The Operational Penumbra: Some Ontological Aspects

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Abstract

Feferman's explicit mathematics and operational set theory are two important examples of families of theories providing an operational approach to mathematics. My aim here is to survey some central developments in these two fields, to sketch some of Fefeman's main achievements, and to relate them to the work of others. The focus of my approach is on ontological questions.

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1 Introduction

I first met Solomon Feferman at the 1978 Logic Colloquium meeting in Mons, Belgium. He gave a survey talk about various approaches to constructive mathematics and presented his own constructive theory of functions and classes. The written form [12] of his talk is published in the proceedings volume of that conference and is one of the three landmark papers about explicit mathematics. This was the time when I was working for my dissertation and, of course, Sol was already known to me very well by his many influential papers on proof theory and the foundations of mathematics. After that I had the privilege to learn from Sol in direct personal contact when we both spent the academic year 1979-1980 at the University of Oxford. We have been in close scientific and personal contact since then, including my visit as an assistant professor at Stanford University in the academic year 1982-1983.

Sol's influence on my scientific development has been manifold. One very important aspect is that he widened the range of my proof-theoretic interests and led me to work on new topics dealing with foundational questions, different from those I had previously studied. Maybe the most typical example along these lines is explicit mathematics, a subject that has never left me since then. It was characteristic of Sol that he was always asking for conceptual clarity and insisting on a clear methodological approach, not obscured by a self-satisfying technical machinery.

A general operational approach has been extremely successful in connection with the λ-calculus and combinatory logic leading, for example, to a variety of functional programming languages. The formal operational approach in mathematics, on the other hand, has not been so popular although many working mathematicians freely make use of operations and the operational machinery whenever convenient, but typically informally and without caring for its foundations. Church's approach to base the foundations of mathematics entirely on operations turned out to be inconsistent.

Feferman's explicit mathematics changed the picture. Motivated by the desire to set up a proper formal framework for Bishop's *Foundations of Constructive Analysis* [2] he proposed a new kind of formalism, baptized *explicit mathematics*. Bishop's book had enormous influence on the discussion of the foundations of mathematics. Bishop showed in his book, putting aside all ideological considerations, that most of the important theorems in real analysis can be established up to equivalence by constructive methods. The success of this book stimulated many logicians to develop formal frameworks for Bishop's approach, and Feferman's system (or family of systems) turned out to pave a very influential way. At the same time, his framework provided a way to account for predicative mathematics and descriptive set theory as well, which could not be done in the other approaches.

Soon after the first presentation of explicit mathematics in Feferman [9], its relevance for other parts of proof theory became evident. For example, systems of explicit mathematics – based on classical or intuitionistic logic – have their natural place in reductive proof theory and constitute a natural setting for studying various forms of abstract computability and recursion in higher type functionals.

In this article I will try to sketch some of the main lines in the research about explicit mathematics. A textbook by Solomon Feferman, Gerhard Jäger, and Thomas Strahm on the foundations of explicit mathematics is in preparation, aiming at providing a systematic approach to the topics mentioned above. In addition to that, Ulrik Buchholtz has set up an online bibliography of explicit mathematics and related topics at `http://home.inf.unibe.ch/~ltg/em_bibliography`.

The second main topic of this article is *operational set theory*, a further central stream in Feferman's operational approach. It goes back to Feferman [18] and is further elaborated in Feferman [19], where also much about the original ideology of operational set theory is explained. Further advances and technical results will be presented in Section 5 of this article.

Feferman's *unfolding program* is a third field under the operational perspective. However, we will not treat it in this article since Strahm's contribution for this volume is dedicated to unfolding. In addition, the reader will find a useful introduction to all three fields in Feferman [22].

My aim here is to survey some developments in explicit mathematics and operational set theory from a common operational perspective, to sketch some of Feferman's main achievements in these to fields, and to relate them to the work of others. The focus of my approach is on ontological questions, a point of view that has been neglected so far. But I am convinced that such ontological questions will play a crucial role in the further development of a general operational penumbra.

2 The general operational framework

Before turning to systems of explicit mathematics and operational set theory we set up the general operational framework. However, in contrast to Church (cf. [7, 8]), who wanted to base the foundations of mathematics solely on operations and whose approach turned out to be inconsistent, we confine ourselves to a consistent and relatively weak core operational theory. The basic idea is simple: The universe of discourse is a partial combinatory algebra; its elements are operations and share the following properties:

- Operations may be partial, they may freely be applied to each other, and self-application of operations is permitted.
- As a consequence, the general theory of operations is type-free. If needed sets or classes of operations can be added with the purpose to partly structure the universe.
- Operations are intensional objects; extensionality of operations is only assumed or claimed axiomatically in very special situations.

Since we will be dealing with possibly undefined objects, it is convenient to work with Beeson's logic of partial terms, see Beeson [1], rather than ordinary classical or intuitionistic logic. Terms are formed in this logic from the variables and constants of the language by simple term application, and we have atomic formulas of the form $(t\downarrow)$ to express that the term t has a value or is defined.

In his first articles [9, 11, 12] about explicit mathematics, Feferman did not make use of the logic of partial terms but worked with a three place relation $App[x, y, z]$ to express that operation x applied to y has value z.

Scott's [58] presents one of several alternative possibilities of dealing with existence and partiality in a logical context. In this E-logic we have a specific relation symbol E, where $E(t)$ has the intuitive interpretation "t exists". In the Beeson/Feferman approach all constants have a value and the free variables range over existing objects, in contrast to Scott's approach where they can also stand for possibly non-existing objects. In both approaches, quantifiers are supposed to range over existing objects only. In spite of this different philosophically attitude, both approaches are technically more or less equivalent; see Troelstra and van Dalen [61].

Any operational language L comprises the following primitive first order symbols:

(PS.1) Countably many individual variables $a, b, c, f, g, h, u, v, w, x, y, z$ (possibly with subscripts) and countably many individual constants, including k, s (combinators), $\mathsf{p}, \mathsf{p_0}, \mathsf{p_1}$ (pairing and unpairing).

(PS.2) The binary function symbol $\circ$ for (partial) term application.

(PS.3) For every natural number n a countable (possibly empty) set of relation symbols, including the unary relation symbol $\downarrow$ for definedness and the binary relation symbol $=$ for equality.

(PS.4) The logical symbols $\neg$ (negation), $\vee$ (disjunction), and $\exists$ (existential quantification).

The *individual terms* $(r, s, t, r_0, s_0, t_0, \ldots)$ of an operational language L are inductively generated as follows:

(T.1) The individual variables and individual constants of L are individual terms of L.

(T.2) If r and s are individual terms of L, then so also is $\circ(r, s)$.

In the following $\circ(r, s)$ is usually written as $(r \circ s)$, as (rs), or – if no confusion arises – simply as rs. The convention of association to the left is also adopted so that $r_1 r_2 \ldots r_n$ stands for $(\ldots(r_1 r_2)\ldots)$, and we often also write $s(r_1, \ldots, r_n)$ for $s r_1 \ldots r_n$. General n-tupling is defined by induction on $n \geq 1$ as follows:

$$<r_1> := r_1 \quad \text{and} \quad <r_1, \ldots, r_{n+1}> := \mathsf{p}(<r_1, \ldots, r_n>, r_{n+1}).$$

Finally, the *formulas* $(A, B, C, A_0, B_0, C_0, \ldots)$ of L are inductively generated by the following three clauses:

(F.1) All expressions $(r\downarrow)$, $(rs\downarrow)$, and $(r = s)$ are (atomic) formulas of L.

(F.2) If L contains additional n-ary relation symbols R, then all expressions $R(r_1, \ldots, r_n)$ are further (atomic) formulas of L.

(F.3) If A and B are formulas of L, then so also are $(\neg A)$, $(A \vee B)$, and $\exists x A$.

In this article we confine ourselves to classical logic. Hence the remaining logical connectives and the universal quantifier can be defined as usual. Also, $(r \neq s)$ is short for $\neg(r = s)$.

We will often omit parentheses and brackets whenever there is no danger of confusion. Moreover, we frequently make use of the vector notation $\vec{\mathcal{E}}$ as shorthand for a finite string $\mathcal{E}_1, \ldots, \mathcal{E}_n$ of expressions whose length is not important or is evident from the context. The set of free variables of a formula A is defined in the standard way. An L formula without free variables is called a closed L formula; the closed L terms are those without variables.

Suppose now that $\vec{a} = a_1, \ldots, a_n$ and $\vec{r} = r_1, \ldots, r_n$, where $a_1, \ldots, a_n$ are pairwise (syntactically) different variables. Then $A[\vec{r}/\vec{a}]$ is the L formula that is obtained from the L formula A by simultaneously replacing all free occurrences of the variables $\vec{a}$ by the L terms $\vec{r}$; in order to avoid collision of variables, a renaming of bound variables may be necessary. In case the L formula A is written as $A[\vec{a}]$, we often simply write $A[\vec{r}]$ instead of $A[\vec{r}/\vec{a}]$. Further variants of this notation will be obvious. The substitution of L terms for variables in L terms is treated accordingly.

As deduction system for the logic of partial terms we make use of a so-called *Hilbert calculus*, consisting of the following axioms and rules of inference.

Propositional axioms and propositional rules. These comprise the usual axioms and rules of inference of some sound and complete Hilbert calculus for classical propositional logic

Quantifier axioms and quantifier rules. The axioms for the existential quantifier consist of all L formulas

$$A[r] \land r{\downarrow} \;\rightarrow\; \exists x A[x],$$

where r may be an arbitrary L term. The rules of inference for the existential quantifier, on the other hand, are all configurations

$$\frac{A \rightarrow B}{\exists x A \rightarrow B}$$

for which the variable x does not occur free in B. Because of axiom (DE.1) below it is not necessary to claim in the premise that a is defined.

Definedness and equality axioms. For all constants r, all L terms s, all variables a, b, and all atomic formulas $A[u]$ of L:

(DE.1) $r{\downarrow} \land a{\downarrow}$.

(DE.2) $A[s] \;\rightarrow\; s{\downarrow}$.

(DE.3) $(a = a)$.

(DE.4) $(a = b) \land A[a] \;\rightarrow\; A[b]$.

The axioms (DE.2) are often referred to as *strictness axioms.* Important special cases are, for example, all assertions

$$(s = t) \;\rightarrow\; s{\downarrow} \wedge t{\downarrow} \quad \text{and} \quad (st){\downarrow} \;\rightarrow\; s{\downarrow} \wedge t{\downarrow},$$

stating that two terms can be equal only in case both have a value and that a compound term has a value only in case all its subterms have values as well. Thus the determination of the value of a term follows a *call-by-value* strategy. Observe that the axioms (DE.3) and (DE.4) are formulated for variables only. We must not claim $(r = r)$ in general since r may not have a value. However, we can introduce the notion of *partial equality* $\simeq$ à la Kleene,

$$(r \simeq s) \;:=\; (r{\downarrow} \vee s{\downarrow}) \rightarrow (r = s),$$

and obtain for all formulas $A[u]$ and terms r, s of L that $(r \simeq s)$ and $A[r]$ imply $A[s]$.

As mentioned above, it is an important aspect of the logic of partial terms that constants are defined and variables only range over defined objects. To point this out explicitly, we include axiom (DE.1). But observe that assertion $a{\downarrow}$ follows from (DE.2) and (DE.3).

The semantics of the logic of partial terms is based on *partial structures* consisting of a non-empty universe, interpretations of all constants within this universe, interpretations of all n-ary relation symbols as n-ary relations over this universe, and a partial binary function on this universe to take care of application. It is not difficult to shown that the above Hilbert system is sound and complete with respect to this semantics.

The basic theory $\mathsf{BO}(\mathrm{L})$ of operations for the language L comprises these axioms and rules of the logic of partial terms and axioms formalizing that the universe is a partial combinatory algebra and that pairing and projections are as expected.

Combinatory axioms, pairing and projections

(Co.1) $\mathsf{k} \neq \mathsf{s}$.

(Co.2) $\mathsf{k}ab = a$.

(Co.3) $\mathsf{s}ab{\downarrow} \;\wedge\; \mathsf{s}abc \simeq (ac)(bc)$.

(Co.4) $\mathsf{p}_0{<}a, b{>} = a \;\wedge\; \mathsf{p}_1{<}a, b{>} = b$.

In general, totality of application is not assumed; but if it is required in a special situation we add the statement

(Tot) $$\forall x \forall y (xy{\downarrow}).$$

Two fundamental principles are an immediate consequence of the combinatory axioms: λ-abstraction and the fixed point theorem. In more detail:

With any L term r, we associate an L term $(\lambda x.r)$ whose variables are those of r excluding x, such that $\mathsf{BO}(\mathrm{L})$ proves

$$(\lambda x.r)\downarrow \ \wedge\ (\lambda x.r)x \simeq r \ \wedge\ (s\downarrow \ \rightarrow\ (\lambda x.r)s \simeq r[s/x]).$$

As usual we can generalize λ-abstraction to several arguments by simply iterating abstraction for one argument. In addition, we have the following fixed point theorems for the partial and the total case.

Theorem 1 (Fixed points)**.** *There exist closed* L *terms* fix *and* fix_t *such that* $\mathsf{BO}(\mathrm{L})$ *proves for any* f *and* a*:*

1. $\mathsf{fix}f\downarrow \ \wedge\ \mathsf{fix}(f,a) \simeq f(\mathsf{fix}f, a)$.
2. $(\mathsf{Tot}) \ \rightarrow\ \mathsf{fix}_t f = f(\mathsf{fix}_t f)$.

This basic operational framework is an adaptation of λ-calculus and combinatory algebra to the partial case. According to my knowledge it has been set up in this form and in all details for the first time in Feferman [9].

We end this section with mentioning an interesting ontological relationship between full definition by cases and operational extensionality. For this purpose we assume that L contains an individual constant d and consider the additional axiom

$$(\mathsf{d}) \qquad (u = v \ \rightarrow\ \mathsf{d}(a,b,u,v) = a) \ \wedge\ (u \neq v \ \rightarrow\ \mathsf{d}(a,b,u,v) = b).$$

This is "full definition by cases" since it tests for arbitrary elements of the universe whether they are equal. Later we will also introduce restricted versions of (d). Clearly, $\mathsf{BO}(\mathrm{L}) + (\mathsf{d})$ is consistent.

Operational extensionality is the principle that claims that two operations are identical in case they have the same "input-output" behavior,

$$(\mathsf{Op\text{-}Ext}) \qquad \forall f \forall g(\forall x(fx = gx) \ \rightarrow\ f = g).$$

Also $\mathsf{BO}(\mathrm{L}) + (\mathsf{Op\text{-}Ext})$ is consistent. However, (d) and $(\mathsf{Op\text{-}Ext})$ as well as (d) and (Tot) are incompatible with each other.

Theorem 2. *Let* L *be an operational language with the constant* d*. Then* $\mathsf{BO}(\mathrm{L}) + (\mathsf{d}) + (\mathsf{Op\text{-}Ext})$ *and* $\mathsf{BO}(\mathrm{L}) + (\mathsf{d}) + (\mathsf{Tot})$ *are inconsistent.*

Proof. To show the first inconsistency, set $r := \mathsf{fix}(\lambda yx.\mathsf{d}(\mathsf{k},\mathsf{s},y,\lambda z.\mathsf{s}))$. In view of the fixed point theorem we then have $r\downarrow$ and

$$(*) \qquad \forall x(rx \ \simeq\ \mathsf{d}(\mathsf{k},\mathsf{s},r,\lambda z.\mathsf{s})).$$

From $r = \lambda z.\mathsf{s}$ we would be able to deduce by (d) and (*) that $\forall x(rx = \mathsf{k})$, in contradiction to the assumption $r = \lambda z.\mathsf{s}$. Hence $r \neq \lambda z.\mathsf{s}$. Since r and $\lambda z.\mathsf{s}$ are defined, (d) and (*) now give us $\forall x(rx = \mathsf{s})$. But then operational extensionality $(\mathsf{Op\text{-}Ext})$ yields $r = \lambda z.\mathsf{s}$; again a contradiction.

To establish the second inconsistency, we work with the term fix_t and let r be the term $\mathsf{fix}_t(\lambda x.\mathsf{d}(\mathsf{k},\mathsf{s},x,\mathsf{s}))$. Now a simple calculation shows that $r = \mathsf{s}$ implies $r = \mathsf{k}$, and $r \neq \mathsf{s}$ yields $r = \mathsf{s}$; again a contradiction. □

3 Applicative theories

Now I do not follow the historic timeline. Originally, Feferman's interest in the operational approach was triggered by his work on explicit mathematics to which we will turn in the following section. Most approaches to explicit mathematics choose a sort of second order operational approach that permits the formation of classes of operations and includes class formation principles of various strengths.

In this section we set a slower pace, stay first order and carefully extend the basic theory $\mathsf{BO}(\mathrm{L})$ by some elementary axioms for the natural numbers. Then we consider various forms of induction on the natural numbers, and later the numerical choice operator μ and the Suslin operator E_1. These theories constitute the first order part of explicit mathematics, and we call them *applicative theories*.

Let L_1 be an operational language that in addition to the primitive first order symbols mentioned on page 4 comprises constants 0 (zero), $\mathsf{s_N}$ (numerical successor), $\mathsf{p_N}$ (numerical predecessor), $\mathsf{d_N}$ (definition by numerical cases), $\mathsf{r_N}$ (primitive recursion), μ (unbounded search), E_1 (Suslin operator), and the unary relation symbol N for the collection of all natural numbers. Then we often use $(r \in \mathsf{N})$ interchangeably with $\mathsf{N}(r)$ and set

$$(r : \mathsf{N}^k \to \mathsf{N}) \;:=\; (\forall x_1, \ldots, x_k \in \mathsf{N})(r(x_1, \ldots, x_k) \in \mathsf{N}),$$

where k is supposed to be a positive natural number. Furthermore, in the following we generally write $(r : \mathsf{N} \to \mathsf{N})$ for $(r : \mathsf{N}^1 \to \mathsf{N})$ and r' for $\mathsf{s_N} r$.

The *basic theory of operations and numbers* BON is the extension of $\mathsf{BO}(\mathrm{L}_1)$ by the following groups of axioms, dealing with the natural numbers.

Natural numbers.

(Nat.1) $0 \in \mathsf{N} \;\wedge\; (a \in \mathsf{N} \;\to\; a' \in \mathsf{N})$.

(Nat.2) $a \in \mathsf{N} \;\to\; (a' \neq 0 \;\wedge\; \mathsf{p_N}(a') = a)$.

(Nat.3) $(a \in \mathsf{N} \;\wedge\; a \neq 0) \;\to\; (\mathsf{p_N} a \in \mathsf{N} \;\wedge\; (\mathsf{p_N} a)' = a)$.

Definition by numerical cases.

(Nat.4) $(a, b \in \mathsf{N} \;\wedge\; a = b) \;\to\; \mathsf{d_N}(u, v, a, b) = u$.

(Nat.5) $(a, b \in \mathsf{N} \;\wedge\; a \neq b) \;\to\; \mathsf{d_N}(u, v, a, b) = v$.

Primitive recursion.

(Nat.6) $(a \in \mathsf{N} \;\wedge\; f : \mathsf{N}^2 \to \mathsf{N}) \;\to\; \mathsf{r_N}(a, f) : \mathsf{N} \to \mathsf{N}$.

(Nat.7) $(a, b \in \mathsf{N} \;\wedge\; f : \mathsf{N}^2 \to \mathsf{N} \;\wedge\; g = \mathsf{r_N}(a, f)) \;\to$
$(g0 = a \;\wedge\; g(b') = f(b, gb))$.

Axioms for the the constants μ and E_1 follow later. Thus far no induction principles are available, and this is the reason that the axioms (Nat.6) and (Nat.7) are needed for representing all primitive recursive functions within BON. But with these axioms at hand, it is straightforward to prove the following.

Theorem 3 (Primitive recursive functions)**.** *For every k-ary primitive recursive function $\mathcal{F}$ there exists a closed term $\mathsf{prf}_{\mathcal{F}}$ of L_1 such that BON proves $\mathsf{prf}_{\mathcal{F}} : \mathsf{N}^k \to \mathsf{N}$ as well as the (canonical translations of the) defining equations of $\mathcal{F}$.*

Several forms of induction have been considered over BON. The weakest form, called *basic induction*, applies induction only to operations that are known to be total from N to N.

Basic induction on N (B-$\mathsf{I_N}$).

$$(f : N \to N \;\wedge\; f0 = 0 \;\wedge\; (\forall x \in \mathsf{N})(fx = 0 \;\to\; f(x') = 0)) \;\to$$
$$(\forall x \in \mathsf{N})(fx = 0).$$

The assumption $f : \mathsf{N} \to \mathsf{N}$ is central in this formulation and responsible for its relative weakness (see below): Basic induction allows us to prove properties of total operations from N to N; however, in general it cannot be employed to show that certain operations are total from N to N. Basic induction is, of course, a special case of the schema of induction on the natural numbers for arbitrary L_1 formulas.

L_1 induction on N (L_1-$\mathsf{I_N}$). For all L_1 formulas $A[u]$,

$$A[0] \wedge (\forall x \in \mathsf{N})(A[x] \;\to\; A[x']) \;\to\; (\forall x \in \mathsf{N})A[x].$$

The canonical model of $\mathsf{BON} + (\mathrm{L}_1\text{-}\mathsf{I_N})$ has the natural numbers $\mathbb{N}$ as universe and interprets application $\circ$ as the partial function $\circ_{\mathbb{N}}$ from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ such that, for all $e, n \in \mathbb{N}$,

$$(e \circ_{\mathbb{N}} n) \;\simeq \{e\}(n),$$

where $\{e\}$ for $e = 0, 1, \ldots$ is the usual indexing of the partial recursive functions on $\mathbb{N}$. There are also partial and total term models of $\mathsf{BON} + (\mathrm{L}_1\text{-}\mathsf{I_N})$, see, e.g., Beeson [1], Feferman [12], and Troelstra and van Dalen [62]. Probably the simplest way to set up a model satisfying operational extensionality is to start off from a term model of the $\lambda\eta$-calculus (extended by reduction rules for the additional constants of L_1) and to use the standard translation of combinatory logic into the λ-calculus.

In the following theorem we summarize several consistency and inconsistency results concerning BON. The two stated inconsistencies are direct consequences of Theorem 2 since in the presence of $\forall x(x \in \mathsf{N})$ definition by numerical cases is full definition by cases.

Theorem 4. *We have the following consistency and inconsistency results:*

1. $\mathsf{BON} + (\mathsf{Tot}) + (\mathsf{Op\text{-}Ext}) + (\mathrm{L}_1\text{-}\mathsf{I}_\mathsf{N})$ *is consistent.*

2. $\mathsf{BON} + \forall x(x \in \mathsf{N}) + (\mathsf{d}) + (\mathrm{L}_1\text{-}\mathsf{I}_\mathsf{N})$ *is consistent.*

3. $\mathsf{BON} + \forall x(x \in \mathsf{N}) + (\mathsf{Op\text{-}Ext})$ *is inconsistent.*

4. $\mathsf{BON} + \forall x(x \in \mathsf{N}) + (\mathsf{Tot})$ *is inconsistent.*

The axioms of BON take care of the induction-free part of primitive recursive arithmetic PRA, equipped with the combinatorial machinery of $\mathsf{BO}(\mathrm{L}_1)$, which does not contribute to proof-theoretic strength. Depending on what form of induction we add to BON we thus obtain systems equivalent to primitive recursive arithmetic PRA or Peano arithmetic PA; there are also intermediate forms of induction that we omit.

Theorem 5. $\mathsf{BON} + (\mathsf{B\text{-}I_N})$ *is proof-theoretically equivalent to* PRA *and* $\mathsf{BON} + (\mathrm{L}_1\text{-}\mathsf{I}_\mathsf{N})$ *to* PA*.*

Adding, for example, the assertion $\forall x(x \in \mathsf{N})$ would not spoil these two equivalences. However, the situation becomes much more interesting as soon as further axioms for the type-2 functionals μ and E_1 are taken into consideration. The numerical choice operator μ is characterized by the following two axioms.

Axioms for μ.

(μ.1) $f : \mathsf{N} \to \mathsf{N} \;\leftrightarrow\; \mu f \in \mathsf{N}$

(μ.2) $(f : \mathsf{N} \to \mathsf{N} \wedge (\exists x \in \mathsf{N})(fx = 0)) \;\to\; f(\mu f) = 0.$

μ is a non-constructive but predicatively acceptable operator, closely related to the well-known operator E_0 for quantification over the natural numbers. The relationship between μ and E_0 on the basis of BON has been studied in Kahle [47] in full detail: E_0 can be defined within BON from μ; for deriving μ from E_0 Kahle extends BON by specific (proof-theoretically irrelevant) strictness assertions.

The non-constructive operator μ and the functional E_0 are a well-studied objects in higher recursion theory, cf., for example, Feferman [10] and Hinman [27] for a comprehensive survey. It is known that the 1-sections of μ and E_0 are identical; they coincide with the set of natural numbers in the constructible hierarchy up to the first non-recursive ordinal ω_1^{ck}, and, hence, with the collection of hyperarithmetic sets of natural numbers. Consequently, the structure $(\mathbb{N}, 1\text{-}sec(\mathsf{E}_1), \ldots)$ is the minimal standard model of Δ^1_1 comprehension.

If one wants to speak about well-foundedness in this context, the natural step is to add the Suslin operator E_1 that tests for well-foundedness of binary relations on the natural numbers. For the formulation of the axioms of E_1 it is convenient to introduce the descending chain condition $DCC[f]$,

$$DCC[f] := (\exists g \in (\mathsf{N} \to \mathsf{N}))(\forall x \in \mathsf{N})(f(g(x'), gx) = 0),$$

stating that there exists a total operation g from N to N describing a descending chain $g0, g1, \ldots$ with respect to the binary relation coded by f.

Axioms for E_1.

(E_1.1) $f : \mathsf{N}^2 \to \mathsf{N} \;\leftrightarrow\; \mathsf{E}_1 f \in \mathsf{N}$.

(E_1.2) $f : \mathsf{N}^2 \to \mathsf{N} \;\to\; (DCC[f] \;\leftrightarrow\; \mathsf{E}_1 f = 0)$.

The recursion theory of E_1 is well established; see, for example, Hinman [27]. An important result states that the 1-section of E_1 coincides with the set of natural numbers in the constructible hierarchy up to the first recursively inaccessible ordinal ι_0. This ordinal is also the least ordinal not recursive in E_1. Also, Gandy showed that the 1-section of E_1 builds the least standard model of Δ^1_2 comprehension.

From the ontological point of view, the operators μ and E_1 behave as expected. μ takes care of quantification over the natural numbers. Therefore, if μ and the axioms for μ are available, every arithmetically definable set of natural numbers can be represented by a total operation from N to N. If, in addition, E_1 and the axioms for E_1 are at our disposal, we can operationally check for well-foundedness and thus the Π^1_1 normal form theorem allows us to represent all Π^1_1 sets of natural numbers as total operations from N to N.

In the following we write $\mathsf{BON}(\mu)$ for the extension of BON by the axioms (μ.1) and (μ.2). In spite of its “recursion-theoretic strength”, $\mathsf{BON}(\mu) + (\mathsf{B}\text{-}\mathsf{I}_\mathsf{N})$ is fairly weak proof-theoretically as shown in Feferman and Jäger [23].

Theorem 6. *We have the following proof-theoretic equivalences:*

1. $\mathsf{BON}(\mu) + (\mathsf{B}\text{-}\mathsf{I}_\mathsf{N}) \;\equiv\; \mathsf{PA} \;\equiv\; \mathsf{ACA}_0 \;\equiv\; \Delta^1_1\text{-}\mathsf{CA}_0$.
2. $\mathsf{BON}(\mu) + (\mathrm{L}_1\text{-}\mathsf{I}_\mathsf{N}) \;\equiv\; \Pi^0_1\text{-}\mathsf{CA}_{<\varepsilon_0} \;\equiv\; \Delta^1_1\text{-}\mathsf{CA}$.

In this theorem and whenever we mention subsystems of second order arithmetic or set theory later we follow the standard nomenclature and refrain from further explanations; see, for example, Buchholz, Feferman, Pohlers, and Sieg [3] or Simpson [59].

The theory $\mathsf{BON}(\mu) + (\mathsf{B}\text{-}\mathsf{I}_\mathsf{N})$ is particularly interesting in connection with Feferman's philosophical analysis of Weyl's *Das Kontinuum* and his reconstruction of the axiom system of *Das Kontinuum* in modern terms; see Feferman [16, 17]. One of his key results is that Weyl's approach can be developed within a conservative extension of PA. This system W can be easily reduced to $\mathsf{BON}(\mu) + (\mathsf{B}\text{-}\mathsf{I}_\mathsf{N})$.

Now we briefly turn to the proof theory of E_1. The applicative theory for E_1 is called SUS and consists of $\mathsf{BON}(\mu)$ and the additional axioms (E_1.1) and (E_1.2). The proof-theoretic analysis of SUS plus various forms of induction is carried through in detail in Jäger and Strahm [43] and Jäger and Probst [41].

Theorem 7. *We have the following proof-theoretic equivalences:*

1. $\mathsf{SUS} + (\mathsf{B}\text{-}\mathsf{I}_\mathsf{N}) \equiv \Pi^1_1\text{-}\mathsf{CA}_0 \equiv \Delta^1_2\text{-}\mathsf{CA}_0$.
2. $\mathsf{SUS} + (\mathrm{L}_1\text{-}\mathsf{I}_\mathsf{N}) \equiv \Pi^1_1\text{-}\mathsf{CA}_{<\varepsilon_0} \equiv \Delta^1_2\text{-}\mathsf{CA}$.

For the lower bounds of SUS plus various forms of induction we exploit the fact that the Suslin operator has the power to deal with Π^1_1 comprehension, provably in $\mathsf{SUS} + (\mathsf{B}\text{-}\mathsf{I}_\mathsf{N})$. Upper bounds are established in Jäger and Strahm [43] by making use of a very specific positive Δ^1_2 inductive definition in the framework of theories of admissible sets and by interpreting the application operation by a Σ definable fixed point of this inductive definition. A more direct approach to the computation of the upper bounds in question is provided in Jäger and Probst [41]; several theories featuring the Suslin operator are embedded into ordinal theories tailored to dealing with non-monotone inductive definitions that enable a smooth definition of the application relation.

4 Explicit mathematics

As already mentioned above, Feferman [9] is the starting point of explicit mathematics. The two other "big elephants" are Feferman [11], in which explicit mathematics is discussed in the context of recursion theory, and Feferman [12], which discusses the relationship between explicit mathematics and several alternative approaches to constructive mathematics.

Originally, explicit mathematics was formulated in a single sorted first order language with a unary relation symbol Cl and a binary relation symbol η, where $Cl(u)$ expressed that u is a class and $(v\ \eta\ u)$ that v has the property described by u in case $Cl(u)$ holds. Later it turned out to be more convenient to formulate explicit mathematics in an extension of the logic of partial terms with class variables; see Jäger [29].

The underlying ontological idea is that we have two sorts of objects: the individuals as in the case of applicative theories and collections of such objects, called *classes*. The individuals form a partial combinatory algebra and are conceived as being given intensionally and explicitly as before, whereas the classes are subsets of the applicative universe and may even be considered to exist in a Platonic sense; this is purposely left open. Membership of individuals in classes is as usual, and we have extensionality on the level of classes.

But also the classes can be addressed explicitly, though in an indirect way: We add a new binary relation $\Re$ to express that the individual x *represents* or *names* class X, written $\Re(x, X)$. Classes are explicitly generated with reference to their names in an operational way, and this process is made uniform in the parameters. For example, we will have a constant nat that names the class of natural numbers and a constant un such that $\mathsf{un}(u, v)$ is the name of the union of the classes U and V provided that u is the name of U and v the name of V.

A suitable language for our purpose is the extension L_2 of L_1 by class variables $U, V, W, X, Y, Z, \dots$ (possibly with subscripts), two new binary relation symbols $\in$ (membership) and $\Re$ (naming, representation) and the new individual constants nat (natural numbers), id (identity), co (complement), un (union), dom (domain), inv (inverse image), j (join), and i (inductive generation). The atomic formulas of L_2 are all expressions $r\downarrow$, $(r = s)$, $\mathsf{N}(r)$, $(r \in U)$, $(U = V)$, and $\Re(r, U)$, where r and s are individual terms of L_2.

The *formulas* $(A, B, C, A_0, B_0, C_0, \dots)$ of L_2 are generated from the atomic L_2 formulas by closing under the propositional connectives and quantification in both sorts. An L_2 formula is called *elementary* if it contains neither the relation symbol $\Re$ nor bound class variables. The *stratified* formulas are those L_2 formulas that do not contain the relation symbol $\Re$.

Some individual terms represent (or name) classes, and we write $(r \in \Re)$ to express that r is a name,

$$(r \in \Re) \;:=\; \exists X \Re(r, X).$$

If r names class X, then X can be regarded as the *extension* of r, and in this sense we can transfer an element relation and extensional equality to the level of individuals:

$$(r \,\dot{\in}\, s) \;:=\; \exists X (\Re(s, X) \,\wedge\, r \in X),$$
$$(r \doteq s) \;:=\; \exists X (\Re(r, X) \,\wedge\, \Re(s, X)).$$

Clearly, $(r \notin U)$ and $(r \,\dot{\notin}\, s)$ are short for $\neg(r \in U)$ and $\neg(r \,\dot{\in}\, s)$, respectively. Since we have extensionality on the level of classes, the subclass relation on classes is as usual with the corresponding notion on the level of individual terms

$$(U \subseteq V) \;:=\; \forall x (x \in U \,\rightarrow\, x \in V),$$
$$(r \,\dot{\subseteq}\, s) \;:=\; \exists X \exists Y (\Re(r, X) \,\wedge\, \Re(s, Y) \,\wedge\, X \subseteq Y).$$

Finally, if $\vec{r}$ is the string $r_1, \dots, r_n$ of individual terms and $\vec{U}$ the string $U_1, \dots, U_n$ of class variables of the same length, we set

$$\Re(\vec{r}, \vec{U}) \;:=\; \bigwedge_{i=1}^{n} \Re(r_i, U_i) \quad \text{and} \quad (\vec{r} \in \Re) \;:=\; \bigwedge_{i=1}^{n} (r_i \in \Re).$$

Observe that all formulas $\Re(\vec{r}, \vec{U})$, $(\vec{r} \in \Re)$, $(r \mathrel{\dot{\in}} s)$, $(r \doteq s)$, and $(r \mathrel{\dot{\subseteq}} s)$ are not stratified.

4.1 Elementary explicit comprehension

The logic of the first order part of our systems of explicit mathematics is still Beeson's logic of partial terms as in the previous sections, of course formulated now for the language L_2. In particular, the definedness axioms extend to atomic L_2 formulas and, therefore, $(r \in U)$ and $\Re(r, U)$ imply that the term r has a value. The logic for the second order part of the systems of explicit mathematics is classical predicate logic with equality.

The non-logical axioms of the *elementary theory* EC *of classes and names* comprises the non-logical axioms of BON plus the following two groups of class axioms for classes.

Explicit representation and extensionality.

(Cl.1) $\exists x \Re(x, U)$.

(Cl.2) $\Re(r, U) \wedge \Re(r, V) \;\rightarrow\; U = V$.

(Cl.3) $\forall x(x \in U \;\leftrightarrow\; x \in V) \;\rightarrow\; U = V$.

These axioms state that each class has a name, that there are no homonyms and that equality of classes is extensional. The second group of axioms for classes ensures the build-up of some basic classes, in parallel with a uniform naming process.

Basic class existence axioms.

(Cl.4) $\mathsf{nat} \in \Re \;\wedge\; \forall x(x \mathrel{\dot{\in}} \mathsf{nat} \;\leftrightarrow\; \mathsf{N}(x))$.

(Cl.5) $\mathsf{id} \in \Re \;\wedge\; \forall x(x \mathrel{\dot{\in}} \mathsf{id} \;\leftrightarrow\; \exists y(x = \langle y, y \rangle))$.

(Cl.6) $r \in \Re \;\rightarrow\; (\mathsf{co}(r) \in \Re \wedge \forall x(x \mathrel{\dot{\in}} \mathsf{co}(r) \;\leftrightarrow\; x \mathrel{\dot{\notin}} r))$.

(Cl.7) $r, s \in \Re \;\rightarrow\; (\mathsf{un}(r, s) \in \Re \wedge \forall x(x \mathrel{\dot{\in}} \mathsf{un}(r, s) \;\leftrightarrow\; (x \mathrel{\dot{\in}} r \;\vee\; x \mathrel{\dot{\in}} s)))$.

(Cl.8) $r \in \Re \;\rightarrow\; (\mathsf{dom}(r) \in \Re \wedge \forall x(x \mathrel{\dot{\in}} \mathsf{dom}(r) \;\leftrightarrow\; \exists y(\langle x, y \rangle \mathrel{\dot{\in}} r)))$.

(Cl.9) $r \in \Re \;\rightarrow\; (\mathsf{inv}(r, f) \in \Re \wedge \forall x(x \mathrel{\dot{\in}} \mathsf{inv}(r, f) \;\leftrightarrow\; fx \mathrel{\dot{\in}} r))$.

These axioms formalize that the natural numbers form a class and that there is the identity class; furthermore, classes are closed under complements, unions, domains and inverse images. It is important that the axioms (C.4) – (C.9) provide a finite axiomatization of uniform elementary comprehension.

Theorem 8 (Elementary comprehension)**.** *For every elementary formula* $A[u, \vec{v}, \vec{W}]$ *with at most the indicated free variables there exists a closed term* t_A *such that* EC *proves:*

1. $\vec{z} \in \Re \rightarrow t_A(\vec{y}, \vec{z}) \in \Re$,

2. $\Re(\vec{z}, \vec{Z}) \rightarrow \forall x(x \mathrel{\dot{\in}} t_A(\vec{y}, \vec{z}) \leftrightarrow A[x, \vec{y}, \vec{Z}])$.

Immediate obvious consequences of this theorem are, for example, the existence of the empty class $\emptyset$ and the universal class $\mathbf{V}$, the closure of the collection of all classes under complements, finite unions, finite intersections, finite Cartesian products, and the finitely iterated formation of function spaces.

By a model construction following Feferman [9, 12] it can be shown that EC is consistent with stratified comprehension, whereas a simple Russell-style argument shows that it is inconsistent with comprehension for arbitrary L_2 formulas.

Interesting induction principles in the context of EC are $(\mathsf{B}\text{-}\mathsf{I}_\mathsf{N})$ and $(\mathrm{L}_1\text{-}\mathsf{I}_\mathsf{N})$ as before plus two new forms of induction: class induction and the schema of induction for all L_2 formulas.

Class induction on N $(\mathsf{C}\text{-}\mathsf{I}_\mathsf{N})$.

$$\forall X(0 \in X \wedge (\forall x \in \mathsf{N})(x \in X \rightarrow x' \in X) \rightarrow (\forall x \in \mathsf{N})(x \in X)).$$

L_2 induction on N $(\mathrm{L}_2\text{-}\mathsf{I}_\mathsf{N})$. For all L_2 formulas $A[u]$,

$$A[0] \wedge (\forall x \in \mathsf{N})(A[x] \rightarrow A[x']) \rightarrow (\forall x \in \mathsf{N})A[x].$$

All combinations of EC and its extension $\mathsf{EC}(\mu)$ by the type-2 functional μ with these forms of first and second order induction have been analyzed proof-theoretically; a detailed presentation will be given in Feferman, Jäger, and Strahm [24]. As illustration we mention three results.

Theorem 9. *We have the following proof-theoretic equivalences:*

1. $\mathsf{EC} + (\mathsf{C}\text{-}\mathsf{I}_\mathsf{N}) \equiv \mathsf{BON} + (\mathrm{L}_1\text{-}\mathsf{I}_\mathsf{N}) \equiv \mathsf{ACA}_0 \equiv \mathsf{PA}$.

2. $\mathsf{EC} + (\mathrm{L}_2\text{-}\mathsf{I}_\mathsf{N}) \equiv \mathsf{ACA}$.

3. $\mathsf{EC}(\mu) + (\mathsf{B}\text{-}\mathsf{I}_\mathsf{N}) \equiv \mathsf{BON}(\mu) + (\mathsf{B}\text{-}\mathsf{I}_\mathsf{N}) \equiv \mathsf{PA}$,

In Feferman [10, 12, 16] it is convincingly argued that EC-like systems provide a natural framework for dealing with large parts of predicative mathematics. In particular, the theory $\mathsf{EC}(\mu) + (\mathsf{B}\text{-}\mathsf{I}_\mathsf{N})$ is a natural extension of $\mathsf{BON}(\mu) + (\mathsf{B}\text{-}\mathsf{I}_\mathsf{N})$ and as such perfectly suited for developing Weyl's approach to the continuum. It is also shown in Feferman [10] that the intensional and extensional variants of finite type theories find their natural place within EC.

4.2 Join and inductive generation

Of course, the theorem about elementary comprehension tells us that in EC the classes are closed under the formation of finite unions and intersections. But in order to form the unions, intersections, and Cartesian products of general possibly infinite families of classes, Feferman introduced a further axiom, and here the constant j comes into play.

Join axiom.

$$(\mathsf{J}) \quad (a \in \Re \wedge (\forall x \,\dot{\in}\, a)(f(x) \in \Re)) \;\to\; (\mathsf{j}(a,f) \in \Re \wedge DU[a,f,\mathsf{j}(a,f)]),$$

where the formula $DU[a,f,b]$ is short for

$$\forall x(x \,\dot{\in}\, b \;\leftrightarrow\; x = <(x)_0,(x)_1> \wedge\, (x)_0 \,\dot{\in}\, a \,\wedge\, (x)_1 \,\dot{\in}\, f((x)_0)).$$

This axiom states that given a class named by a and an operation f from this class to names, $\mathsf{j}(a,f)$ is the name of the disjoint union of the classes named by these $f(x)$ with $x \,\dot{\in}\, a$. Clearly, the generation of $\mathsf{j}(a,f)$ is uniform in a and f.

Finally, let us turn to inductive generation and introduce an auxiliary abbreviation. Given an L_2 formula $A[u]$ we write $Prog[a,b,A]$ for

$$(\forall x \,\dot{\in}\, a)(\forall y(<y,x> \,\dot{\in}\, b \;\to\; A[y]) \;\to\; A[x]).$$

Moreover, $Prog[a,b,c]$ stands for $Prog[a,b,C]$ with $C[u]$ being the formula $(u \,\dot{\in}\, c)$. If we think of b coding a binary relation on the class named a, then $Prog[a,b,A]$ states that formula $A[u]$ is progressive on a with respect to b. Feferman's axioms about inductive generation guarantee the existence of accessible parts of classes with respect to binary relations.

Axioms for inductive generation.

$(\mathsf{IG}.1) \quad a,b \in \Re \;\to\; (\mathsf{i}(a,b) \in \Re \wedge Prog[a,b,\mathsf{i}(a,b)]).$

$(\mathsf{IG}.2) \quad (a,b \in \Re \wedge Prog[a,b,A]) \;\to\; (\forall x \,\dot{\in}\, \mathsf{i}(a,b))A[x]$

for all L_2 formulas $A[u]$. Let a and b be names. According to (IG.1) then $\mathsf{i}(a,b)$ names a class and is progressive on a with respect to b. (IG.2) is an induction principle and states that the class named $\mathsf{i}(a,b)$ is minimal with respect to this property.

The most famous theory of explicit mathematics is called T_0 and extends EC by join, inductive generation and full induction on the natural numbers for arbitrary L_2 formulas,

$$\mathsf{T}_0 \;:=\; \mathsf{EC} + (\mathsf{J}) + (\mathsf{IG}.1) + (\mathsf{IG}.2) + (\mathrm{L}_2\text{-}\mathsf{I}_\mathsf{N}).$$

Many subsystems of T_0 - obtained, for example, by restricting the induction principles or omitting inductive generation - have been introduced and studied in Chapter II (written by Feferman and Sieg), of Buchholz, Feferman, Pohlers, and Sieg [3].

As far as T_0 itself is concerned, [3] also provides an argument that it can be embedded into the system $\Delta^1_2\text{-}\mathsf{CA} + (\mathsf{BI})$ of second order arithmetic. Then Jäger and Pohlers [40] determined the upper bound of the proof-theoretic strength the latter system via the theory KPi of iterated admissible sets, and Jäger [28] showed by a well-ordering proof within (even the intuitionistic version of) T_0 that this bound is sharp.

Theorem 10. $\mathsf{T}_0 \ \equiv\ \Delta^1_2\text{-}\mathsf{CA} + (\mathsf{BI}) \ \equiv\ \mathsf{KPi}.$

Recently Sato presented an interesting reduction of $\Delta^1_2\text{-}\mathsf{CA} + (\mathsf{BI})$ to T_0 without employing a well-ordering proof; see [56].

Feferman [9] also introduces the extension of T_0 by the non-constructive μ and baptizes it T_1. He makes a point that T_0 provides an elegant framework for Borelian and hyperarithmetic mathematics. In particular, he advocates studying a generalization of hyerarithmetic model theory by means of formalization in T_0. Feferman [13] contains further conceptual work and technical results along similar lines.

Glaß and Strahm [26] mentiones that T_0 and T_1 are equiconsistent and determines the proof-theoretic strengths of many subsystems of T_1. Finally, ongoing work of Probst is about extensions of T_1 by the Suslin operator E_1. One of his observations is that the second order framework provides several ways of formulating E_1-like operators that may turn out not to be equivalent.

4.3 Monotone inductive definitions

A further interesting principle is introduced in Feferman [15]. It expresses that every monotone operation from classes to classes has a least fixed point. Define

$$\begin{aligned} Mon[f] \ &:=\ (\forall x, y \in \Re)(x \,\dot{\subseteq}\, y \ \rightarrow\ fx \,\dot{\subseteq}\, fy), \\ Lfp[f,a] \ &:=\ fa \doteq a \ \wedge\ (\forall x \in \Re)(fx \,\dot{\subseteq}\, x \ \rightarrow\ a \,\dot{\subseteq}\, x). \end{aligned}$$

In view of our definition of $(u \,\dot{\subseteq}\, v)$, $Mon[f]$ implies that f maps names to names; similarly, $Lfp[f,a]$ implies that a is a name. Then (MID) is the axiom stating that every monotone operation has a least fixed point,

$$(\mathsf{MID}) \qquad \forall f(Mon[f] \ \rightarrow\ \exists a Lfp[f,a]).$$

The analysis (MID) turned out to be very interesting. Adding (MID) to T_0 or a (natural) subsystem of T_0 leads to an enormous increase of its proof-theoretic strength. A first result in Takahashi [60] says that $\mathsf{T}_0 + (\mathsf{MID})$ is interpretable in $\Pi^1_2\text{-}\mathsf{CA} + (\mathsf{BI})$. Later Rathjen in a series of articles [50, 51, 53, 52] and Glass, Rathjen, Schlüter [25] managed to provide a complete proof-theoretic analysis of (MID) and the uniform version (UMID) of this principle over T_0 and some of its natural subsystems. They were able to determine the exact relationship between these systems of explicit mathematics and systems of second order arithmetic with Π^1_2 comprehension.

4.4 Universes

Universes have been introduced into explicit mathematics in Feferman [14], Marzetta [48], Jäger, Kahle and Studer [38], and Jäger and Strahm [42] as a powerful method for increasing its expressive and proof-theoretic strength. Informally speaking, universes play a similar role in explicit mathematics as admissible sets in weak set theory and the sets V_κ (for regular cardinals κ) in full classical set theory; explicit universes are also closely related to universes in Martin-Löf type theory. More formally, universes in explicit mathematics are classes which consist of names only and reflect the theory $\mathsf{EC} + (\mathsf{J})$.

Let $\mathfrak{C}[U, a]$ be the closure condition that is formed by the disjunction of the following L_2 formulas:

(1) $a = \mathsf{nat} \lor a = \mathsf{id}$,

(2) $\exists x(a = \mathsf{co}(x) \land x \in U)$.

(3) $\exists x \exists y(a = \mathsf{un}(x, y) \land x \in U \land y \in U)$,

(4) $\exists x(a = \mathsf{dom}(x) \land x \in U)$.

(5) $\exists x \exists f(a = \mathsf{inv}(x, f) \land x \in U)$,

(6) $\exists x \exists f(a = \mathsf{j}(x, f) \land x \in U \land (\forall y \,\dot{\in}\, x)(fx \in U))$.

Thus the formula $\forall x(\mathfrak{C}[U, x] \to x \in U)$ states that U is a class that is closed under (the finite axiomatization of) elementary comprehension and join. If, in addition, all elements of U are names, we call U a *universe* and write $Univ[U]$ to express this fact,

$$Univ[U] \;:=\; \forall x(\mathfrak{C}[U, x] \to x \in U) \land (\forall x \in U)(x \in \Re).$$

Also, $\mathbb{U}[a]$ states that the individual a is a name of a universe,

$$\mathbb{U}[a] \;:=\; \exists X(\Re(a, X) \land Univ[X]).$$

It is an immediate consequence of the closure properties of universes that they satisfy elementary comprehension and join. The first important axiom in connection with universes is the *limit axiom.* We assume that L_2 contains a fresh individual constant ℓ and express this by

(Lim) $$a \in \Re \;\to\; \mathbb{U}[\ell a] \land a \,\dot{\in}\, \ell a.$$

Hence this axiom states that the individual ℓ uniformly picks for each name x of a class the name ℓx of a universe containing x. Since universes are the explicit analogue of admissible sets, the axiom (Lim) is the explicit analog of the limit axiom in admissible set theory which enforces that any set is contained in an admissible set. The limit axiom (Lim) together with $\mathsf{EC} + (\mathsf{J})$ provides the explicit analogue of (recursive) inaccessibility.

There is a very natural way in explicit mathematics to go a step further and couch (recursive) Mahloness into this framework. To simplify the notation we set

$$(f \in \Re \to \Re) \ := \ \forall x(x \in \Re \ \to \ fx \in \Re),$$
$$(f \mathrel{\dot\in} a \to a) \ := \ \forall x(x \mathrel{\dot\in} a \ \to \ fx \mathrel{\dot\in} a)$$

and let m be a further fresh individual constant of L_2. Then the *Mahlo axiom* is the assertion

(Mahlo) $$a \in \Re \ \wedge \ f \in (\Re \to \Re) \ \ \to \\ \mathbb{U}[\mathsf{m}(a,f)] \ \wedge \ a \mathrel{\dot\in} \mathsf{m}(a,f) \ \wedge \ f \in (\mathsf{m}(a,f) \to \mathsf{m}(a,f)).$$

This means that given a name a and an operation f from names to names the individual m uniformly picks a universe $\mathsf{m}(a,f)$ that contains a and is closed under f.

For the proof-theoretic analysis of (Lim) and (Mahlo) over the relevant metapredicative and impredicative systems of explicit mathematics we refer to Jäger, Kahle, and Studer [38] and Jäger and Strahm [24]. In all cases there is a direct correspondence to systems of iterated admissible sets, but space does not permit to go into details here. Jäger and Strahm [44] even explains how stronger reflection principles can be formulated within the explicit framework.

4.5 Names of classes and universes

One of the very central ontological observations is that the names of a class never form a class, no matter how simple this class may be. This theorem follows immediately from Jäger [31] and is proved in full detail in Jäger [30] and Feferman, Jäger, and Strahm [24].

Theorem 11. $\mathsf{EC} \ \vdash \ \forall X \neg \exists Y \forall z (z \in Y \ \leftrightarrow \ \Re(z, X))$.

In Section 2 we introduced the notion of operational extensionality. Clearly, there is also a corresponding notion of class extensionality:

(Cl-Ext) $$(\forall x, y \in \Re)(x \doteq y \ \to \ x = y),$$

claiming that two names are identical provided that they name the same class. Although at a first glance this principle may appear to be acceptable or even natural, we have to dismiss it since it is inconsistent with EC. The following theorem is a consequence of Theorem 11 above. An alternative proof, due to Gordeev, of a similar result is presented in Beeson [1].

Corollary 12. (Cl-Ext) *is inconsistent with* EC.

Proof. Pick, for example, the class of natural numbers. From (Cl-Ext) we could conclude that all names of this class are identical to nat and thus form a class (in view of elementary comprehension), contradicting Theorem 11. □

Hence T_0 + (Cl-Ext) is inconsistent as well, thus answering a question raised in Feferman [12]. Although the names of a class never form a class, it is consistent to claim that there exists the class of all names. This can be seen by extending the model construction for EC that is presented in detail in Feferman [12] and Feferman, Jäger, and Strahm [24].

Theorem 13. *The assertion* $\exists X \forall x(x \in X \leftrightarrow x \in \Re)$ *is consistent with* EC*, but not provable in* EC.

With some additional effort even a strengthening of this result is possible: We can consistently assume in EC that all objects are names.

Let us now take a look at power classes. In principle, one could think of two forms of power classes. The *strong power class axiom* states that for every class X there exists a class Y such that Y contains exactly the names of <u>all</u> subclasses of X,

(SP) $$\forall X \exists Y \forall z(z \in Y \leftrightarrow \exists Z(\Re(z, Z) \land Z \subseteq X)).$$

On the other hand, the *weak power class axiom* asks for less. Then we only claim that for each class X there exists a class Y such that each element of Y names a subclass of X and for any subclass of X <u>at least one</u> of its names belongs to Y,

(WP) $$\forall X \exists Y((\forall z \in Y)(\exists Z \subseteq X)(\Re(z, Z)) \land (\forall Z \subseteq X)(\exists z \in Y)\Re(z, Z)).$$

Clearly, each of these can be formulated uniformly by adjunction of suitable constants. Neither the strong nor the weak power class axiom is provable in EC. Much worse, by Theorem 11 we know that in EC the names of the empty class cannot form a class, and thus the strong power class of the empty class cannot exist.

Corollary 14. (SP) *is inconsistent with* EC.

As the following remark shows, the weak power class axiom is less problematic in this respect. Its consistency with EC is a consequence of Theorem13.

Corollary 15. $\mathsf{EC} + \exists X \forall x (x \in X \leftrightarrow x \in \Re)$ *proves*

$$\exists f (\forall a \in \Re)((\forall b \mathrel{\dot\in} fa)(b \mathrel{\dot\subseteq} a) \wedge (\forall b \mathrel{\dot\subseteq} a)(\exists c \mathrel{\dot\in} fa)(b \doteq c).$$

Hence the (uniform version of the) weak power class axiom is provable in $\mathsf{EC} + \exists X \forall x (x \in X \leftrightarrow x \in \Re)$ *and thus consistent with* EC.

Proof. Let Z be the class of all names and let z be a name of Z. Also, let r be the closed term $\lambda xy.\mathsf{co}(\mathsf{un}(\mathsf{co}(x), \mathsf{co}(y)))$. This means that for all names a and b, $r(a, b)$ is a name of the intersection of the classes represented by a and b. Now we consider the elementary formula

$$A[u, v, W] \ := \ (\exists x \in W)(u = r(v, x))$$

and choose t_A according to Theorem 8. Then $t_A(v, w)$ is a name in case w is a name, and we have

$$\Re(w, W) \ \rightarrow \ \forall u (u \mathrel{\dot\in} t_A(v, w) \ \leftrightarrow \ (\exists x \in W)(u = r(v, x))).$$

Since Z is supposed to be the class of all names and z one of its names, this implies

$$\forall u (u \mathrel{\dot\in} t_A(v, z) \ \leftrightarrow \ (\exists x \in \Re)(u = r(v, x))).$$

Put $s := \lambda v.t_A(v, z)$. Clearly, $r(a, b) \mathrel{\dot\subseteq} a$ for all $a, b \in \Re$ and $s(a, b) \doteq b$ for any $b \mathrel{\dot\subseteq} a$. Hence s is a witness for the existential assertion we have to prove. □

However, we have to be careful. The join axiom (J) is incompatible with the weak power class axioms.

Theorem 16. $\mathsf{EC} + (\mathsf{J})$ *proves the negation of* (WP). *Also, in* $\mathsf{EC} + (\mathsf{J})$ *the names cannot form a class.*

Proof. Working in $\mathsf{EC} + (\mathsf{J})$, we let a be a name of the universal class $\mathbf{V}$ and assume (WP). Then there exists an element $b \in \Re$ – namely a name of a weak power class of $\mathbf{V}$ – such that:

$$(\forall x \mathrel{\dot\in} b)(x \mathrel{\dot\subseteq} a), \tag{1}$$

$$(\forall x \in \Re)(\exists y \mathrel{\dot\in} b)(x \doteq y). \tag{2}$$

Assertion (1) implies that all elements of (the class represented by) b are names. Now we apply (J) to b and the operation $\lambda z.z$ and obtain that $\mathsf{j}(b, \lambda z.z) \in \Re$ and

$$\forall x (x \mathrel{\dot\in} \mathsf{j}(b, \lambda z.z) \ \leftrightarrow \ (\exists y_1 \mathrel{\dot\in} b) \exists y_2 (x = \langle y_1, y_2 \rangle \wedge y_2 \mathrel{\dot\in} y_1).$$

By elementary comprehension we can thus form a class X satisfying

$$\forall x (x \in X \ \leftrightarrow \ \langle x, x \rangle \mathrel{\dot\notin} \mathsf{j}(b, \lambda z.z)).$$

According to (2), X has a name $u \dot{\in} b$. However, this implies

$$u \in X \;\leftrightarrow\; \neg(u \dot{\in} b \wedge u \dot{\in} u) \;\leftrightarrow\; u \dot{\notin} u \;\leftrightarrow\; u \notin X.$$

This is a contradiction. Hence **V** cannot have a weak power class, and (WP) has been refuted. Therefore, it is also clear in view of the previous corollary that the names must not form a class. □

Now we turn to some remarkable ontological properties of universes. A first observation, proved in Marzetta [48], reveals that no universe may contain one of its names. We have mentioned already that the names of a class do not form a class. In connection with universes, a stronger result is possible: Each class has so many names that not all of them can be contained in a single universe; in other words, no universe is large enough to contain all names of a given type. For a proof of this result see Jäger, Kahle, and Studer [38] or Minari [49]. This result implies that in the presence of the limit axiom (Lim), a name a cannot have the same extensions as the universe represented by ℓa. Also, the operation ℓ does not preserve extensional equality; see [38] for details.

Theorem 17.

1. $\mathsf{EC} \vdash \; \mathit{Univ}[U] \wedge \Re(a, U) \;\rightarrow\; a \notin U.$
2. $\mathsf{EC} + (\mathsf{J}) \vdash \; \mathit{Univ}[U] \;\rightarrow\; \exists x(\Re(x, V) \wedge x \notin U).$
3. $\mathsf{EC} + (\mathsf{J}) + (\mathsf{Lim}) \vdash \; (\forall x \in \Re)(x \neq \ell x) \wedge (\exists x, y \in \Re)(x \doteq y \wedge \ell x \neq \ell x).$

In this section several important ontological properties of explicit mathematics have been collected. For more along these lines consult Feferman [12], Jäger, Kahle, and Studer [38], and Jäger and Zumbrunnen [46].

5 Operational set theory

Feferman's original motivation for operational set theory was to provide a setting for the operational formulation of large cardinal statements directly over set theory in a way that seemed to him to be more natural mathematically than the metamathematical formulations using reflection and indescribability principles, etc. He saw operational set theory as a natural extension of the von Neumann approach to axiomatizing set theory. Another principal motivation was to relate formulations of classical large cardinal statements to their analogues in admissible set theory. However, in view of Jäger and Zumbrunnen [45] this aim of operational set theory has to be analyzed further; see below.

The central systems of present day operational set theory can be considered as an applicative (based) reformulation of systems of classical set theory ranging in strength from Kripke-Platek set theory to von Neumann-Bernays-Gödel set theory and a bit beyond.

The basic system OST has been introduced in Feferman [18] and further discussed in Feferman [19] and Jäger [32, 33, 34, 35]. For a gentle introduction into operational set theory and some general motivation we refer to these articles, in particular to [19].

There is also an interesting relationship between some more constructive variants of operational set theory and constructive or semi-constructive set theory, but we will not discuss this line of research here. For a profound discussion of this topic and some interesting technical results see Cantini and Crosilla [5, 6], Cantini [4], and Feferman [20].

5.1 The central systems

Let $\mathcal{L}$ be a typical language of first order set theory with the binary symbols $\in$ and $=$ as its only relation symbols and countably many set variables $a, b, c, f, g, u, v, w, x, y, z, \dots$ (possibly with subscripts). We further assume that $\mathcal{L}$ has a constant ω for the collection of all finite von Neumann ordinals. The formulas of $\mathcal{L}$ are defined as usual.

The language $\mathcal{L}^\circ$ of operational set theory extends $\mathcal{L}$ by the binary function symbol $\circ$ for partial term application, the unary relation symbol $\downarrow$ for definedness and a series of constants: (i) the combinators k and s, (ii) $\top$, $\bot$, **el**, **non**, **dis**, **e**, and **E** for logical operations, (iii) $\mathbb{D}$, $\mathbb{U}$, $\mathbb{S}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{P}$ for set-theoretic operations. The meaning of these constants will be specified by the axioms below.

$\mathcal{L}^\circ$ is an operational language in the sense of Section 2, and we define the terms and formulas of $\mathcal{L}^\circ$ exactly as there. To increase readability, we freely use standard set-theoretic terminology. For example, if $A[x]$ is an $\mathcal{L}^\circ$ formula, then $\{x : A[x]\}$ denotes the collection of all sets satisfying A; it may be (extensionally equal to) a set, but this is not necessarily the case. Special instances are

$$\mathbf{V} \;:=\; \{x : x\downarrow\}, \quad \emptyset \;:=\; \{x : x \neq x\}, \quad \text{and} \quad \mathbf{B} \;:=\; \{x : x = \top \lor x = \bot\}$$

so that $\mathbf{V}$ denotes the collection of all sets (it is not a set itself), $\emptyset$ stands for the empty collection, and $\mathbf{B}$ for the unordered pair consisting of the truth values $\top$ and $\bot$ (it will turn out that $\emptyset$ and $\mathbf{B}$ are sets in OST). The following shorthand notation, for n an arbitrary natural number greater than 0,

$$(f : a^n \to b) \;:=\; (\forall x_1, \dots, x_n \in a)(f(x_1, \dots, x_n) \in b)$$

expresses that f, in the operational sense, is an n-ary mapping from a to b. It does not say, however, that f is an n-ary function in the set-theoretic sense. In this definition the set variables a and b may be replaced by $\mathbf{V}$ and $\mathbf{B}$. So, for example, $(f : a \to \mathbf{V})$ means that f is total on a, and $(f : \mathbf{V} \to b)$ means that f maps all sets into b.

As in the case of explicit mathematics, all systems of operational set theory start off from the basic theory $\mathsf{BO}(\mathcal{L}^\circ)$. The additional non-logical axioms of OST comprise some basic set-theoretic axioms, the representation of elementary logical connectives as operations, and operational set existence axioms.

Basic set-theoretic axioms. They comprise: (i) the usual extensionality axiom; (ii) assertions that give the appropriate meaning to the constant ω; (iii) $\in$-induction for arbitrary formulas $A[u]$ of $\mathcal{L}^\circ$,

$$\forall x((\forall y \in x)A[y] \to A[x]) \to \forall x A[x].$$

Logical operations axioms.

(L.1) $\top \neq \bot$,

(L.2) $(\mathbf{el} : \mathbf{V}^2 \to \mathbf{B}) \wedge \forall x \forall y(\mathbf{el}(x, y) = \top \leftrightarrow x \in y)$,

(L.3) $(\mathbf{non} : \mathbf{B} \to \mathbf{B}) \wedge (\forall x \in \mathbf{B})(\mathbf{non}(x) = \top \leftrightarrow x = \bot)$,

(L.4) $(\mathbf{dis} : \mathbf{B}^2 \to \mathbf{B}) \wedge (\forall x, y \in \mathbf{B})(\mathbf{dis}(x, y) = \top \leftrightarrow (x = \top \vee y = \top))$,

(L.5) $(f : a \to \mathbf{B}) \to (\mathbf{e}(f, a) \in \mathbf{B} \wedge (\mathbf{e}(f, a) = \top \leftrightarrow (\exists x \in a)(fx = \top)))$.

Set-theoretic operations axioms.

(S.1) Unordered pair: $\mathbb{D}(a, b)\downarrow \wedge \forall x(x \in \mathbb{D}(a, b) \leftrightarrow x = a \vee x = b)$.

(S.2) Union: $\mathbb{U}(a)\downarrow \wedge \forall x(x \in \mathbb{U}(a) \leftrightarrow (\exists y \in a)(x \in y))$.

(S.3) Separation for definite operations:

$$(f : a \to \mathbf{B}) \to (\mathbb{S}(f, a)\downarrow \wedge \forall x(x \in \mathbb{S}(f, a) \leftrightarrow (x \in a \wedge fx = \top))).$$

(S.4) Replacement:

$$(f : a \to \mathbf{V}) \to (\mathbb{R}(f, a)\downarrow \wedge \forall x(x \in \mathbb{R}(f, a) \leftrightarrow (\exists y \in a)(x = fy))).$$

(S.5) Choice: $\exists x(fx = \top) \to (\mathbb{C}f\downarrow \wedge f(\mathbb{C}f) = \top)$.

This finishes our description of the system OST. $\mathsf{OST}(\mathbb{P})$ is $\mathsf{OST} + (\mathbb{P})$ and $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ is $\mathsf{OST} + (\mathbb{P}) + (\mathbf{E})$, where $(\mathbb{P})$ and $(\mathbf{E})$ are axioms providing for the operational form of power set and unbounded existential quantification, respectively:

$(\mathbb{P})$ $$(\mathbb{P} : \mathbf{V} \to \mathbf{V}) \wedge \forall x \forall y(x \in \mathbb{P}y \leftrightarrow x \subset y),$$

$(\mathbf{E})$ $$(f : \mathbf{V} \to \mathbf{B}) \to (\mathbf{E}(f) \in \mathbf{B} \wedge (\mathbf{E}(f) = \top \leftrightarrow \exists x(fx = \top))).$$

Finally, $\mathsf{OST}^r(\mathbf{E}, \mathbb{P})$ is obtained from $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ by restricting the schema of $\in$-induction for arbitrary $\mathcal{L}^\circ$ formulas to $\in$-induction for sets.

Theorem 18. *We have the following proof-theoretic equivalences:*

1. $\mathsf{OST} \equiv \mathsf{KP}$.
2. $\mathsf{OST}(\mathbb{P}) \equiv \mathsf{KP(P)}$.
3. $\mathsf{OST}^r(\mathbf{E}, \mathbb{P}) \equiv \mathsf{ZFC}$.
4. $\mathsf{OST}(\mathbf{E}, \mathbb{P}) \equiv \mathsf{NBG}^+$.

KP is Kripke-Platek set theory with infinity, KP(P) is Power Kripke-Platek set theory as in Rathjen [55], ZFC is Zermelo-Fraenkel set theory with the axiom of choice, and NBG^+ is von Neumann-Bernays-Gödel theory NBG for sets and classes extended by a suitable form of (Σ^1_1-AC) for classes and $\in$-induction for all formulas.

For proofs of the first equivalence see Feferman [18, 19] and Jäger [32], the second equivalence is due to Rathjen (see his [54, 55] and private communication); it should also be provable via an adaptation of the method in Sato and Zumbrunnen [57]. The third equivalence is proved in Jäeger [32], and the fourth follows from Jäger [33] together with Jäger and Krähenbühl [39].

5.2 Operational closure

With respect to ontological properties, it is a natural question to ask what it means for a set to be operationally closed. As it turns out, this has a very direct relationship to the concept of stability. More precisely, operationally closed sets behave like Σ_1 substructures of the universe. The detailed proof-theoretic analysis of the concept of operational closure is carried through in Jäger [35].

Definition 19.

1. *A set d is called* operationally closed*, in symbols $Opc[d]$, iff d is transitive, contains the constants of $\mathcal{L}^\circ$ as elements, and satisfies*

$$(\forall x, f \in d)(fx\downarrow \;\rightarrow\; fx \in d).$$

2. *The* operational limit axiom *states that every set is an element of an operational closed set,*

$$(\mathsf{OLim}) \qquad\qquad \forall x \exists y(x \in y \,\wedge\, Opc[y]).$$

An immediate consequence of this definition is that all closed terms of $\mathcal{L}^\circ$ that have a value are contained in every operationally closed set. Also, if we have a λ-term of $\mathcal{L}^\circ$ whose free variables belong to an operationally closed d, then this term belongs to d as well. The strength of the concept of operational closure and its connection to Σ_1 substructures becomes evident by the following observation.

Theorem 20. *For any Δ_0 formula $A[\vec{u}, v]$ of the language $\mathcal{L}$ with at most the variables $\vec{u}, v$ free, the theory* OST *proves that*

$$Opc[d] \wedge \vec{a} \in d \wedge \exists x A[\vec{a}, x] \rightarrow (\exists x \in d) A[\vec{a}, x].$$

Recall that a transitive set d with $\omega \in d$ is called a Σ_1-*elementary substructure of the transitive class* $\mathbf{M}$ iff $d \in \mathbf{M}$ and for all Σ_1 formulas $A[\vec{u}]$ with parameters $\vec{u}$ and all $\vec{a} \in d$,

$$d \models A[\vec{a}] \iff \mathbf{M} \models A[\vec{a}].$$

Hence the preceding theorem says that any operationally closed set is an Σ_1-elementary substructure of the universe $\mathbf{V}$. Also it implies that all instances of

$$(\Sigma_1\text{-}\mathsf{Sep}) \qquad \forall x \exists y \forall z (z \in y \rightarrow z \in x \wedge A[z]),$$

where $A[u]$ is a Σ_1 formula of $\mathcal{L}$, are provable in $\mathsf{OST} + (\mathsf{OLim})$.

Theorem 21. $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep})$ *is contained in* $\mathsf{OST} + (\mathsf{OLim})$.

On the other hand, an ordinal α is called *stable* (in symbols $Stab[\alpha]$) iff $\mathbf{L}_\alpha$ is a Σ_1-elementary substructure of the constructible universe $\mathbf{L}$. Then $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L}) + (\Sigma_1\text{-}\mathsf{Sep})$ proves that every ordinal α is majorized by a stable ordinal,

$$\mathsf{KP} + (\mathbf{V}{=}\mathbf{L}) + (\Sigma_1\text{-}\mathsf{Sep}) \vdash \forall \alpha \exists \beta (a < \beta \wedge Stab[\beta]).$$

Since by means of the inductive model construction presented in Jäger and Zumbrunnen [45] the theory $\mathsf{OST} + (\mathsf{OLim})$ can be reduced to $\mathsf{KP} + (\mathbf{V}{=}\mathbf{L}) + \forall \alpha \exists \beta (a < \beta \wedge Stab[\beta])$, and adding $(\mathbf{V}{=}\mathbf{L})$ to $\mathsf{KP}{+}(\Sigma_1\text{-}\mathsf{Sep})$ does not increase its proof-theoretic strength, we obtain the following characterization.

Theorem 22. *We have the following proof-theoretic equivalences:*

$$\mathsf{OST} + (\mathsf{OLim}) \equiv \mathsf{KP} + (\mathbf{V}{=}\mathbf{L}) + \forall \alpha \exists \beta (\alpha < \beta \wedge Stab[\beta]) \equiv \mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep}).$$

In Jäger [35] it is also shown that $\mathsf{OST} + \exists x\, Opc[x]$ is equiconsistent to KP plus parameter-free Σ_1 separation on ω.

So we notice that the concept of operational closure is proof-theoretically very powerful, lifting OST to a new dimension. However, from a operational perspective, this notion is somewhat problematic: The uniform version of (OLim), with a new constant OC,

$$\forall x (x \in \mathsf{OC}(x) \wedge Opc[\mathsf{OC}(x)]),$$

is easily seen to lead to inconsistency.

5.3 Relativizing operational set theory

A further motivation for operational set theory, formulated in Feferman [18, 19], was to use his general applicative framework for explaining the admissible analogues of various large cardinal notions. Everything works out fine as along as only one (classically or recursively) regular universe is concerned. However, in view of Jäger and Zumbrunnen [45] this aim of OST had to be analyzed further. It is shown in [45] that a direct relativization of operational reflection leads to theories that are significantly stronger than theories formalizing the admissible analogues of classical large cardinal axioms. This refutes the conjecture 14(1) on p. 977 of Feferman [19].

The main reason is that simply restricting quantifiers to specific sets and operations to operations from and to those sets does not affect the global application relation and thus substantial strength may be imported – so to say – through the back door. Hence relativizing operational set theory requires a more cautious approach.

In a nutshell: The applicative structure must also be relativized when explaining the notion of relativized regularity in the context of OST. In contrast to the usual way of relativizing formulas with respect to a given set d, we now relativize our formulas A with respect to a set d and a set $e \subseteq d^3$ to formulas $A^{(d,e)}$; then d is the new universe and e takes care of application in the sense described below. This way of relativizing operational set theory is worked out in all details in Jäger [37].

First we add to $\mathcal{L}^\circ$ a fresh binary relation symbol Reg to express relativized regularity and a fresh constant **reg** for the operational representation of Reg in the sense of the following axiom that has to be added to the logical operations axioms,

(L.6) $(\mathbf{reg} : \mathbf{V}^2 \to \mathbf{B}) \wedge \forall x \forall y((\mathbf{reg}(x,y) = \top \leftrightarrow \mathsf{Reg}(x,y))$.

Then we turn to relativizing application: For all $\mathcal{L}^\circ$ terms r and variables e we define the formula $(r \,\partial\, e)$ by induction on the complexity of r as follows:

1. If r is a variable or a constant of $\mathcal{L}^\circ$, then $(r \,\partial\, e)$ is the formula $(r = r)$.
2. If r is the $\mathcal{L}^\circ$ term $r_1 r_2$, then choose some variable x not appearing in r_1, r_2 and different from e and let $(r \,\partial\, e)$ be the formula

$$(r_1 \,\partial\, e) \wedge (r_2 \,\partial\, e) \wedge \exists x(\langle r_1, r_2, x\rangle \in e).$$

Think of e as a ternary relation; then $(r \,\partial\, e)$ formalizes that the term r is defined if application within r is treated according to e. For us only such relations are interesting that are compatible with the real term application. To single those out, we set

$$Comp[e] \;:=\; \forall x \forall y \forall z(\langle x, y, z\rangle \in e \,\to\, xy = z).$$

Clearly, if $Comp[e]$ and $(r \mathrel{\partial} e)$, then $r\downarrow$. However, observe that in general we may have $Comp[e]$ and $r\downarrow$, but not $(r \mathrel{\partial} e)$; so it is possible that term r has a value without being defined in the sense of e.

In a next step this form of relativizing application via e is combined with restricting the universe of discourse to d. For all $\mathcal{L}^\circ$ formulas A we define the relativized formula $A^{(d,e)}$ by induction on the complexity of A as follows:

$$\begin{aligned}
(r = s)^{(d,e)} &:= (r \mathrel{\partial} e) \wedge (s \mathrel{\partial} e) \wedge r = s, \\
(r \in s)^{(d,e)} &:= (r \mathrel{\partial} e) \wedge (s \mathrel{\partial} e) \wedge r \in s, \\
(r\downarrow)^{(d,e)} &:= (r \mathrel{\partial} e) \wedge r \in d, \\
\mathsf{Reg}(r,s)^{(d,e)} &:= (r \mathrel{\partial} e) \wedge (s \mathrel{\partial} e) \wedge \mathsf{Reg}(r,s), \\
(\neg A)^{(d,e)} &:= \neg A^{(d,e)}, \\
(A \vee B)^{(d,e)} &:= (A^{(d,e)} \vee C^{(d,e)}), \\
((\exists x \in r)A)^{(d,e)} &:= (r \mathrel{\partial} e) \wedge (\exists x \in r)A^{(d,e)}, \\
(\exists x A)^{(d,e)} &:= (\exists x \in d)A^{(d,e)},
\end{aligned}$$

Now the relation Reg comes into play. $\mathsf{Reg}(d,e)$ is supposed to state that set d is regular with respect to e, and has the following intuitive interpretation: (i) d is a transitive set containing all constants of $\mathcal{L}^\circ$ as elements and e is a ternary relation on d compatible with the general application relation; (ii) if application is interpreted in the sense of e, then d satisfies the axioms of OST; (iii) we claim a linear ordering of those pairs $\langle d,e\rangle$ for which $\mathsf{Reg}(d,e)$ holds. To make this precise, we add to OST additional so-called Reg-axioms. Here $TranCon[d]$ is short for the $\mathcal{L}^\circ$ formula stating that d is transitive and contains all constants of $\mathcal{L}^\circ$.

Axioms for Reg.

(Reg.1) $\mathsf{Reg}(d,e) \;\rightarrow\; (TranCon[d] \wedge e \subseteq d^3 \wedge Comp[e])$.

(Reg.2) If A is an applicative axiom, logical operations axiom, or set-theoretic operations axiom with at most the variables $\vec{x}$ free such that neither the variables d, e do not appear in the list $\vec{x}$, then

$$\mathsf{Reg}(d,e) \;\rightarrow\; (\forall \vec{x} \in d)A^{(d,e)}.$$

(Reg.3) $\mathsf{Reg}(d_1,e_1) \wedge \mathsf{Reg}(d_2,e_2) \;\rightarrow\; d_1 \in d_2 \vee d_1 = d_2 \vee d_2 \in d_1$.

(Reg.4) $\mathsf{Reg}(d_1,e_1) \wedge \mathsf{Reg}(d_2,e_2) \wedge d_1 \in d_2 \;\rightarrow\; e_1 \in d_2 \wedge e_1 \subseteq e_2$.

In the following we write $\mathsf{OST(LR)}$ for the extension of OST by the axioms (Reg.1)-(Reg.4) and the limit axiom for (relativized) regular sets,

(Lim-Reg) $\forall x \exists y \exists z (x \in y \wedge \mathsf{Reg}(y,z)).$

One of the central results of Jäger [37] is that OST(LR) is proof-theoretically equivalent to the theory KPi of iterated admissible sets and thus describes an recursively inaccessible universe from an operational perspective.

Theorem 23. OST(LR) $\equiv$ KPi.

As can be seen from the proof of this equivalence, our notion of relativized regularity is the operational analogue of admissibility and thus provides a first essential step in capturing recursive analogues of large cardinal assertions. There is no intrinsic reason to stop at inaccessibility, and it seems that we can deal with, for example, Mahloness in an analogous way. The hope is that also the recursive versions of very strong forms of reflection can be handled in this way.

6 Future work

Explicit mathematics and operational set theory are couched in an operational framework and as such have a lot in common. However, there are also significant differences. The article Jäger and Zumbrunnen [46] tries to clarify this relationship more systematically, especially from an ontological perspective.

A basic and significant difference is that in explicit mathematics we deal with individuals and classes, whereas operational set theory is completely first order. Hence it is an interesting question whether there exist natural operational theories of sets and classes. Feferman's draft notes [21] present some first ideas and Jäger [36] discusses several technical and conceptual problems; it also presents a "technically working" system that, however, does not satisfy the criterion of naturalness.

In explicit mathematics we can take a given applicative structure and build the universe of classes above this structure without being forced to change the underlying applicative structure; no new individuals are created. In an operational theory of sets and classes the situation is different: Again we may start off from the applicative universe, which now models set-theoretic axioms. However, building classes above this universe may force us to generate new sets, in particular if we want the "Aussonderungsprinzip" to be satisfied: given a set x and a class Y, the intersection $x \cap Y$ is a set. Therefore, a sort of strong impredicativity makes the interplay between sets and classes very delicate.

In spite of such difficulties it is worthwhile to search for "good" operational theories of sets and classes, even if they can only cope with systems of very high consistency strength. If successful, this framework is likely to be very useful in studying strong reflection principles from an operational perspective.

The analysis of strong forms of reflection is also a topic in explicit mathematics. This together with the development of a convincing operational descriptive set theory are major tasks for the future.

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