# Corrigendum of “Determinacy of Wadge classes and subsystems of second order arithmetic”

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## 1 Introduction

In [3], it is claimed to be proved that $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$ determinacy in the Cantor space (in the notation in [3], $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)\text{-}\mathsf{Det}^*$) holds in $\Pi^1_1\text{-}\mathsf{TR}_0$. However, the proof given there is not correct. This paper will corrects it.

All of the definitions and notations appearing in this paper without mention can be found in [3].

## 2 The mistake of the previous proof

In [3], the following lemma is given.

**Lemma 6.4** *For any $\Pi^0_2$ formula $\varphi(f)$ with a distinguished function variable $f \in \{0,1\}^{\mathbb{N}}$, we can find, in $\mathsf{RCA}_0$, a $\Pi^0_0$ formula $\theta(x)$ in which $n$ does not occur such that $\forall f \in \{0,1\}^{\mathbb{N}}(\varphi(f) \leftrightarrow \forall n \exists m > n\theta(f[m]))$.*

A proof can be found in [2]. Using this lemma, a tree $T_{\theta_0,\theta_1}$ is defined by

$$T_{\theta_0,\theta_1} = \{x \in (\{0,1\}^{<\mathbb{N}})^{<\mathbb{N}} : x(0) \subsetneq x(1) \subsetneq ... \subsetneq x(|x|-1), \theta_0(x(k)) \text{ for each even } k < |x| \text{ and } \theta_1(x(k)) \text{ for each odd } k < |x|\}$$

for a pair $(\theta_0(x), \theta_1(x))$ of $\Pi^0_0$ formulae such that, for all $f \in \{0,1\}^{\mathbb{N}}$, $\forall n(\exists m > n)\theta_0(f[m])$ if and only if $\neg\forall n \exists m > n\theta_1(f[m])$. We may assume that $\{s \in \{0,1\}^{<\mathbb{N}} : \theta_0(s) \wedge \theta_1(s)\} = \emptyset$ and $\theta_0(\langle\rangle)$ by replacing $\theta_0(x)$ with $\theta'_0(x) \equiv (\theta_0(x) \wedge \neg\theta_1(x)) \vee (x = \langle\rangle)$ and $\theta_1(x)$ with $\theta'_1(x) \equiv \theta_1(x) \wedge \neg\theta_0(x) \wedge (x \neq \langle\rangle)$ if necessary. The above tree $T_{\theta_0,\theta_1}$ is well-founded, for if $F$ were an infinite path of $T_{\theta_0,\theta_1}$, $f \in \{0,1\}^{\mathbb{N}}$ defined by $f(n) = (F(n))(n)$ would satisfy both $\forall n(\exists m > n)\theta_0(f[m])$ and $\forall n(\exists m > n)\theta_0(f[m])$.

On this $T_{\theta_0,\theta_1}$, [3] claims as follows:

> For every $f \in \{0,1\}^{\mathbb{N}}$, we can prove that there exists $x \in T$ such that $x$ has no proper extension in $T$ and $x(|x|-1) \subseteq f$ as follows. If $\forall n(\exists m > n)\theta_i(f[m])$, there exist $p$ and $r > p$ with $\forall q > p\neg\theta_{1-i}(f[q])$ and $\theta_i(f[r])$. If $i = 0$, $\langle f[r]\rangle$ enjoys the desired property. If $i = 1$, $\langle\langle\rangle, f[r]\rangle$ enjoys the desired property.

However, there is no obvious reason to belive that $\langle f[r]\rangle$ or $\langle\langle\rangle, f[r]\rangle$ has no proper extension in $T_{\theta_0,\theta_1}$. Actually, if the above assertion were true, for any $f \in \{0,1\}^{\mathbb{N}}$, there would exist $n_0 < n_1 < ... < n_k$ such that $\langle f[n_0], f[n_1], ..., f[n_k]\rangle \in T_{\theta_0,\theta_1}$ with no proper extension in $T_{\theta_0,\theta_1}$. Then, by weak König's lemma, there is $l$ such that, for all $f \in \{0,1\}^{\mathbb{N}}$, we can determine whether $\forall n(\exists m > n)\theta_0(f[m])$ or $\forall n(\exists m > n)\theta_1(f[m])$ by checking $f[l]$. We see that this is impossible. Take a formula $\psi(f) \equiv \exists n(f(n) = 1)$. Then

$$\psi(f) \leftrightarrow \forall n(\exists m > n)(\exists k < m)f(k) = 1,$$
$$\neg\psi(f) \leftrightarrow \forall n(\exists m > n)(\forall k < m)f(k) = 0.$$

However, there is no $l$ such that we can determine whether $\psi(f)$ or $\neg\varphi(f)$ for all $f \in \{0,1\}^{\mathbb{N}}$ by checking $f[l]$.

# 3 A correct proof

For each $s \in \{0,1\}^{<\mathbb{N}}$, $(s)$ denotes the set $\{t \in \{0,1\}^{<\mathbb{N}} : s \subseteq t\}$. An *s-strategy for player I* (resp. II) is a function $\sigma : (s) \cap \{t \in \{0,1\}^{<\mathbb{N}} : |t| \text{ is even (resp. odd)}\} \to \{0,1\}$. For an $s$-strategy $\sigma$ for player I and an $s$-strategy $\tau$ for player II, $\sigma \otimes \tau$ denotes the sequence $f$ such that $f(i) = s(i)$ for all $i < |s|$, $f(2i) = \sigma(f[2i])$ for all $2i \geq |s|$, $f(2i+1) = \tau(f[2i+1])$ for all $2i+1 \geq |s|$, in other words, $\sigma \otimes \tau$ is the play, starting from $s$, in which player I follows $\sigma$ and player II follows $\tau$. For a game $\varphi(f)$ in $X^{\mathbb{N}}$, an $s$-strategy $\sigma$ for player I (resp. II) is winning if $\varphi(\sigma \otimes \tau)$ (resp. $\neg\varphi(\tau \otimes \sigma)$) for all $s$-strategies $\tau$ for player II (resp. I). *Player I (resp. II) wins at s in* $\varphi(f)$ if there is a winning $s$-strategy for player I (resp. II). Note that although "player I wins at $s$" is defined in another way in [3], these two definitions make no difference.

We need several lemmata.

**Lemma 1.** *Let $1 < n < \omega$. Let $\varphi(x,f)$ be a $\Sigma^0_n$ game in $\{0,1\}^{\mathbb{N}}$. Assume that player I (or player II) wins $\varphi(x,f)$ at $s$ for all $(s,x) \in W \subseteq \{0,1\}^{<\mathbb{N}} \times \mathbb{N}$. Then,* $\mathsf{RCA}_0$ *proves that* $\Sigma^0_n$-$\mathsf{Det}^*$ *in* $\{0,1\}^{\mathbb{N}}$ *yields a sequence* $\langle \sigma_s : s \in W\rangle$ *of winning s-strategies for player I (resp. II) in* $\varphi(x,f)$.

*Proof.* We work in $\mathsf{RCA}_0$. Assume $\Sigma^0_n$-$\mathsf{Det}^*$. Let $\varphi(x,f)$ be a $\Sigma^0_n$ game in $\{0,1\}^{\mathbb{N}}$ and $W \subseteq \{0,1\}^{<\mathbb{N}} \times \mathbb{N}$. Assume that player I wins $\varphi(f)$ at each $s \in W$. Fix an enumeration $e : \mathbb{N} \to \{0,1\}^{<\mathbb{N}} \times \mathbb{N}$. For any $m$, let $(m_0, m_1) = e(m)$. Consider the following game $\varphi'(f)$:

First, player II chooses $(s,x) \in \{0,1\}^{<\mathbb{N}}$. If $(s,x) \in W$, players starts game $\varphi(x,g)$ from $s$ and player I wins when $\varphi(x,g)$ holds. Otherwise, player I wins.

Let $f$ be a play. Such a game is realized as follows:

- Player II choose $m \in \mathbb{N}$ at by playing 0 at her first $m$ turns and playing 1 at her $(n+1)$-th turn. If $e(m) \notin W$, player I wins.
- If $e(m) \in W$ and $|e(m)|$ is even, then player I wins if $\varphi(m_1, m_0 * (f \ominus 2m+2))$.
- If $e(m) \in W$ and $|s|$ is odd, then player I wins if $\varphi(m_1, m_0 * (f \ominus (2m+3)))$.

Formally, $\varphi'(f)$ is defined as follows:

$$\begin{aligned}\varphi'(f) \equiv \exists m(\forall i < m)(f(2i+1) = 0 \wedge f(2m+1) = 1 \wedge e(m) \in W) \to \\ \exists m \exists k((\forall i < m)(f(2i+1) = 0 \wedge f(2m+1) = 1) \wedge \\ ((|m_0| = 2k \wedge \varphi(f \ominus (2m+2))) \vee (m_k| = 2k+1 \wedge \varphi(f \ominus (2m+3)))))\end{aligned}$$

Since $n > 1$, $\Sigma^0_n$-$\mathsf{Det}^*$ proves that one of the players has a winning strategy in $\varphi'(f)$. We can check that player II has no winning strategy in $\varphi'(f)$. For contradiction, suppose that player II had a winning strategy $\tau$. Consider such a play $f$:

- Player I first play 0 until player II plays 1.
- Player II follows $\tau$.

Note that player II must play 1 at some turn, i.e., $f(2m+1) = 1$ for some $m$, and $e(m) \in W$, otherwise player II loses. If $\tau$ were a winning strategy, then it would yield a winning $m_0$-strategy for player II in $\varphi(m_1, f)$. Since player I wins $\varphi(m_1, f)$ at $m_0$, this is impossible. Hence player I has a winning strategy $\sigma$ in $\varphi'(f)$.

For $(s,x) \in \{0,1\}^{<\mathbb{N}}$, let $\overline{e}(s,x)$ be the least $k$ with $e(k) = (s,x)$ and $s_x$ the $(2\overline{e}(s,x)+2)$-length sequence $t$ such that $t(2k+1) = 0$ for each $2k+1 < 2\overline{e}(s,x)$, $t(2\overline{e}(s,x)+1) = 1$, and $t(2k) = \sigma(t[2k])$ for each $2k < 2\overline{e}(s,x)+2$. In other words, $\overline{s}$ is the finite play in which player II has just chosen a sequence $s \in \{0,1\}^{\mathbb{N}}$ and player I followed $\sigma$.

Then, for $(s,x) \in W$, define an $s$-strategy $\sigma_{s,x}$ in $\varphi(x,s)$ for player I by $\sigma_{s,x}(s * t) = \sigma(s_x * t)$ for even-length $s \in W$ and $\sigma_{s,x}(s * t) = \sigma(s_x * \langle \sigma(s_x)\rangle * t)$ for odd-length $s \in W$. Clearly $\sigma_{s,x}$ is a winning $s$-strategy for player I in $\varphi(f)$ for each $(s,x) \in W$.

The assertion for player II can be proved similarly. □

In descriptive set theory, Hausdorff proved (cf. [1, §37. III. Theorem]) that a $\mathbf{\Delta}^0_2$ set can be represented as a boolean combination of transfinitely many $\Pi^0_1$ sets, i.e., for any $\mathbf{\Delta}^0_2$ set $A$ of Polish space $\mathcal{X}$, there exist an ordinal $\gamma < \omega_1$ and a decreasing sequence $\langle A_\alpha : \alpha < \gamma\rangle$ of $\mathbf{\Pi}^0_1$ sets such that

$$A = \{x \in A_0 : \min\{\alpha : x \notin A_\alpha\} \text{ is odd}\}.$$

The following lemma is a formalization of the above theorem in $\mathsf{Z}_2$.

**Lemma 2.** *For any $\Sigma^0_2$ formula $\psi_0(f)$ and $\Pi^0_2$ formula, we can find a $\Pi^0_0$ formula $\theta(x, i, f)$ suth that $\mathsf{ACA}_0$ prove the following.*

$$\begin{aligned}
&\forall f \in \{0,1\}^{\mathbb{N}}(\psi_0(f) \leftrightarrow \psi_1(f)) \to\\
&\quad(\exists X(WO(X, <_X)) \wedge (\forall f \in \{0,1\}^{\mathbb{N}})(((y, j) <^*_X (x, i) \wedge \forall n\theta(x, i, f[n])) \to \forall n\theta(y, j, n))\wedge\\
&\quad\quad(\forall f \in \{0,1\}^{\mathbb{N}})(\psi_0(f) \leftrightarrow \exists x \in X(\forall n\theta(x, 0, f[n]) \wedge \neg\forall n\theta(x, 1, f[n])))\wedge\\
&\quad\quad\quad(\forall f \in \{0,1\}^{\mathbb{N}})(\neg\psi_0(f) \leftrightarrow \exists x \in X(\forall n\theta(x, 1, f[n]) \wedge \neg\forall n\theta(x', 0, f[n])))), \quad (\star)
\end{aligned}$$

*where $WO(X, <_X)$ is a $\Pi^1_1$ formula which asserts that $(X, <_X)$ is a well ordering, where $x'$ is the $<_X$-successor of $x$, and where $(X \times \{0, 1\}, <^*_X)$ is a well ordering defined by*

$$(x, i) <^*_X (y, j) \leftrightarrow x <_X y \vee (x = y \wedge i < j).$$

*Proof.* See Theorem 5.1 of [4]. □

**Theorem 1.** $\Pi^1_1$-$\mathsf{TR}_0$ *proves* $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$-$\mathsf{Det}^*$.

*Proof.* Let $\varphi(f)$ be a game in the Cantor space such that $\forall f \in \{0,1\}^{\mathbb{N}}(\varphi(f) \leftrightarrow (\psi(f)\wedge\eta_0(f))\vee(\neg\psi(f)\wedge\eta_1(f)))$ for some $\Sigma^0_2$ formulae $\psi(f)$ and $\eta_1(f)$ and $\Pi^0_2$ formula $\eta_0(f)$. Assume that $\forall f(\psi(f) \leftrightarrow \psi'(f))$ for some $\Pi^0_2$ formula $\psi'(f)$. Note that, for all $f \in \{0,1\}^{\mathbb{N}}$, $\neg\varphi(f) \leftrightarrow (\psi(f) \wedge \neg\eta_0(f)) \vee (\neg\psi(f) \wedge \neg\eta_1(f))$. By applying Lemma 2, taking $\psi(f)$ and $\psi'(f)$ as $\psi_0(f)$ and $\psi_1(f)$ respectively, we can find a $\Pi^0_1$ formula $\forall n\theta(x, i, f[n])$ such that $(\star)$ holds. Then there exists a well ordering $(X, <_X)$ and for all $f \in \{0,1\}^{\mathbb{N}}$, $\psi(f) \leftrightarrow \exists x(\forall n\theta(x, 0, f[n])\wedge\neg\forall n\theta(x, 1, f[n]))$ and $\neg\psi(f) \leftrightarrow \exists x(\forall n\theta(x, 1, f[n])\wedge\neg\forall n\theta(x', 0, f[n]))$ hold, where $x'$ is the $<_X$-successor of $x$ in $X$. Then, by $\Pi^1_1$ transfinite recursion, define $V_{x,i}$ and $W_{x,i}$ and ($x \in X$ and $i < 2$) as follows:

$$\begin{aligned}
V_{x,0} =&\{s : |s| \text{ is even}, \exists t \subseteq s\neg\theta(x, 1, t) \text{ and player II wins } \eta'_0(x, f) \text{ at } s\},\\
W_{x,0} =&\{s : |s| \text{ is even}, \exists t \subseteq s\neg\theta(x, 1, t) \text{ and } s \notin V_{x,0}\},\\
W_{x,1} =&\{s : |s| \text{ is even}, \exists t \subseteq s\neg\theta(x', 0, t) \text{ and player I wins } \eta'_1(x, f) \text{ at } s\},\\
V_{x,1} =&\{s : |s| \text{ is even}, \exists t \subseteq s\neg\theta(x', 0, t) \text{ and } s \notin W_{x,1}\},
\end{aligned}$$

where $\eta'_0(x, f) \equiv ((\forall n\theta(x, 0, f[n]) \wedge \eta_0(f)) \vee \exists nf[n] \in W_{<_X(x,i)})$, where $W_{<_X(x,i)} = \bigcup_{(y,j)<_X(x,i)} W_{y,i}$ and where $\eta'_1(x, f) \equiv ((\forall n\theta(x, 1, f[n]) \wedge \eta_1(f)) \vee \exists nf[n] \in W_{<_X(x,i)})$.

Set a new game $\varphi^*(f) \equiv \exists n((\forall m < n)(f[m] \notin \bigcup_{x\in X,i<2} V_{x,i}) \wedge f[n] \in \bigcup_{x\in X,i<2} W_{x,i})$. We can check that $\neg\varphi^*(f) \leftrightarrow \exists n((\forall m < n)(f[m] \notin \bigcup_{x\in X,i<2} W_{x,i}) \wedge f[n] \in \bigcup_{x\in X,i<2} V_{x,i})$.

First, we see $\bigcup_{x\in X,i<2} W_{X,i}\cap\bigcup_{x\in X,i<2} V_{X,i} = \emptyset$. Assume $s \in W_{X,i}$. We may assume that $(x, i)$ is the $<^*_X$-least such one. Then $s \notin \bigcup_{(y,j)<^*_X(x,i)} W_{y,j}\cup\bigcup_{(y,j)<^*_X(x,i)} V_{y,j}$ and $s \notin V_{x,i}$. If $\exists t \subseteq s\neg\theta(y, 1, t)$ for some $(y, 1)$ with $(x, i) <^*_X (y, 1)$, then player I wins $\eta'_0(y, f)$ at $s$ and so $s \notin V_{y,0}$. If $\exists t \subseteq s\neg\theta(y', 0, t)$ for some $(y', 0)$ with $(x, i) <^*_X (y', 0)$, then player I wins $\eta'_1(y, f)$ at $s$ and so $s \notin V_{y,1}$. Thus $\bigcup_{x\in X,i<2} W_{x,i} \cap \bigcup_{x\in X,i<2} V_{x,i} = \emptyset$.

Since, for all $f \in \{0,1\}^{\mathbb{N}}$, either $\exists x \in X(\forall n\theta(x, 0, f[n]) \wedge \neg\forall n\theta(x, 1, f[n]))$ or $\exists x \in X(\forall n\theta(x, 1, f[n]) \wedge \neg\forall n\theta(x', 0, f[n]))$ holds, and since $W_{x,0} \cup V_{x,0} = \{s \in \{0,1\}^{<\mathbb{N}} : \exists t \subseteq s\neg\theta(x, 1, t)\}$ and $W_{x,1} \cup V_{x,1} = \{s \in \{0,1\}^{<\mathbb{N}} : \exists t \subseteq s\neg\theta(x', 0, t)\}$ hold, for all $f \in \{0,1\}^{\mathbb{N}}$, there exists $n$ such that $f[n] \in \bigcup_{x\in X,i<2} W_{x,i} \cup \bigcup_{x\in X,i<2} V_{x,i}$. Since $\bigcup_{x\in X,i<2} W_{x,i}\cap\bigcup_{x\in X,i<2} V_{x,i} = \emptyset$, the $\subseteq$-least such $f[n]$ is in exactly one of $\bigcup_{x\in X,i<2} W_{x,i}$ and $\bigcup_{x\in X,i<2} V_{x,i}$.

Therefore, for any $f \in \{0,1\}^{\mathbb{N}}$, exactly one of the following holds:

- $\varphi^*(f) \equiv \exists n((\forall m < n)(f[m] \notin \bigcup_{x\in X, i<2} V_{x,i}) \wedge f[n] \in \bigcup_{x\in X, i<2} W_{x,i})$,
- $\neg\varphi^*(f) \leftrightarrow \exists n((\forall m < n)(f[m] \notin \bigcup_{x\in X, i<2} W_{x,i}) \wedge f[n] \in \bigcup_{x\in X, i<2} V_{x,i})$,

The next claim completes the proof.

**Claim 1.** *The player who wins $\varphi^*(f)$ also wins in $\varphi(f)$.*

First, assume that player I has a winning strategy $\sigma^*$ in $\varphi^*(f)$. By Lemma 1, take a sequence $\langle \sigma_s : s \in \bigcup_{x\in X, i<2} W_{x,i}\rangle$ such that if $(x, i)$ is the $<^*_X$-least element of $X \times \{0, 1\}$ with $s \in W_{x,i}$, then $\sigma_s$ is a winning $s$-strategies for player I in $\eta'_i(x, f)$. Then, for any $s \in \bigcup_{x\in X, i<2} W_{x,i}$, define an $s$-strategy $\sigma^*_s$ for player I by transfinite recursion along $(X \times \{0, 1\}, <^*_X)$ as follows:

$$\sigma^*_s(t) = \begin{cases} \sigma^*_u(t) & \text{if } u \text{ is the } \subseteq\text{-least initial segment of } t \text{ with } s \subsetneq u \text{ and } u \in W_{<^*_X(x,i)}, \\ \sigma_s(t) & \text{if there is no such } u. \end{cases}$$

Now we prove, by $\Pi^1_1$ transfinite induction on $(X \times \{0, 1\}, <^*_X)$, for any $s \in \bigcup_{x\in X, i<2} W_{x,i}$, $\sigma^*_s$ is a winning $s$-strategy for player I in $\varphi(f)$. Assume that $\sigma^*_t$ is a winning $t$-strategy for player I in $\varphi(f)$ for all $(y, j) <^*_X (x, i)$ and for all $t \in W_{y,j}$. Take $s \in V_{x,i}$ and an $s$-strategy $\rho$ for player II. If there is $k$ such that $t = (\sigma^*_s \otimes \rho)[k] \in W_{y,j}$ for some $(y, j) <^*_X (x, i)$, take the least such $k$. Then $\sigma^*_s \otimes \rho = \sigma^*_t \otimes \rho'$, where $\rho' = \rho \restriction (t)$, and so $\varphi(\sigma^*_s \otimes \rho)$ holds by induction hypothesis. If there is no such $k$ and if $i = 0$, then $\forall n\theta(x, 0, (s * f)[n]) \wedge \eta_0(s * f)$ hold. Since $s \in V_{x,i}$, there is $t \subseteq s$ with $\neg\theta(x, 1, t)$, and so $\varphi(f)$. If there is no such $k$ and if $i = 1$, we can similarly prove that $\varphi(f)$ holds.

It is now easy to check that $\sigma$ defined by

$$\sigma(t) = \begin{cases} \sigma^*_u(t) & u \text{ is the } \subseteq\text{-least initial segment of } t \text{ with } s \subsetneq u \text{ and } u \in \bigcup_{x\in X, i<2} W_{x,i} \\ \sigma^*(t) & \text{if there is no such } u \end{cases}$$

is a winning strategy $\sigma$ for player I in $\varphi(f)$.

Let us turn to the case in which player II has a winning strategy $\tau^*$ in $\varphi^*(f)$. As in the previous case, take a sequence $\langle \tau_s : s \in \bigcup_{x\in X, i<2} V_{x,i}\rangle$ of winning $s$-strategies for player II in $\eta'_i(x, f)$ and define a sequence of strategies $\langle \tau^*_s : s \in \bigcup_{x\in X, i<2}\rangle$ by

$$\tau^*_s(t) = \begin{cases} \tau^*_u(u) & \text{if } u \text{ is the } \subseteq\text{-least initial segment of } t \text{ with } s \subsetneq u \text{ and } u \in V_{<^*_X(x,i)} = \bigcup_{(y,j)<^*_X(x,i)} V_{y,j}, \\ \tau_s(t) & \text{otherwise.} \end{cases}$$

Then, we can prove that, for any $s \in \bigcup_{x\in X, i<2} V_{x,i}$, $\tau^*_s$ is a winning $s$-strategy for player II in $\varphi(f)$ by $\Pi^1_1$ transfinite induction. Assume that $\tau^*_t$ is a winning $t$-strategy for player II in $\varphi(f)$ for any $(y, j) <^*_X (x, i)$ and for any $t \in \bigcup_{(y,j)<^*_X(x,i)} V_{y,j}$. Take $s \in V_{x,i}$ and an $s$-strategy $\nu$ for player I. If there is $k$ such that $t = (\nu \otimes \tau^*_s)[k] \in \bigcup_{(y,j)<^*_X(x,i)} V_{y,j}$, then $\nu \otimes \tau^*_s = \nu' \otimes \tau^*_t$, where $nu' = \nu \restriction (t)$, and so $\varphi(\nu \otimes \tau^*_s)$ holds by induction hypothesis. Next we consider the case in which there is no such $k$. We may assume $i = 0$, because the case $i = 1$ can be proved similarly. If $\exists n\neg\theta(x, 0, (\nu \otimes \tau^*_s)[n])$ holds, then, by the property of $\theta$, $(\nu \otimes \tau^*_s)[n] \in \bigcup_{(y,j)<^*_X(x,0)}(V_{y,j} \cup W_{y,j})$. By the fact that $\tau^*_t$ is a winning strategy for player II in $\eta'_0(x, f)$, $(\nu \otimes \tau^*_s)[n]$ is not in $\bigcup_{(y,j)<^*_X(x,i)} W_{y,j}$, and, by the assumption, $(\nu \otimes \tau^*_s)[n]$ is not in $\bigcup_{(y,j)<^*_X(x,i)} V_{y,j}$, which is a contradiction. Therefore $\forall n\theta(x, 0, (\nu\otimes\tau^*_s)[n])$ holds, and so $\nu\otimes\tau^*_s$ satisfies both $\forall n\theta(x, 0, (\nu \otimes \tau^*_s)[n]) \wedge \exists n\neg\theta(x, 1, (\nu \otimes \tau^*_s)[n])$ and $\neg\eta_0(\nu \otimes \tau^*_s)$, which means that player II wins $\varphi(f)$. □

# References


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