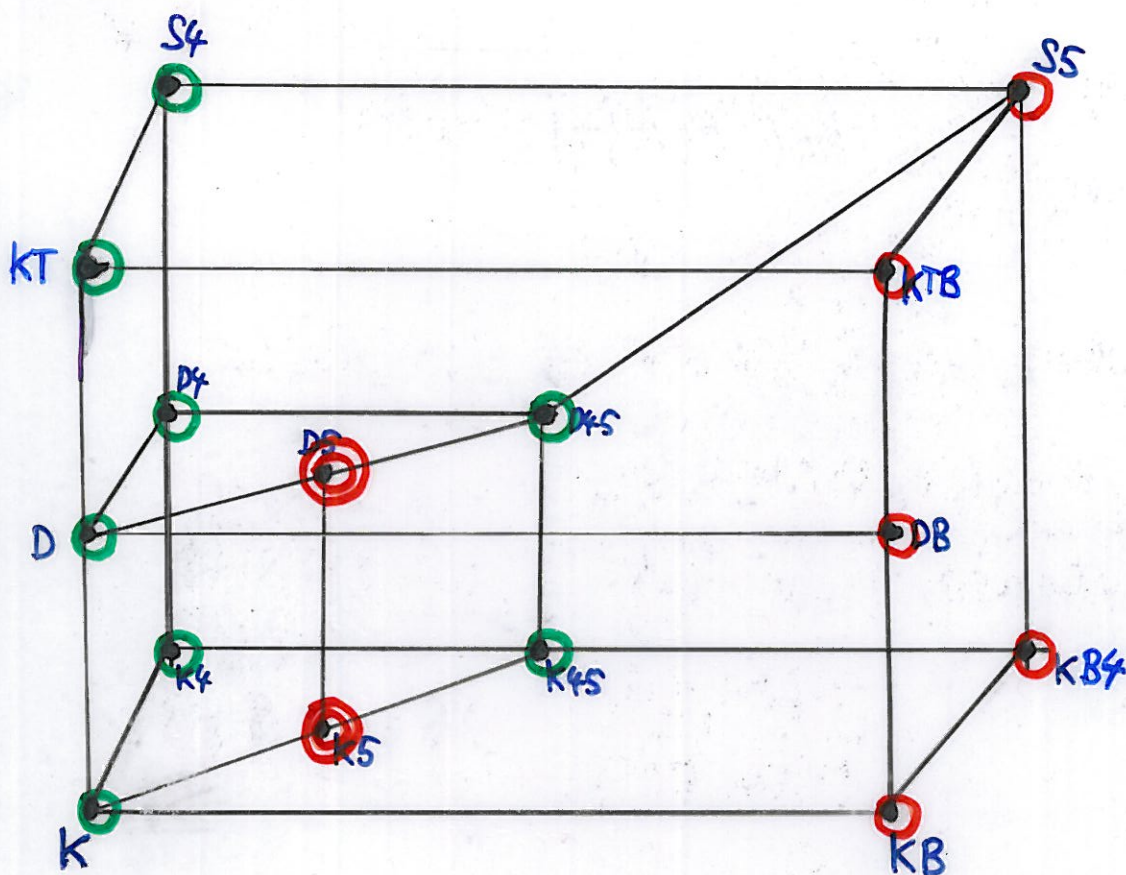


NESTED SEQUENTS

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THE MODAL CUBE



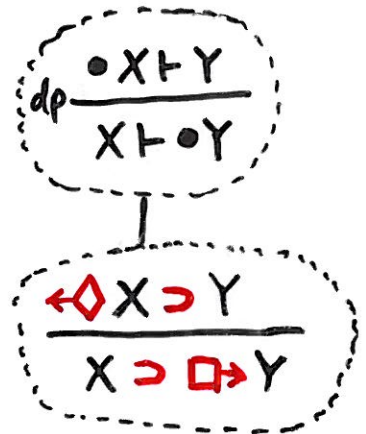
- ⊙ there is a cut-free sequent calculus
- ⊙ no cut-free sequent calculus
- ⊙ not even analytic cut

TWO PROMINENT EXTENSIONS OF THE SEQUENT CALCULUS

I. THE DISPLAY CALCULUS

$$dp \frac{X \circ Y \vdash Z}{Y \vdash (*X) \circ Z}$$

$$dp \frac{**X \vdash Y}{X \vdash Y}$$



II. LABELLED SEQUENTS

$$\square \frac{\Gamma, tRs \vdash s:A, \Delta}{\Gamma \vdash t: \square A, \Delta}$$

(s is fresh)

$$4 \frac{\Gamma, sRt, tRu, sRu \vdash \Delta}{\Gamma, sRt, tRu \vdash \Delta}$$

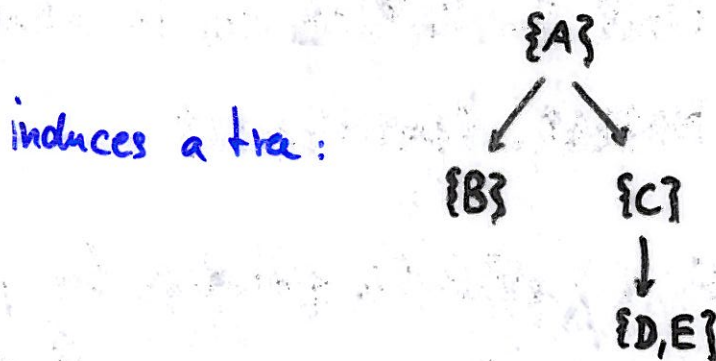
DEEP SEQUENTS (or NESTED)

Def A sequent is a multiset of formulas and boxed sequents.

A boxed sequent is an expression $[\Gamma]$ where Γ is a sequent.

Example

$A, [B], [C, [D, E]]$ is a sequent



Def A sequent context, denoted $\Gamma \{ \}$ is a sequent with exactly one occurrence of the symbol $\{ \}$.

The sequent $\Gamma \{ \Delta \}$ is obtained by replacing $\{ \}$ by Δ in $\Gamma \{ \}$.

NESTED SEQUENTS

HISTORY

- R.A. Bull 1992
- R. Kashima 1994
- myself 2006 "deep sequents"
- F. Poggiolesi 2007 "tree-hypersequents"

SYSTEM K

$\Gamma\{a, \bar{a}\}$

$$\wedge \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}}$$

$$\vee \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

$$\text{cl} \frac{\Gamma\{\Delta, \Delta\}}{\Gamma\{\Delta\}}$$

$$\square \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}}$$

$$\diamond \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [A]\}}$$

SYSTEM K

$$\Gamma\{a, \bar{a}\} \quad \wedge \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \quad \vee \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \quad \text{ch} \frac{\Gamma\{\Delta, \Delta\}}{\Gamma\{\Delta\}}$$

$$\square \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \quad \diamond \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [A]\}}$$

FRAME CONDITIONS - \Diamond -RULES

$$\diamond d \frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \quad \diamond f \frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}} \quad \diamond b \frac{\Gamma\{A, [A]\}}{\Gamma\{[\Diamond A, A]\}}$$

$$\diamond 4 \frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [A]\}} \quad \diamond 5 \frac{\Gamma\{\emptyset\} \{ \Diamond A \}}{\Gamma\{\Diamond A\} \{ \emptyset \}}$$

(20)

SYSTEM K

$$\Gamma\{a, \bar{a}\} \quad \wedge \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \quad \vee \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \quad \text{ctr} \frac{\Gamma\{\Delta, \Delta\}}{\Gamma\{\Delta\}}$$

$$\square \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \quad \diamond \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [A]\}}$$

FRAME CONDITIONS AS STRUCTURAL RULES

$$d \frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}} \quad + \frac{\Gamma\{[\Sigma]\}}{\Gamma\{\Sigma\}} \quad b \frac{\Gamma\{[\Delta, [\Sigma]]\}}{\Gamma\{\Sigma, [\Delta]\}}$$

$$4 \frac{\Gamma\{[\Delta], [\Sigma]\}}{\Gamma\{[\Delta, [\Sigma]]\}} \quad 5 \frac{\Gamma\{\emptyset\} \{[\Delta]\}}{\Gamma\{[\Delta]\} \{\emptyset\}} \quad \text{⑦⑩}$$

SYSTEM K

$$\Gamma\{a, \bar{a}\} \quad \wedge \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \quad \vee \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \quad \text{ctr} \frac{\Gamma\{\Delta, \Delta\}}{\Gamma\{\Delta\}}$$

$$\square \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \quad \diamond \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

FRAME CONDITIONS AS STRUCTURAL RULES

$$d \frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}} \quad + \frac{\Gamma\{[\Sigma]\}}{\Gamma\{\Sigma\}} \quad b \frac{\Gamma\{[\Delta, [\Sigma]]\}}{\Gamma\{\Sigma, [\Delta]\}}$$

$$4 \frac{\Gamma\{[\Delta], [\Sigma]\}}{\Gamma\{[\Delta, [\Sigma]]\}} \quad 5 \frac{\Gamma\{\emptyset\} \{[\Delta]\}}{\Gamma\{[\Delta]\} \{\emptyset\}}$$

(20)

THEOREM

For each $X \in \{d, t, b, 4, 5\}$ $K+X$ admits cut-elimination, and thus is complete.

SYNTACTIC CUT ELIMINATION

$$\text{cut} \frac{\Gamma \{A\} \quad \Gamma \{\bar{A}\}}{\Gamma \{\emptyset\}}$$

$$\begin{array}{c} \text{1} \\ \hline \Gamma \{\Delta, A\} \\ \hline \Gamma \{\Delta, \textcircled{A}\} \\ \text{cut} \\ \hline \Gamma \{\Delta\} \end{array} \quad \begin{array}{c} \text{2} \\ \hline \Gamma \{\Delta, \bar{A}, \textcircled{A}\} \\ \hline \Gamma \{\Delta, \textcircled{A}\} \end{array}$$

$$\begin{array}{c} \text{1} \\ \hline \Gamma \{\Delta, A\} \\ \hline \Gamma \{\Delta, A, \textcircled{A}\} \\ \text{wk}^2 \\ \text{cut} \\ \hline \Gamma \{\Delta, \textcircled{A}\} \end{array} \quad \begin{array}{c} \text{1} \\ \hline \Gamma \{\Delta, A\} \\ \hline \Gamma \{\Delta, \bar{A}, \textcircled{A}\} \\ \text{wk} \\ \text{cut} \\ \hline \Gamma \{\Delta, \textcircled{A}\} \end{array} \quad \begin{array}{c} \text{2} \\ \hline \Gamma \{\Delta, \bar{A}, \textcircled{A}\} \end{array}$$

$$\text{cut} \frac{\Gamma \{\Delta, \textcircled{A}\} \quad \Gamma \{\Delta, \textcircled{A}\}}{\Gamma \{\Delta\}}$$