on membership of $(Z)^n$ with Σ^0_1 formula. Player I does not win this game, because if he had a winning strategy, then it would yield a winning strategy for player II in the same game, which is impossible. Therefore player II has a winning strategy. For any player II's winning strategy τ, it is also provable that τ always gives a correct information on the membership of Z, i. e., $\tau(\langle n, i, 0\rangle) = \langle n, i, 2\rangle$ iff $\exists f \forall m \theta(\langle n, i\rangle, f[m], (Z)^i[m])$, which means Z is yielded by τ by arithmetical comprehension. □

Corollary 3.4. *Over* RCA_0, Π^1_1-TRω *and* $(\Sigma^0_1)_\omega$-Det *are equivalent.*

4 Related & future works

As mentioned in Introduction, in [8], it is proved that Π^0_1-Detω is equivalent to $(\Sigma^0_1)_\omega$-Det*, i. e., ACA_0^+ over RCA_0. In [2], a system Π^0_β-CA_0 which consists of RCA_0 plus "β is well-ordered" and Π^0_1-TRβ for a presentation β of computable ordinal. It is natural to expect that the results in [8] and this article are generalized to such systems and $(\Sigma^0_1)_\beta$ determinacy as follows:

Conjecture Let β be a presentation of limit computable ordinal. In the system RCA_0+"β is well-ordered", the following are provable.

1. Π^0_1-TRβ is equivalent to $(\Sigma^0_1)_\beta$-Det*.
2. Π^1_1-TRβ, $(\Sigma^0_1)_\beta$-Det and $\mathsf{Bisep}((\Sigma^0_1)_\beta, \Sigma^0_2)$-Det* are pairwise equivalent.

For the definition of Bisep(-, -), see [6].

References

[1] C. Kuratowski. *Topology, vol. 1.* Academic Press, 1966.

[2] A. Marcone and A. Montalbán. The Veblen function for computability theorists. To appear in J. Symbolic Logic.

[3] M. O. MedSalem and K. Tanaka. Δ^0_3 determinacy, comprehension and induction. *J. Symbolic Logic*, 2(72):452–462, 2007.

[4] M. O. MedSalem and K. Tanaka. Weak determinacy and iterations of inductive definitions. computational prospects of infinity. In *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap*, volume 15, pages 333–353. World Sci. Publ., 2008.

[5] A. Montalbán and R. Shore. The limits of determinacy in second order arithmetic. Submitted for publication.

[6] T. Nemoto. Determinacy of Wadge classes and subsystems of second order arithmetic. *MLQ Math. Log. Q*, 55(2):154–176, 2009.

[7] T. Nemoto. Corrigendum to "determinacy of Wadge classes and subsystems of second order arithmetic". preprint, 2011.

[8] T. Nemoto. Determinacy and transfinite recursion. preprint, 2011.

[9] R. A. Shore. Invariants, Boolean algebras and ACA_0^+. *Trans. Amer. Math. Soc.*, 358(3):989–1014, 2005. electronic.

[10] S. G. Simpson. *Subsystems of second order arithmetic*. Springer Verlag, 1999.

[11] M. O. Medsalem T. Nemoto and K. Tanaka. Infinite games in the Cantor space and subsystems of second order arithmetic. *MLQ Math. Log. Q*, 53(3):226–236, 2007.

[12] K. Tanaka. Weak axiom of determinacy and subsystems of analysis I: Δ^0_2 games. *Z. Math. Logik Grundlag. Math.*, 6(36):481–491, 1990.

[13] K. Tanaka. Weak axiom of determinacy and subsystems of analysis II: Σ^0_2 games. *Ann. Pure Appl. Logic*, 52(1-2):181–193, 1991.