

# A sequent calculus demonstration of Herbrand's Theorem

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## Abstract

Herbrand's theorem is often presented as a corollary of Gentzen's sharpened Hauptsatz for the classical sequent calculus. However, the midsequent gives Herbrand's theorem directly only for formulae in prenex normal form. In the Handbook of Proof Theory, Buss claims to give a proof of the full statement of the theorem, using sequent calculus methods to show completeness of a calculus of *Herbrand proofs*, but as we demonstrate there is a flaw in the proof.

In this note we give a correct demonstration of Herbrand's theorem in its full generality, as a corollary of the full cut-elimination theorem for LK. The major difficulty is to show that, if there is an Herbrand proof of the premiss of a contraction rule, there is an Herbrand proof of its conclusion. We solve this problem by showing the admissibility of a *deep contraction rule*.

## 1 Introduction

Herbrand's fundamental theorem [4] gives that provability in the predicate calculus may be reduced to propositional provability: specifically, given any formula  $A$  in the language of first-order logic, we can compute, given a proof of  $A$ , a valid quantifier-free formula built from substitution instances of subformulae of  $A$ . Herbrand's theorem most easily stated for  $\exists$ -formulae or  $\forall\exists$ -formulae, and this form of the theorem is sufficient for applications. The most general form of the theorem that most students of logic will see is for a disjunction of prenex formulae, as this follows as an almost immediate consequence of Gentzen's midsequent theorem (or sharpened Hauptsatz) [3]. The following is the midsequent theorem for **GS**, a one-sided sequent system with multiplicative context handling, as shown in Table 1:

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	$\frac{}{\vdash a, \bar{a}} \text{Ax}$	
$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \vee\text{R}$		$\frac{\vdash \Gamma, A \quad \vdash \Gamma', B}{\vdash \Gamma, \Gamma', A \wedge B} \wedge\text{R}$
$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \text{CR}$		$\frac{\vdash \Gamma}{\vdash \Gamma, A} \text{WR}$
$\frac{\vdash \Gamma, A(t)}{\vdash \Gamma, \exists y. A} \exists\text{R}$		$\frac{\vdash \Gamma, A(z)}{\vdash \Gamma, \forall z. A} \forall\text{R} \quad z \notin \text{free}(\Gamma)$

Table 1: System **GS**

**Theorem 1.** *Suppose that  $\Gamma$  is a sequence of prenex formulae provable in the system **GS**. Then there is some quantifier-free sequent  $\Gamma'$  and a proof in **GS** of the form*

$$\begin{array}{c} \parallel N \\ \vdash \Gamma' \\ \parallel M \\ \vdash \Gamma \end{array}$$

where the derivation  $M$  necessarily contains only propositional rules, and where  $N$  contains only introductions of quantifiers and structural rules. The sequent  $\Gamma'$  is then called the midsequent.

From this one may easily extract a form of Herbrand's theorem for prenex sequents, see for example [7].

The original theorem, as stated by Herbrand, was more general, and stated in terms of a system of proofs for first-order classical logic. The opening chapter of the Handbook of Proof Theory, by Buss [6], gives a readable presentation of a variant of this system called "Herbrand proofs". The general version of Herbrand's theorem can be rendered thus: a formula is valid if and only if it has an Herbrand proof. Buss gives a proof of this statement, which relies on the following incorrect lemma: the system **GS** given above is complete when the contraction rule is restricted to quantifier-free formulae and formulae whose main connective is an existential quantifier. To see that this does not hold, consider the sequent

$$\vdash \forall x. A \wedge \forall x. B, (\exists x. \bar{A} \vee \exists x. \bar{B}) \wedge (\exists x. \bar{A} \vee \exists x. \bar{B})$$

The application of any rule of **GS** other than contraction on the rightmost conjunction yields an invalid sequent.

Of course, Herbrand's theorem does hold in the form stated by Buss. In this note we give a repaired proof of Herbrand's theorem which, like Buss's attempt, derives the theorem from the cut-free completeness of **GS**. We give

a construction yielding, from a cut-free **GS** proof of a formula  $\Gamma$ , an Herbrand proof of  $\Gamma$ ; thus the general Herbrand's theorem is shown to be a corollary of the *general* cut-elimination for the first-order classical sequent calculus, rather than of the midesequent theorem. We prove this by showing that each rule of **GS** is *admissible* in the Herbrand proofs system: given an Herbrand proof of the premises one may obtain an Herbrand proof of the conclusion. The only non-trivial case is that of contraction, where the admissibility of contraction for formulae of rank  $< n$  is not enough to demonstrate admissibility of contraction on rank  $n$  formulae; instead, we show that a more general *deep* contraction rule is admissible.

## 1.1 Conventions

Formulae of first-order logic are always written in negation normal form (that is, negation is primitive only at the level of atoms, with the negation of a general formula being given by the De Morgan laws). The *rank* of a formula is its depth as a tree. We consider formulae of first-order logic modulo the renaming of bound variables ( $\alpha$ -equivalence). A formula  $A$  will be said to be *alpha-normal* if there is at most one occurrence of a quantifier  $q.x$  in  $A$ , where  $x$  is a variable and  $q$  either  $\forall$  or  $\exists$ . Every formula  $A$  is  $\alpha$ -equivalent to an alpha-normal formula. A formula  $B$  is in *prenex normal form* if it has the form  $Q.M$ , where  $M$  contains no quantifiers and  $Q$  is a sequence of quantifiers. In that case, we call  $M$  the *matrix* of  $B$ .

We assume a particular form of variable use for sequent proofs in **GS**. Variables should be used strictly: each universal rule binds a unique eigenvariable, and that eigenvariable occurs only in the subproof above the rule which binds it. Further, we enforce a Barendregt-style convention on the use of variables: the sets of bound and free variables appearing in a proof should be disjoint.

## 2 Herbrand proofs

We give first the definition of Herbrand proofs as formulated by Buss [2].

**Remark 1.** We consider, for cleanness of presentation, only pure first-order logic over a signature of relation symbols and function symbols, containing at least one constant symbol. Extending our approach to one dealing theories containing equality or with nonempty sets of nonlogical axioms may be done with no change in the shape of our argument.

We begin with three key definitions:

**Definition 2.** *Let  $A$  be a formula in negation normal form. An  $\forall$ -expansion of  $A$  is any formula obtained from  $A$  by a finite number of applications of the following operation:*

*If  $B$  is a subformula of an  $\forall$ -expansion  $A'$  of  $A$ , replacing  $B$  in  $A'$  with  $B \vee B$  produces another  $\forall$ -expansion of  $A$ .*

A strong  $\forall$ -expansion of  $A$  is defined similarly, except that now the formula  $B$  is restricted to be a subformula with outermost connective an existential quantifier.

(Note that by this definition,  $A$  is a (strong- $\forall$ ) expansion of itself.) From now on, we will abbreviate “strong  $\forall$ -expansion” to “expansion”. An expansion  $\hat{\Gamma}$  of a sequent  $\Gamma = A_1 \dots A_n$  is a sequence  $\hat{A}_1 \dots \hat{A}_n$  of expansions of the members of  $\Gamma$ .

**Definition 3.** Let  $A$  be an alpha-normal formula. A prenexification of  $A$  is a formula  $B$  in prenex normal form derived from  $A$  by successive applications of the operations

$$qx.A * B \rightsquigarrow qx.(A * B) \quad A * qx.B \rightsquigarrow qx.(A * B)$$

(where  $q$  is either  $\forall$  or  $\exists$ , and  $*$  is either  $\wedge$  or  $\vee$ ). If  $\bigvee \Gamma$  is alpha-normal, a prenexification of  $\Gamma$  is a prenexification of  $\bigvee \Gamma$ .

**Definition 4.** Let  $A$  be a valid alpha-normal first-order formula in prenex normal form. If  $A$  contains  $r \geq 0$  existential quantifiers, then  $A$  is of the following form, with  $B$  quantifier free:

$$(\forall x_1 \dots \forall x_{n_1})(\exists y_1)(\forall x_{n_1+1} \dots \forall x_{n_2})(\exists y_2) \dots (\exists y_r)(\forall x_{n_r+1} \dots \forall x_{n_{r+1}})B(\bar{x}, \bar{y})$$

with  $0 \leq n_1 \leq n_2 \leq \dots \leq n_{r+1}$ . A witnessing substitution for  $A$  is a sequence of terms  $t_1, \dots, t_r$  such that (1) each  $t_i$  contains arbitrary free variables but only bound variables from  $x_1, \dots, x_{n_i}$ , and (2) the formula  $B(\bar{x}, t_1, \dots, t_r)$  is a tautology.

We are now ready to define Herbrand proofs:

**Definition 5 (Buss).** An Herbrand proof of a first-order formula  $A$  consists of a prenexification  $A^*$  of a strong  $\forall$ -expansion of  $A$ , plus a witnessing substitution  $\sigma$  for  $A^*$ .

We will need the more general notion of an Herbrand proof of a sequent:

**Definition 6.** An Herbrand proof of a sequent  $\Gamma$  is a triple consisting of a strong  $\forall$ -expansion  $\hat{\Gamma}$  of  $\Gamma$ , a prenexification  $\Gamma^*$  of  $\hat{\Gamma}$ , and a witnessing substitution  $\sigma$  for  $\Gamma^*$ .

### 3 The proof of Herbrand’s theorem

We show next that the system of Herbrand proofs as given above is complete — each valid sequent has an Herbrand proof. We prove that each rule of the system **GS** is *admissible*; that is, whenever we have an Herbrand proof or proofs of the premises, we have an Herbrand proof of the conclusion. Since **GS** is complete for first-order classical logic, this will be enough to show completeness of the Herbrand proofs system. Proving admissibility is trivial for most of the rules of **GS**, and we leave the proof as an exercise:

**Proposition 7.** *Let  $\rho \in \{\text{AX}, \wedge\text{R}, \vee\text{R}, \forall\text{R}, \exists\text{R}\}$ . Then, for any instance of  $\rho$ , if there is are Herbrand proofs of the premisses, there is an Herbrand proof of the conclusion.*

The admissibility of weakening relies on the presence of a constant in the signature over which we work:

**Proposition 8.** *Let  $A$  be a formula of first-order logic. Then if  $\Gamma$  has an Herbrand proof, so does  $\Gamma, A$ .*

*Proof.* Let  $(\hat{\Gamma}, Q.C, \sigma)$  be an Herbrand proof of  $\Gamma$ . Let  $Q'.D$  be a prenexification of  $A$  sharing no bound variables with  $Q.C$ . Then we form an Herbrand proof

$$((\hat{\Gamma}, A), Q'.Q.(C \vee D), \sigma')$$

of  $\Gamma, A$ , where  $\sigma'$  assigns the same term as  $\sigma$  to existential quantifiers in  $Q$ , and assigns a constant term  $c$  to all existentially bound variables in  $Q'$ .  $\square$

The only rule to pose some difficulty is contraction. We would like to prove contraction admissible by induction on the rank of a formula to be contracted, but this induction hypothesis is not strong enough. To see this, suppose that we have shown contraction admissible for all formulae of rank  $\leq n$ , and let  $A \wedge B$  have rank  $n + 1$ . Given an Herbrand proof

$$((\hat{\Gamma}, \hat{A}_1 \wedge \hat{B}_1, \hat{A}_2 \wedge \hat{B}_2), \Gamma^*, \sigma)$$

of  $\Gamma, A \wedge B, A \wedge B$ , how do we use our induction hypothesis to produce a proof of  $\Gamma, A \wedge B$ ? We can get close by using the valid implication

$$(A \wedge B) \vee (C \vee D) \Rightarrow (A \vee C) \wedge (B \vee D). \quad (9)$$

**Remark 2.** This implication plays an important role in the proof-theoretic formalism known as *deep inference*, where it is known as *medial*. It is used as an inference rule in Brännler's system SKS [1] to reduce contraction to atomic form. Its use here is similar.

**Lemma 10.** *If*

$$((\hat{\Gamma}, \hat{A}_1 \wedge \hat{B}_1, \hat{A}_2 \wedge \hat{B}_2), Q.C, \sigma)$$

*is an Herbrand proof of  $\Gamma, A \wedge B, A \wedge B$ , then*

$$((\hat{\Gamma}, (\hat{A}_1 \vee \hat{A}_2) \wedge (\hat{B}_1 \vee \hat{B}_2)), Q.C', \sigma)$$

*is an Herbrand proof of  $\Gamma, (A \vee A) \wedge (B \vee B)$ , where  $Q.C'$  is the unique (up to associativity of  $\vee$ ) prenexification of  $\hat{\Gamma}, (\hat{A}_1 \vee \hat{A}_2) \wedge (\hat{B}_1 \vee \hat{B}_2)$  with quantifier prefix  $Q$ .*

*Proof.* It is clear that  $\hat{\Gamma}, (\hat{A}_1 \vee \hat{A}_2) \wedge (\hat{B}_1 \vee \hat{B}_2)$  is an expansion of  $\Gamma, (A \vee A) \wedge (B \vee B)$ , and has a prenexification of the form  $Q.C'$ . We must check that  $\sigma$  is a witnessing substitution for  $Q.C'$ . Since it is a witnessing substitution for  $Q.C$ , it satisfies

the condition on free variables of substituting terms, and we need only check that  $\sigma(C')$  is a tautology. Let  $A_i^*$  be the matrix of  $\hat{A}_i$ ,  $B_i^*$  be the matrix of  $\hat{B}_i$ , and  $G$  be the matrix of  $\hat{\Gamma}$ . Then we know, since  $\sigma$  is a witnessing substitution for  $Q.C$ , that

$$\sigma(G) \vee (\sigma(A_1^*) \wedge \sigma(B_1^*)) \vee (\sigma(A_2^*) \wedge \sigma(B_2^*))$$

is a tautology. Applying (9), we conclude that

$$\sigma(C') = \sigma(G) \vee (\sigma(A_1^*) \vee \sigma(A_2^*)) \wedge (\sigma(B_1^*) \vee \sigma(B_2^*))$$

is a tautology.  $\square$

Of course in the sequent calculus one can only apply contractions across a comma, so even in this case we may not apply our induction hypothesis. To move forward we will need to show admissibility of a “deep” contraction rule, which can act on arbitrary subformulae in the ednsequent. Admissibility of ordinary, “shallow”, contraction follows immediately. We will need the following definitions:

**Definition 11.** (a) A one-hole-context is a sequent with precisely one positive occurrence of the special atom  $\{\}$  (the hole). We write  $\Gamma\{\}$  to denote a one hole context.

(b) An  $n$ -hole-context is a sequent with precisely one positive occurrence each of the  $n$  special atoms  $\{\}_1 \dots \{\}_n$ . We write  $\Gamma\{\} \dots \{\}$  to denote an  $n$  hole context, where by convention  $\{\}_1$  is the leftmost hole in the sequent etc..

(c) If  $\Gamma\{\}$  is a one hole context, we write  $\Gamma\{A\}$  for the sequent given by replacing the hole with  $A$ . Similarly for  $n$  hole contexts.

The following easy lemma will be crucial.

**Lemma 12.** An expansion of a sequent  $\Gamma\{A\}$  has the form  $\hat{\Gamma}\{A_1\} \dots \{A_n\}$ , where  $A_1 \dots A_n$  are expansions of  $A$  and  $\hat{\Gamma}\{\} \dots \{\}$  is an expansion of  $\Gamma\{\}$

**Lemma 13.** The deep contraction rule

$$\frac{\Gamma\{A \vee A\}}{\Gamma\{A\}} \text{DeepC}$$

is admissible for Herbrand proofs.

*Proof.* By induction on the structure of  $A$ :

- Suppose we have an Herbrand proof

$$(\hat{\Gamma}\{a \vee a\} \dots \{a \vee a\}, Q.C, \sigma)$$

of  $\Gamma\{a \vee a\}$ . Then clearly there is an Herbrand proof

$$(\hat{\Gamma}\{a\} \dots \{a\}), Q.C', \sigma)$$

of  $\Gamma\{a\}$ .

Now suppose, for each remaining case, that deep contraction is admissible for formulae of rank  $\leq n$ , and that  $A$  has rank  $n + 1$ .

- Suppose  $A = B \vee C$ , and that we have an Herbrand proof

$$(\hat{\Gamma}\{(\hat{B}_{11} \vee \hat{C}_{11}) \vee (\hat{B}_{12} \vee \hat{C}_{12})\} \dots \{(\hat{B}_{n1} \vee \hat{C}_{n1}) \vee (\hat{B}_{n2} \vee \hat{C}_{n2})\}, Q.C, \sigma)$$

of  $\Gamma\{(C \vee D) \vee (C \vee D)\}$ . Then

$$(\hat{\Gamma}\{(\hat{B}_{11} \vee \hat{B}_{12}) \vee (\hat{C}_{11} \vee \hat{C}_{12})\} \dots \{(\hat{B}_{n1} \vee \hat{B}_{n2}) \vee (\hat{C}_{n1} \vee \hat{C}_{n2})\})$$

is an expansion of  $\Gamma\{(C \vee C) \vee (D \vee D)\}$ , with a prenexification  $Q.C'$ ; these two, plus  $\sigma$ , give us an Herbrand proof of  $\Gamma\{(C \vee C) \vee (D \vee D)\}$ . Apply the induction hypothesis to obtain an Herbrand proof of  $\Gamma\{C \vee D\}$

- Suppose  $A = \exists x.B$ , with  $B$  of rank  $n$ . An Herbrand proof of  $\Gamma\{\exists x.B \vee \exists y.B\}$  has the form

$$(\hat{\Gamma}\{\hat{A}_{11} \vee \hat{A}_{12}\} \dots \{\hat{A}_{m1} \vee \hat{A}_{m2}\}, Q.C, \sigma).$$

But this is also an Herbrand proof of  $\Gamma\{\exists x.B\}$ , since if  $A_1$  and  $A_2$  are expansions of an existential formula  $\exists x.B$ , then so is  $A_1 \vee A_2$ .

- Suppose  $A = \forall x.B$ , with  $B$  of rank  $n$ . An Herbrand proof of  $\Gamma\{\forall x.B \vee \forall y.B\}$  has the form

$$(\hat{\Gamma}\{\forall x_1.\hat{B}_{11} \vee \forall y_1.\hat{B}_{12}\} \dots \{\forall x_m.\hat{B}_{m1} \vee \forall y_m.\hat{B}_{m2}\}, Q.C, \sigma).$$

Suppose we are given such a proof. We generate a new sequence  $Q'$  of quantifiers as follows: let  $z_i$  stand for the first occurrence a member of  $\{x_i, y_i\}$  in  $Q$  and  $w_i$  for the second. Let  $Q'$  be the result of deleting each occurrence of  $\forall w_i$  from  $Q$  (so that, from each pair  $\{\forall x_i, \forall y_i\}$ , we keep the first and discard the second.)  $Q'$  contains the same existential variables  $x_1, \dots, x_r$  as  $Q$ , and in the same order. Let  $\sigma = t_1, \dots, t_r$ . Let  $t'_i = t_i[w_1 := z_1] \dots [w_m := z_m]$ , and let  $\sigma' = t'_1, \dots, t'_r$ .

The following is then an Herbrand proof of  $\Gamma\{\forall z.(B \vee B)\}$ :

$$(\hat{\Gamma}\{\forall z_1((\hat{B}_{11} \vee \hat{B}_{12})[w_1 := z_1])\} \dots \{\forall z_m((\hat{B}_{m1} \vee \hat{B}_{m2})[w_m := z_m])\}, Q'.C', \sigma'),$$

By the induction hypothesis, we derive an Herbrand proof of  $\Gamma\{\forall z.B\}$ .

- Finally, suppose that  $A = B \wedge C$ . Then, if we have a Herbrand proof of  $\Gamma\{(B \wedge C) \vee (B \wedge C)\}$ , it consists of an expansion of the form

$$\hat{\Gamma}\{(\hat{B}_{11} \wedge \hat{C}_{11}) \vee (\hat{B}_{12} \wedge \hat{C}_{12})\} \dots \{(\hat{B}_{m1} \wedge \hat{C}_{m1}) \vee (\hat{B}_{m2} \wedge \hat{C}_{m2})\}$$

a prenexification  $Q.C$  and a substitution  $\sigma$ . The formula

$$\hat{\Gamma}\{(\hat{B}_{11} \vee \hat{B}_{12}) \wedge (\hat{C}_{11} \vee \hat{C}_{12})\} \dots \{(\hat{B}_{m1} \vee \hat{B}_{m2}) \wedge (\hat{C}_{m1} \vee \hat{C}_{m2})\}$$

is an expansion of  $\Gamma\{(B \vee B) \wedge (C \vee C)\}$ , and it can be easily seen that it has a prenexification of the form  $Q.C'$ . By the same reasoning used to prove Lemma 10,  $\sigma(C')$  is a tautology, and therefore  $\sigma$  is a witnessing substitution for  $Q.C'$ . This gives an Herbrand proof of  $\Gamma\{(B \vee B) \wedge (C \vee C)\}$ . Apply the induction hypothesis twice to obtain an Herbrand proof of  $\Gamma\{B \wedge C\}$

□

**Corollary 14.** *Contraction is admissible for Herbrand proofs.*

**Theorem 15.** *A formula of first-order logic is valid if and only if it has an Herbrand proof.*

*Proof.* Follows immediately from Propositions 7 and 8, Corollary 14 and the cut-free completeness of **GS**. □

## 4 Conclusions

As we have seen, Herbrand’s theorem in its full generality can be seen as a consequence of cut-elimination for the sequent calculus (and not, as usually claimed, of the midsequent theorem). To show this, we had to consider an extended sequent calculus with a deep contraction rule, and show that each proof in that extended calculus gives rise to an Herbrand proof. This raises some potentially interesting questions: are there Herbrand proofs which arise from a proof with deep contraction, but not from any shallow proof? If so, is there an easy condition separating the “shallow” Herbrand proofs from the “deep”?

For the special case of the prenex Herbrand theorem, the author has studied in [5] the elimination of *cuts* in Herbrand’s theorem, giving a notion of Herbrand proofs with cut and showing a syntactic cut-elimination theorem. This works because of a strong connection between the structure of prenex Herbrand proofs and the corresponding midsequent-factored sequent proofs. The Herbrand proofs we present in this paper have a similar strong connection to proofs with deep contraction. It is unlikely that we can find a similar cut-elimination result for general Herbrand proofs without a cut-elimination result for the system with deep contractions. Syntactic cut elimination for the system with deep contraction seems to be a very challenging problem.

This note began with the observation that restricting the contraction rule to existential and quantifier-free formulae broke completeness. The crucial observation is that contraction on  $A \wedge B$  does not follow inductively from contraction on  $A$  and contraction on  $B$ . Instead of moving to deep contraction, we can instead simply add back contraction for conjunctions (so now we only disallow contraction on disjunctions and universal quantifications). This gives rise to an Herbrand-like theorem in which first-order provability is reduced to provability in a well-behaved fragment of multiplicative linear logic. This is ongoing work.

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