Assumption-Based Reasoning with Algebraic Clauses*

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1 INTRODUCTION

Ten years ago, de Kleer introduced **assumption-based truth maintenance systems** (ATMS) as a powerful tool for different applications in the domain of uncertain reasoning (de Kleer, 1986). A traditional ATMS is based on propositional logic, but it is limited to Horn clauses and to simple queries. An ATMS requires that there is a subset of propositions declared as **assumptions**. Assumptions are needed to express uncertainty. The fundamental ATMS problem is to identify combinations of assumptions (labels) such that a given query holds. The weak point of the traditional ATMS concept is the restriction to Horn clauses. In many cases, also non-Horn clauses are needed to express the knowledge. The advantage of ATMS lies in its efficient computations.

Subsequently, Reiter and de Kleer proposed **clause management systems** (CMS) as an extension of the original ATMS framework (Reiter & de Kleer, 1987). CMS do not distinguish between propositions and assumptions and they are not restricted to Horn clauses. The idea of CMS has been worked out by many other authors, for example in (Siegel, 1987), (Laskey & Lehner, 1989), (Provan, 1990), or (Inoue, 1991). Later, Kohlas proposed the concepts of **assumption-based systems** (ABS) and **probabilistic ABS** (Kohlas & Monney, 1993; Kohlas & Monney, 1995). ABS are obtained from CMS by introducing de Kleer's original idea of assumptions. The main problem in ABS is to compute **minimal quasi-supports**. If probabilities are assigned to the assumptions, then the concept of ABS leads to probabilistic ABS, which provides a link between numerical and symbolic uncertainty models. The problem in probabilistic ABS is to determine numerical **degrees of support**. Such degrees correspond to the notion of **belief** in Dempster-Shafer's theory of evidence (Dempster, 1967; Shafer, 1976).

This paper proposes **algebraic assumption-based systems** which are obtained from common ABS by allowing **algebraic clauses**. An algebraic clause is a disjunction of a common propositional clause with algebraic predicates like equations and inequalities. In this paper, only the case of **simple** algebraic clauses is considered. This improves the expressive and deductive power of ABS significantly. The paper shows how algebraic ABS can be transformed into common ABS by eliminating all numerical variables.

2 ASSUMPTION-BASED SYSTEMS

Propositional logic with uncertain assumptions provides a concise language for describing uncertain information. Let $P = \{p_1, \ldots, p_q\}$ and $A = \{a_1, \ldots, a_r\}$ be two disjoint sets of **propositions**. The elements of A are called **assumptions**. They are used to express uncertainty, e.g. possible interpretations, unknown risks, uncertain events, or unpredictable circumstances. $N = P \cup A$ is the entire set of symbols considered. The corresponding sets of negated propositions are denoted by $\sim P$, $\sim A$, and $\sim N$. $P^{\pm} = P \cup \sim P$, $A^{\pm} = A \cup \sim A$ and $N^{\pm} = N \cup \sim N$ represent the sets of **literals** on P, A, and N respectively. Furthermore, $\mathcal{L}_N$ denotes the set of all well-formed propositional formulae with symbols from N.

A **clause** d is a disjunction $\ell_1 \vee \cdots \vee \ell_n$ of literals $\ell_i \in N^{\pm}$. Often, a clause is also considered as the set $\{\ell_1, \ldots, \ell_n\}$ of its literals. D_N denotes the set of all clauses containing only symbols from N. A set $\Sigma = \{d_1, \ldots, d_s\} \subseteq D_N$ of clauses is called **knowledge base**. It represents the available information. Note that a more general framework

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in which Σ consists of arbitrary propositional formulae can always be transformed into an equivalent form consisting of clauses only. The conjunction $\sigma = d_1 \wedge \cdots \wedge d_s$ is a logical representation of Σ. If Σ is empty, then $\sigma = \top$. A triple $\mathcal{A} = (P, A, \Sigma)$, where P and A are disjoint sets of propositional symbols and $\Sigma \subseteq D_N$ is a set of clauses, is called **assumption-based system** (ABS).

Given an assumption-based system $\mathcal{A} = (P, A, \Sigma)$, one may be interested in certain **hypotheses** h (queries). Such hypotheses can be expressed as logical formulae in $\mathcal{L}_N$. What can be derived from Σ about the possible truth of h? If some assumptions are considered to be true and some to be false, then h can possibly be deduced from Σ. Such combinations (conjunctions) of true and false assumptions (literals) can be regarded as arguments in favor of h.

In view of these remarks consider conjunctions $c = \ell_1 \wedge \cdots \wedge \ell_n$. If the literals in c only consist of assumptions (elements of A), then c is called **argument**. C_A denotes the set of the arguments in which an assumption appears at most once. The main ABS problem consists in finding arguments $c \in C_A$, such that $c \wedge \sigma \models h$, where σ is the conjunction of the clauses contained in Σ, and $h \in \mathcal{L}_N$ represents a hypothesis. Often, we write $c, \Sigma \models h$ instead of $c \wedge \sigma \models h$. Such arguments c allow to deduce h from Σ. They are called **quasi-supports** for h relative to Σ. The set of all quasi-supports

$$QS(h, \Sigma) = \{c \in C_A : \; c, \Sigma \models h\} \subseteq C_A \tag{2.1}$$

is called **quasi-support set** for h relative to Σ. Often, it is more useful to have only non-contradictory arguments. A quasi-support c is called **support** for h relative to Σ if $c, \Sigma \not\models \perp$. The set of all supports

$$SP(h, \Sigma) = \{c \in C_A : \; c, \Sigma \models h \text{ and } c, \Sigma \not\models \perp\} = QS(h, \Sigma) - QS(\perp, \Sigma) \tag{2.2}$$

is called **support set** for h relative to Σ.

Note that in a subset $C \subseteq C_A$ some conjunctions may be absorbed by others and can therefore be dropped. A conjunction c_1 **absorbs** c_2 if $c_2 \models c_1$ or in terms of sets if $c_1 \subseteq c_2$. C is called **minimal** if no conjunction can be dropped in this way. μC denotes the set obtained from an arbitrary C by dropping all absorbed conjunctions. Note that C and μC are logically equivalent. The elements of $\mu QS(h, \Sigma)$ are called **minimal quasi-supports** for h relative to Σ. $\mu QS(h, \Sigma)$ can be seen as an appropriate representation of $QS(h, \Sigma)$. Therefore, the ABS problem is solved if all minimal quasi-supports are found for a given hypothesis.

There are different techniques to solve the ABS problem. Most of them are based on **resolution**. An overview of the existing methods is given in (Kohlas & Haenni, 1996). The author recommends the method based on **variable elimination** (Kohlas & Moral, 1995; Haenni & Lehmann, 1997). The method for algebraic ABS presented in Section 3 also uses the idea of variable elimination.

3 ALGEBRAIC ASSUMPTION-BASED SYSTEMS

This section introduces the concept of algebraic assumption-based systems. Let $V = \{v_1, \ldots, v_s\}$ be a set of variables of domain $\mathbb{R}$. T_V denotes the set of all possible algebraic terms on V, e.g. $-x + 3y$, $\sin(z^2) + 3$, etc. $\Pi = \{=, \neq, <, \leq, \geq, >\}$ is the set of predicate symbols considered. Two terms $t_1, t_2 \in T_V$ and one predicate symbol $\pi \in \Pi$ can be used to build algebraic predicates of the form $t_1 \pi t_2$, e.g. $-x + 3y \leq \sin(z^2) + 3$, etc. R_V denotes the set of all possible algebraic predicates with variables from V.

If $d \in D_N$ is a clause, then a disjunction $k = d \vee R$ with $R \in R_V$ is called **algebraic clause**. Note that only **simple** algebraic clauses with at most one predicate are considered here. If $M = N \cup V = P \cup A \cup V$ is the set of all available symbols, then we use D_M to denote the set of all simple algebraic clauses on M. A set $\Sigma = \{k_1, \ldots, k_s\} \subseteq D_M$ of algebraic clauses is now considered as knowledge base. A quadruple $\mathcal{A} = (P, A, V, \Sigma)$, where P and A are disjoint sets of propositional symbols, V is a set of variables of domain $\mathbb{R}$, and $\Sigma \subseteq D_M$ is a set of algebraic clauses, is called **algebraic assumption-based system** (algebraic ABS).

Given an algebraic ABS $\mathcal{A} = (P, A, V, \Sigma)$, one may again be interested in certain **hypotheses** $h \in \mathcal{L}_N$, or more generally, hypotheses $h \in \mathcal{L}_M$. In this paper, only the case of $h \in \mathcal{L}_N$ is considered. The problem is still to find the quasi-support set $QS(h, \Sigma)$ for h. The method we propose consists of two steps: (1) eliminate all variables $v_i \in V$ from Σ; (2) apply one of the existing methods for common ABS. In the following we show how variables can be eliminated from an algebraic ABS.

Let $\Sigma = \{k_1, \ldots, k_s\}$ be a set of algebraic clauses and $x \in V$ the variable to be eliminated. If $R = t_1 \pi t_2$ is an algebraic predicate containing the variable x, then we assume that x can be isolated on the left side of R, i.e. that $t_1 \pi t_2$ can be transformed into an equivalent predicate $x\pi' t'$. For example, from $-x + 3y \leq \sin(z^2) + 3$ we obtain $x \geq 3y - \sin(z^2) - 3$. Isolating a variable can be seen as a mapping $I_x : R_V \longrightarrow R_V$, e.g. $I_x(-x + 3y \leq \sin(z^2) + 3) = x \geq 3y - \sin(z^2) - 3$. In order to eliminate x from Σ let's decompose Σ into two disjoint sets Σ^x and Σ^*:

$$\Sigma^x = \{k = d \vee R \in \Sigma : \ R \text{ contains } x\}, \quad (3.1)$$

$$\Sigma^* = \Sigma - \Sigma^x. \quad (3.2)$$

The important set for the elimination of x will be Σ^x. If $k = d \vee R$ is an algebraic clause in Σ^x with $d \in D_N$ and $R \in R_V$, then it can be transformed into $k' = d \vee I_x(R)$. Let $k'_1 = d_1 \vee I_x(R_1)$ and $k'_2 = d_2 \vee I_x(R_2)$ be two such clauses derived from Σ^x with $I_x(R_1) = x\pi'_1 t'_1$ and $I_x(R_2) = x\pi'_2 t'_2$. For different combinations of predicates π'_1 and π'_2 it is possible to define a mapping $\rho : D_M \times D_M \longrightarrow D_M$ called **algebraic resolution**. The result of this mapping is called **resolvent** of k_1 and k_2. Note that the resolvent does not contain the variable x. Algebraic resolution is possible for the following cases (symmetric cases are not indicated explicitely):

clause 1	**clause 2**	**resolvent**
$k'_1 = x\pi'_1 t'_1 \vee d_1$	$k'_2 = x\pi'_2 t'_2 \vee d_2$	$\rho(k_1, k_2) = t'_1 \pi' t'_2 \vee d_1 \vee d_2$
$x = t'_1 \vee d_1$	$x = t'_2 \vee d_2$	$t'_1 = t'_2 \vee d_1 \vee d_2$
$x = t'_1 \vee d_1$	$x \neq t'_2 \vee d_2$	$t'_1 \neq t'_2 \vee d_1 \vee d_2$
$x = t'_1 \vee d_1$	$x > t'_2 \vee d_2$	$t'_1 > t'_2 \vee d_1 \vee d_2$
$x = t'_1 \vee d_1$	$x \geq t'_2 \vee d_2$	$t'_1 \geq t'_2 \vee d_1 \vee d_2$
$x = t'_1 \vee d_1$	$x < t'_2 \vee d_2$	$t'_1 < t'_2 \vee d_1 \vee d_2$
$x = t'_1 \vee d_1$	$x \leq t'_2 \vee d_2$	$t'_1 \leq t'_2 \vee d_1 \vee d_2$
$x < t'_1 \vee d_1$	$x > t'_2 \vee d_2$	$t'_1 > t'_2 \vee d_1 \vee d_2$
$x < t'_1 \vee d_1$	$x \geq t'_2 \vee d_2$	$t'_1 > t'_2 \vee d_1 \vee d_2$
$x \leq t'_1 \vee d_1$	$x > t'_2 \vee d_2$	$t'_1 > t'_2 \vee d_1 \vee d_2$
$x \leq t'_1 \vee d_1$	$x \geq t'_2 \vee d_2$	$t'_1 \geq t'_2 \vee d_1 \vee d_2$

Algebraic resolution can be used to eliminate x from Σ. The new set after eliminating x is

$$\Sigma^{\downarrow M - \{x\}} = \mu(\Sigma^* \cup \{\rho(k_1, k_2) : \ k_1, k_2 \in \Sigma^x\}). \quad (3.3)$$

By eliminating subsequently all variables $v \in V$ from Σ we can transform an algebraic ABS $\mathcal{A} = (P, A, V, \Sigma)$ into a common ABS $\mathcal{A}' = (P, A, \Sigma^{\downarrow N})$ without loss of information on the propositions in N. Queries can then be treated by one of the existing methods for common ABS.

4 EXAMPLE

Consider the faulty arithmetical network of Figure 4.1. Instead of the normal outputs $f = 12$ and $g = 12$ the values $f = 10$ and $g = 12$ are observed. The problems of finding an explanation of the system's behavior and of localizing the faulty component can be solved using an algebraic ABS. If $A = \{ok_{M_1}, ok_{M_2}, ok_{M_3}, ok_{A_1}, ok_{A_2}\}$ contains the assumptions that the corresponding components work properly, $P = \{ok\}$ the proposition that the whole systems works properly, and $V = \{a, b, c, d, e, f, g\}$ the numerical values of the system, then the network can be described by the following expressions which can easily be transformed into algebraic clauses:

1) $ok_{M_1} \rightarrow a \cdot c = x$ 2) $ok_{M_2} \rightarrow b \cdot d = y$ 3) $ok_{M_3} \rightarrow c \cdot e = z$
4) $ok_{A_1} \rightarrow x + y = f$ 5) $ok_{A_2} \rightarrow y + z = g$ 6) $ok \leftrightarrow ok_{M_1} \wedge \cdots \wedge ok_{A_2}$

The knowledge base is completed by adding the observed values $a = 3$, $b = 2$, $c = 2$, $d = 3$, $e = 3$, $f = 10$ and $g = 12$. An explanation of the system's behavior is then obtained by computing $SP(\sim ok, \Sigma)$ (i.e. "give me the (minimal) arguments for the hypothesis that the system is not working properly"). Using the method presented in the previous section we obtain

$$\mu SP(\sim ok, \Sigma) = \{\sim ok_{A_1}, \sim ok_{M_1}, \sim ok_{A_2} \wedge \sim ok_{M_2}, \sim ok_{M_2} \wedge \sim ok_{M_3}\}, \tag{4.1}$$

which is obviously the correct set of minimal explanations of the faulty system.

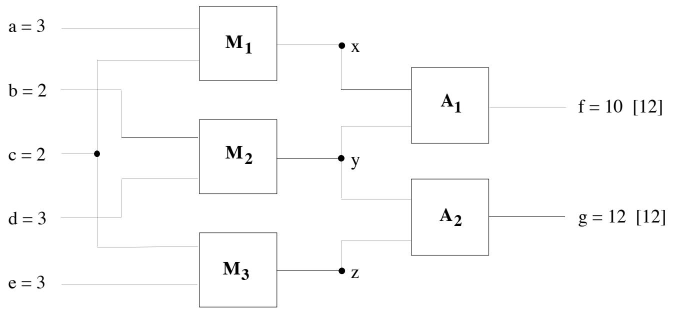

Figure 4.1: Faulty arithmetical network.

References

de Kleer, J. 1986. An Assumption-based TMS. *Artificial Intelligence*, **28**, 127–162.

Dempster, A. 1967. Upper and Lower Probabilities Induced by a Multivalued Mapping. *Ann. Math. Stat.*, **38**, 325–339.

Haenni, R., & Lehmann, N. 1997. Improving the Search of Minimal Quasi-Supports (Labels) in Assumption-Based Systems.

Inoue, K. 1991. An Abductive Procedure for the CMS/ATMS. *Pages 34–53 of:* Martins, J.P., & Reinfrank, M. (eds), *Truth Maintenance Systems, Lecture Notes in A.I.* Springer.

Kohlas, J., & Haenni, R. 1996. *Assumption-Based Reasoning and Probabilistic Argumentation Systems.* Tech. Rep. 96–07. Institute of Informatics, University of Fribourg.

Kohlas, J., & Monney, P.A. 1993. Probabilistic Assumption-Based Reasoning. *In:* Heckerman, & Mamdani (eds), *Proc. 9th Conf. on Uncertainty in Artificial Intelligence.* Kaufmann, Morgan Publ.

Kohlas, J., & Monney, P.A. 1995. *A Mathematical Theory of Hints. An Approach to the Dempster-Shafer Theory of Evidence.* Lecture Notes in Economics and Mathematical Systems, vol. 425. Springer.

Kohlas, J., & Moral, S. 1995. *Propositional Information System.* Working Paper 96–01. Institute of Informatics, University of Fribourg.

Laskey, K.B., & Lehner, P.E. 1989. Assumptions, Beliefs and Probabilities. *Artificial Intelligence*, **41**, 65–77.

Provan, G.M. 1990. A Logic-Based Analysis of Dempster-Shafer Theory. *International Journal of Approximate Reasoning*, **4**, 451–495.

Reiter, R., & de Kleer, J. 1987. Foundations of Assumption-Based Truth Maintenance Systems. *Proceedings of the American Association in AI*, 183–188.

Shafer, G. 1976. *The Mathematical Theory of Evidence.* Princeton University Press.

Siegel, P. 1987. *Représentation et Utilisation de la Connaissance en Calcul Propositionel.* Ph.D. thesis, Université d'Aix-Marseille II. Luminy, France.