A First-order Logic for Reasoning about Higher-order Upper and Lower Probabilities

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Abstract. We present a first-order probabilistic logic for reasoning about the uncertainty of events modeled by sets of probability measures. In our language, we have formulas that essentially say that according to agent Ag, for all x, formula $\alpha(x)$ holds with the lower probability at least $\frac{1}{3}$. Also, the language is powerful enough to allow reasoning about higher order upper and lower probabilities. We provide corresponding Kripke-style semantics, axiomatize the logic and prove that the axiomatization is sound and strongly complete (every satisfiable set of formulas is consistent).

Keywords: probabilistic logic, uncertainty, axiomatization, strong completeness

1 Introduction

Reasoning with uncertainty has gained an important role in computer science, artificial intelligence and cognitive science. These applications require the development of formal models which could capture reasoning through probability [3, 4, 6–9, 11, 13, 17, 19].

We investigate a probabilistic logic approach, considering the situation when there is also uncertainty about probabilities. In this case, the uncertainty is often described using the two boundaries, called *upper probability* and *lower probability* [14, 15]. Those probabilities are previously formalized in logics developed in [12, 20, 21]. Halpern and Pucella [12] give the following example: a bag contains 100 marbles, 30 of them are red and the remaining 70 are either blue or yellow, but we do not know their exact proportion. Thus, we can assign exact probability 0.3 to the event that a randomly picked ball from the bag is red, while for each possible probability p for picking a blue ball, we know that the remaining probability for yellow one is 0.7-p. For the set of possible probability measures obtained in that way, we can assign a pair of functions, the upper and lower probability measure, that assign the supremum and the infimum the probability of an event according to the probability measures in the set.

We use the papers [12, 20, 21] as a starting point and generalize them in two ways:

- We want to reason not only about lower and upper probabilities an agent assigns to a certain event, but also about her uncertain belief about other agent's imprecise probabilities. Thus, we introduce separate lower and upper probability operators for different agents, and we allow nesting of the operators, similarly as it has been done in [6], in the case of simple probabilities[4]. Suppose that an agent a is planning a visit to the city C based on the weather reports from several sources, and she decides to take an action if probability of rain is at most $\frac{1}{10}$, according to all reports she considers. Since she wishes to go together with b, she should be sure with probability at least $\frac{9}{10}$ that b (who might consult different weather reports) has the same conclusion about possibility of rain. In our language, it can be formalized as

$$U^a_{\leq \frac{1}{10}} Rain(C) \wedge L^a_{\geq \frac{9}{10}} (U^b_{\leq \frac{1}{10}} Rain(C)).$$

- We extend both [12, 20, 21] and [6] by allowing reasoning about events expressible in a first-order language. The papers [12, 20] deal with propositinal reasoning, while [21] introduces a logic whose syntax allows only Boolean combinations of formulas in which lower and upper probability operators are applied to first order sentences. On the other hand, here we use the most general approach, allowing arbitrary combination of probability operators and quantifiers, so we can express the statement like "according to the agent a, the lower probability of rain in all cities is at least $\frac{1}{3}$" ($L^a_{\geq \frac{1}{3}} \forall x Rain(x)$), but also "There exists a city in which it will surely not rain" ($(\exists x) U^a_{=0} Rain(x)$).

Formally, if the uncertainty about probabilities is modeled by a set of probability measures P defined on a given algebra H, then the lower probability measure $P_\star$ and the upper probability measure $P^\star$ are defined by $P_\star(X) = \inf\{\mu(X) \mid \mu \in P\}$ and $P^\star(X) = \sup\{\mu(X) \mid \mu \in P\}$, for every $X \in H$. Those two functions are related by the formula $P_\star(X) = 1 - P^\star(X^c)$.

In this paper, we logically formalize such situations using a generalization of Kripke models – for each agent, every world is equipped with a probabilistic space which consists of the accessible worlds, algebra of subsets, and a set of measures. We denote our logic by $\mathcal{L}_{lu}$.

We propose a sound and strongly complete axiomatization of the logic. Since we use different completion technique than the one used in [6, 12], we did not have to incorporate the arithmetical operations in the language. Instead, we use unary operators for upper and lower probability, following [20]. Since, like the other real-valued probabilistic logics, $\mathcal{L}_{lu}$ is not compact, any finitary axiomatic system would be incomplete [22]. In order to achieve completeness, we use two infinitary rules of inference, with countably many premises and one conclusion.

[4] For a discussion on higher-order probabilities we refer the reader to [10].

2 The logic $\mathcal{L}_{lu}$ – syntax and semantics

Let $S = \mathbb{Q} \cap [0,1]$, $Var = \{x, y, z, \ldots\}$ be a denumerable set of variables and let $\Sigma = \{a, b, \ldots\}$ be a finite, non-empty set of agents. The language of the logic $\mathcal{L}_{lu}$ consists of:

- the elements of set Var,
- classical propositional connectives $\neg$ and $\wedge$,
- universal quantifier $\forall$,
- for every integer $k \geq 0$, denumerably many function symbols $F_0^k, F_1^k, \ldots$ of arity k,
- for every integer $k \geq 0$, denumerably many relation symbols $P_0^k, P_1^k, \ldots$ of arity k,
- the list of upper probability operators $U^a_{\geq s}$, for every $s \in S$,
- the list of lower probability operators $L^a_{\geq s}$, for every $s \in S$,
- comma, parentheses.

Functions of arity 0 will be called constants.
Note that we use conjunction and negation as primitive connectives, while $\vee$, $\rightarrow$, $\leftrightarrow$ and $\exists$ are introduced in the usual way. The notions of a term, atomic formula, bound and free variables, sentence and a term free for a variable in formula, can be defined as usual.

Definition 1 (Formula) *The set $For_{\mathcal{L}_{lu}}$ of formulas is the smallest set containing atomic formulas and that is closed under following formation rules: if α, β are formulas, then $L^a_{\geq s}\alpha$, $U^a_{\geq s}\alpha$, $\neg\alpha$, $\alpha \wedge \beta$, $(\forall x)\alpha$ are formulas as well. The formulas from $For_{\mathcal{L}_{lu}}$ will be denoted by $\alpha, \beta, \ldots$*

We use the following abbreviations to introduce other types of inequalities:

- $U^a_{<s}\alpha$ is $\neg U^a_{\geq s}\alpha$, $U^a_{\leq s}\alpha$ is $L^a_{\geq 1-s}\neg\alpha$, $U^a_{=s}\alpha$ is $U^a_{\leq s}\alpha \wedge U^a_{\geq s}\alpha$, $U^a_{>s}\alpha$ is $\neg U^a_{\leq s}\alpha$,
- $L^a_{<s}\alpha$ is $\neg L^a_{\geq s}\alpha$, $L^a_{\leq s}\alpha$ is $U^a_{\geq 1-s}\neg\alpha$, $L^a_{=s}\alpha$ is $L^a_{\leq s}\alpha \wedge L^a_{\geq s}\alpha$, $L^a_{>s}\alpha$ is $\neg L^a_{\leq s}\alpha$.

We also denote $\alpha \vee \neg\alpha$ by $\top$, and $\alpha \wedge \neg\alpha$ by $\bot$.

The semantics for the logic $\mathcal{L}_{lu}$ is based on the possible-world approach.

Definition 2 ($\mathcal{L}_{lu}$-structure) *An $\mathcal{L}_{lu}$-structure is a tuple $\mathcal{M} = \langle W, D, I, LUP\rangle$, where:*

- *W is a nonempty set of worlds,*
- *D associates a non-empty domain $D(w)$ with every world $w \in W$,*
- *I associates an interpretation $I(w)$ with every world $w \in W$ such that:*
  - *$I(w)(F_i^k) : D(w)^k \rightarrow D(w)$, for all i and k,*
  - *$I(w)(P_i^k) \subseteq D(w)^k$, for all i and k,*
- *LUP assigns, to every $w \in W$ and every agent $a \in \Sigma$, a space, such that $LUP(a, w) = \langle W(a, w), H(a, w), P(a, w)\rangle$, where:*
  - *$\emptyset \neq W(a, w) \subseteq W$,*

- *$H(a,w)$ is an algebra of subsets of $W(a,w)$, i.e. a set of subsets of $W(a,w)$ such that:*
  - $W(a,w) \in H(a,w)$,
  - *if* $A, B \in H(a,w)$, *then* $W(a,w) \setminus A \in H(a,w)$ *and* $A \cup B \in H(a,w)$,
- *$P(a,w)$ is a set of finitely additive probability measures defined on $H(a,w)$, i.e. for every $\mu(a,w) \in P(a,w)$, $\mu(a,w) : H(a,w) \longrightarrow [0,1]$ and the following conditions hold:*
  - $\mu(a,w)(W(a,w)) = 1$,
  - $\mu(a,w)(A \cup B) = \mu(a,w)(A) + \mu(a,w)(B)$, *whenever* $A \cap B = \emptyset$.

Definition 3 (Variable valuation) *Let $\mathcal{M} = \langle W, D, I, LUP \rangle$ be an $\mathcal{L}_{lu}$-structure. A variable valuation υ assigns to every variable some element of the corresponding domain to every world $w \in W$, i.e. $\upsilon(w)(x) \in D(w)$. For υ, $w \in W$ and $d \in D(w)$ we define $\upsilon_w[d/x]$ is a valuation same as υ except that $\upsilon_w[d/x](w)(x) = d$.*

Definition 4 *Let $\mathcal{M} = \langle W, D, I, LUP \rangle$ be an $\mathcal{L}_{lu}$-structure and t a term. The* value of a term t, *denoted by $I(w)(t)_\upsilon$ is defined as follows:*

- *if t is a variable x, then $I(w)(x)_\upsilon = \upsilon(w)(x)$, and*
- *if $t = F_i^m(t_1, \dots, t_m)$, then*

$$I(w)(t)_\upsilon = I(w)(F_i^m)(I(w)(t_1)_\upsilon, \dots, I(w)(t_m)_\upsilon).$$

Now we define satisfiability of the formulas from $For_{\mathcal{L}_{lu}}$ in the worlds of $\mathcal{L}_{lu}$-structures.

Definition 5 *The truth value of a formula α in a world $w \in W$ of a model $\mathcal{M} = \langle W, D, I, LUP \rangle$ for a given valuation υ, denoted by $I(w)(\alpha)_\upsilon$ is defined as follows:*

- *if $\alpha = P_i^m(t_1, \dots, t_m)$, then $I(w)(\alpha)_\upsilon = true$ if $\langle I(w)(t_1)_\upsilon, \dots, I(w)(t_m)_\upsilon \rangle \in I(w)(P_i^m)$, otherwise $I(w)(\alpha)_\upsilon = false$,*
- *if $\alpha = \neg\beta$, then $I(w)(\alpha)_\upsilon = true$ if $I(w)(\beta)_\upsilon = false$, otherwise $I(w)(\alpha)_\upsilon = false$,*
- *if $\alpha = \beta \wedge \gamma$, then $I(w)(\alpha)_\upsilon = true$ if $I(w)(\beta)_\upsilon = true$ and $I(w)(\gamma)_\upsilon = true$,*
- *if $\alpha = U^a_{\geq s}\beta$, then $I(w)(\alpha)_\upsilon = true$ if $P^\star(w,a)\{u \in W(w,a) \mid I(u)(\beta)_\upsilon = true\} \geq s$, otherwise $I(w)(\alpha)_\upsilon = false$,*
- *if $\alpha = L^a_{\geq s}\beta$, then $I(w)(\alpha)_\upsilon = true$ if $P_\star(w,a)\{u \in W(w,a) \mid I(u)(\beta)_\upsilon = true\} \geq s$, otherwise $I(w)(\alpha)_\upsilon = false$,*
- *if $\alpha = (\forall x)\beta$, then $I(w)(\alpha)_\upsilon = true$ if for every $d \in D(w)$, $I(w)(\beta)_{\upsilon_w[d/x]} = true$, otherwise $I(w)(\alpha)_\upsilon = false$.*

Recall that $P_\star(w,a)\{u \in W(w,a) \mid I(u)(\beta)_\upsilon = true\} = \inf\{\mu(w,a)(\{u \in W(w,a) \mid I(u)(\beta)_\upsilon = true\}) \mid \mu(w,a) \in P(w,a)\}$, and $P^\star(w,a)\{u \in W(w,a) \mid I(u)(\beta)_\upsilon = true\} = \sup\{\mu(w,a)(\{u \in W(w,a) \mid I(u)(\beta)_\upsilon = true\}) \mid \mu(w,a) \in P(w,a)\}$.

Definition 6 *A formula α holds in a world w from a model $\mathcal{M} = \langle W, D, I, LUP\rangle$, denoted by $\mathcal{M}, w \models \alpha$, if for every valuation v, $I(w)(\alpha)_v = true$. If $d \in D(w)$, we will use $\mathcal{M}, w \models \alpha(d)$ to denote that $I(w)(\alpha(x))_{v_w[d/x]} = true$, for every valuation v.*

A sentence α is satisfiable if there is a world w in an $\mathcal{L}_{lu}$-model $\mathcal{M}$ such that $\mathcal{M}, w \models \alpha$. A sentence α is valid if it is satisfied in every world in every $\mathcal{L}_{lu}$-model $\mathcal{M}$. A set of sentences T is satisfiable if there is a world w in an $\mathcal{L}_{lu}$-model $\mathcal{M}$ such that $\mathcal{M}, w \models \alpha$ for every $\alpha \in T$.

We will consider a class of $\mathcal{L}_{lu}$ models that satisfy:

- all the worlds from a model have the same domain, i.e., for all $v, w \in W$, $D(v) = D(w)$,
- for every sentence α, for every agent $a \in \Sigma$ and every world w from a model $\mathcal{M}$, the set $\{u \in W(w, a) \mid I(u)(\alpha)_v = true\}$ of all worlds from $W(w, a)$ that satisfy α is measurable,
- the terms are rigid, i.e., for every model their meanings are the same in all the worlds.

We will use the notation $[\alpha]^a_w$ for the set $\{u \in W(w, a) \mid I(u)(\alpha)_v = true\}$, and also $\mathcal{L}_{lu_{Meas}}$ to denote the class of all fixed domain measurable models with rigid terms.

The following example shows that Compactness theorem does not hold for the logic $\mathcal{L}_{lu}$, i.e. we can construct a set T such that every finite subset of a set T is satisfiable, but T itself is not.

Example 1 *Consider the set of formulas*

$$T = \{\neg U^a_{=0}\alpha\} \cup \{U^a_{<\frac{1}{n}}\alpha \mid n \quad is \quad a \quad positive \quad integer\}.$$

It is clear that every finite subset of T is $\mathcal{L}_{lu_{Meas}}$-satisfiable, but the set T is not.

3 The axiomatization $Ax_{\mathcal{L}_{lu}}$

In this section we introduce an axiomatic system for the logic $\mathcal{L}_{lu}$. That system will be denoted by $Ax_{\mathcal{L}_{lu}}$. In order to axiomatize upper and lower probabilities, we need to completely characterize them with a small number of properties. There are many complete characterizations in the literature, and the earliest appears to be by Lorentz [16]. We use the characterization result by Anger and Lembcke [1] . It uses the notion of (n, k)-cover.

Definition 7 ((n, k)-cover) *A set A is said to be covered n times by a multiset $\{\{A_1, \ldots, A_m\}\}$ of sets if every element of A appears in at least n sets from $A_1, \ldots, A_m$, i.e., for all $x \in A$, there exists $i_1, \ldots, i_n$ in $\{1, \ldots, m\}$ such that for all $j \leq n$, $x \in A_{i_j}$. An (n, k)-cover of (A, W) is a multiset $\{\{A_1, \ldots, A_m\}\}$ that covers W k times and covers A $n + k$ times.*

Theorem 1 ([1]) *Let W be a set, H an algebra of subsets of W, and f a function $f : H \longrightarrow [0,1]$. There exists a set P of probability measures such that $f = P^\star$ iff f satisfies the following three properties:*

(1) $f(\emptyset) = 0$,
(2) $f(W) = 1$,
(3) *for all natural numbers m, n, k and all subsets $A_1, \ldots, A_m$ in H, if the multiset $\{\{A_1, \ldots, A_m\}\}$ is an (n,k)-cover of (A, W), then $k + nf(A) \leq \sum_{i=1}^{m} f(A_i)$.*

This theorem is also used in the Halpern and Pucella's paper on the logical formalization of upper and lower probabilities [12].

Axiom schemes

(1) all instances of the classical propositional tautologies
(2) $(\forall x)(\alpha \to \beta) \to (\alpha \to (\forall x)\beta)$, where the variable x does not occur free in α
(3) $(\forall x)\alpha(x) \to \alpha(t)$, where $\alpha(t)$ is obtained by substitution of all free occurrences of x in the first-order formula $\alpha(x)$ by the term t which is free for x in $\alpha(x)$
(4) $U^a_{\leq 1}\alpha \wedge L^a_{\leq 1}\alpha$
(5) $U^a_{\leq r}\alpha \to U^a_{<s}\alpha$, $s > r$
(6) $U^a_{<s}\alpha \to U^a_{\leq s}\alpha$
(7) $(U^a_{\leq r_1}\alpha_1 \wedge \ldots \wedge U^a_{\leq r_m}\alpha_m) \to U^a_{\leq r}\alpha$, if $\alpha \to \bigvee_{J\subseteq\{1,\ldots,m\},|J|=k+n} \bigwedge_{j\in J} \alpha_j$ and $\bigvee_{J\subseteq\{1,\ldots,m\},|J|=k} \bigwedge_{j\in J} \alpha_j$ are tautologies, where $r = \frac{\sum_{i=1}^{m} r_i - k}{n}$, $n \neq 0$
(8) $\neg(U^a_{\leq r_1}\alpha_1 \wedge \ldots \wedge U^a_{\leq r_m}\alpha_m)$, if $\bigvee_{J\subseteq\{1,\ldots,m\},|J|=k} \bigwedge_{j\in J} \alpha_j$ is a tautology and $\sum_{i=1}^{m} r_i < k$
(9) $L^a_{=1}(\alpha \to \beta) \to (U^a_{\geq s}\alpha \to U^a_{\geq s}\beta)$

Inference Rules

(1) From α and $\alpha \to \beta$ infer β
(2) From α infer $(\forall x)\alpha$
(3) From α infer $L^a_{\geq 1}\alpha$
(4) From the set of premises

$$\{\alpha \to U^a_{\geq s-\frac{1}{k}}\beta \mid k \geq \frac{1}{s}\}$$

infer $\alpha \to U^a_{\geq s}\beta$
(5) From the set of premises

$$\{\alpha \to L^a_{\geq s-\frac{1}{k}}\beta \mid k \geq \frac{1}{s}\}$$

infer $\alpha \to L^a_{\geq s}\beta$.

The axioms 7 and 8 together capture the condition 3) from the Theorem 1. Indeed, note that $\{\{A_1, \dots, A_m\}\}$ covers a set A n times iff

$$A \subseteq \bigcup_{J\subseteq\{1,\dots,m\},|J|=n} \bigcap_{j\in J} A_j.$$

Hence, the condition that a formula $\alpha \to \bigvee_{J\subseteq\{1,\dots,m\},|J|=k+n} \bigwedge_{j\in J} \alpha_j$ is a tautology gives us that, for every $a \in \Sigma$ and $w \in W$, $[\alpha]^a_w$ is covered $n+k$ times by a multiset $\{\{[\alpha_1]^a_w, \dots, [\alpha_m]^a_w\}\}$, while the condition that $\bigvee_{J\subseteq\{1,\dots,m\},|J|=k} \bigwedge_{j\in J} \alpha_j$ is a tautology ensures that, for every $a \in \Sigma$ and $w \in W$, $W(w,a) = [\top]^a_w$ is covered k times by a multiset $\{\{[\alpha_1]^a_w, \dots, [\alpha_m]^a_w\}\}$.

Rule 4 and Rule 5 are infinitary rules of inference and intuitively says that if upper/lower probability is arbitrary close to a rational number s then it is at least s.

Definition 8 (Inference relation)

- $\vdash \alpha$ *(α is a* theorem*) iff there is an at most denumerable sequence of formulas $\alpha_1, \alpha_2, \dots, \alpha$, such that every α_i is an axiom or it is derived from the preceding formulas by an inference rule;*
- $T \vdash \alpha$ *(α is* derivable *from T) if there is an at most denumerable sequence of formulas $\alpha_1, \alpha_2, \dots, \alpha$, such that every α_i is an axiom or a formula from the set T, or it is derived from the preceding formulas by an inference rule, with the exception that Inference Rule 3 can be applied only to the theorems;*
- T *is* consistent *if there is at least one formula $\alpha \in For_{\mathcal{L}_{lu}}$ that is not deducible from T, otherwise T is inconsistent;*
- T *is* maximally consistent *set if it is consistent and for every $\alpha \in For_{\mathcal{L}_{lu}}$, either $\alpha \in T$ or $\neg\alpha \in T$;*
- T *is* deductively closed *if for every $\alpha \in For_{\mathcal{L}_{lu}}$, if $T \vdash \alpha$, then $\alpha \in T$;*
- T *is* saturated *if it is maximally consistent and satisfies:*
  if $\neg(\forall x)\alpha(x) \in T$, then for some term t, $\neg\alpha(t) \in T$.

Note that T is inconsistent iff $T \vdash \bot$. Also, it is easy to check that every maximally consistent set is deductively closed.

It is straightforward to prove that our axiomatic system is sound with respect to the class of $\mathcal{L}_{lu_{Meas}}$-models.

4 Completeness

Deduction theorem holds for $Ax_{\mathcal{L}_{lu}}$: if T is a set of formulas and α a sentence, then $T \cup \{\alpha\} \vdash \beta$ iff $T \vdash \alpha \to \beta$. This theorem can be proved using the facts that our infinitary inference rules have implicative form, and that the application of Rule 3 is restricted to theorems only.

Now, we show how to extend an arbitrary consistent set of formulas T to a saturated set of formulas $T^\star$. In the end the canonical model $\mathcal{M}_{Can}$ is constructed and after that, it is proved that for every world w and every formula

α, $\alpha \in w$ iff $w \models \alpha$, so the proof of the completeness theorem is an easy consequence.

Theorem 2 (Lindenbaum's theorem) *Every consistent set of formulas can be extended to a saturated set.*

Sketch of the proof. Consider a consistent set T and let $\alpha_0, \alpha_1, \dots$ be an enumeration of all formulas from $For_{\mathcal{L}_{lu}}$. A sequence of sets T_i, $i = 0, 1, 2, \dots$ is defined as follows:

(1) $T_0 = T$,
(2) for every $i \geq 0$,
  (a) if $T_i \cup \{\alpha_i\}$ is consistent, then $T_{i+1} = T_i \cup \{\alpha_i\}$, otherwise
  (b) if α_i is of the form $\beta \to U^a_{\geq s}\alpha$, then $T_{i+1} = T_i \cup \{\neg\alpha_i, \beta \to \neg U^a_{\geq s-\frac{1}{n}}\alpha\}$, for some positive integer n, so that T_{i+1} is consistent, otherwise
  (c) if α_i is of the form $\beta \to L^a_{\geq s}\alpha$, then $T_{i+1} = T_i \cup \{\neg\alpha_i, \beta \to \neg L^a_{\geq s-\frac{1}{n}}\alpha\}$, for some positive integer n, so that T_{i+1} is consistent, otherwise
  (d) if the set T_{i+1} is obtained by adding a formula of the form $\neg(\forall x)\beta(x)$ to the set T_i, then for some $c \in C$ (C is a countably infinite set of new constant symbols), $\neg\beta(c)$ is also added to T_{i+1}, so that T_{i+1} is consistent, otherwise
  (e) $T_{i+1} = T_i \cup \{\neg\alpha_i\}$.
(3) $T^\star = \bigcup_{i=0}^{\infty} T_i$.

Obviously, the set T_0 is consistent. Natural numbers (n), from the steps $2(b)$ and $2(c)$ of the construction exist (this is a direct consequence of the Deduction Theorem), and each T_i is consistent. The maximality of $T^\star$ (either $\alpha \in T$ or $\neg\alpha \in T$) is ensured by the steps (1) and (2) of the above construction. It is clear that $T^\star$ does not contain all the formulas because for a formula $\alpha \in For_{\mathcal{L}_{lu}}$, the set $T^\star$ does not contain both $\alpha = \alpha_i$ and $\neg\alpha = \alpha_j$, since the set $T_{\max\{i,j\}+1}$ is consistent.

It only remains to prove that $T^\star$ is deductively closed. Let $\alpha \in For_{\mathcal{L}_{lu}}$. We will prove by the induction on the length of the inference that if $T^\star \vdash \alpha$, then $\alpha \in T^\star$. Consider the infinitary Rule 5. Let $\alpha_i = \beta \to L^a_{\geq s}\gamma$ be obtained from the set of premises $\{\alpha_i^k = \beta \to L^a_{\geq s_k}\gamma \mid s_k \in S\}$. Using the induction hypothesis, we conclude that $\alpha_i^k \in T^\star$, for every k. If $\alpha_i \notin T^\star$, by step (2)(c) of the construction, there must be some l and j such that $\neg(\beta \to L^a_{\geq s}\gamma)$, $\beta \to \neg L^a_{\geq s-\frac{1}{l}}\gamma \in T_j$. Hence, we have that for some $j' \geq j$: $\beta \wedge \neg L^a_{\geq s}\gamma \in T_{j'}$; $\beta \in T_{j'}$; $\neg L^a_{\geq s-\frac{1}{l}}\gamma$, $L^a_{\geq s-\frac{1}{l}}\gamma \in T_{j'}$.

Therefore, we have that $T^\star$ is deductively closed set, and $T^\star$ does not contain all the formulas, so it is consistent.
The step $(2)(d)$ of the construction guarantees that $T^\star$ is saturated. □

Now we define a canonical model, using the saturated sets of formulas.

Definition 9 (Canonical model) *A canonical model $\mathcal{M}_{Can} = \langle W, D, I, LUP\rangle$ is a tuple such that:*

– *W is the set of all saturated sets of formulas,*
– *D is the set of all variable-free terms,*
– *for every $w \in W$, $I(w)$ is an interpretation such that:*
  - *for every function symbol F_i^m, $I(w)(F_i^m) : D^m \to D$ such that for all variable-free terms $t_1, \ldots, t_m$, $I(w)(F_i^m) : \langle t_1, \ldots, t_m \rangle \mapsto F_i^m(t_1, \ldots, t_m)$,*
  - *for every relation symbol P_i^m, $I(w)(P_i^m) = \{\langle t_1, \ldots, t_m \rangle \mid P_i^m(t_1, \ldots, t_m) \in w\}$, for all variable-free terms $t_1, \ldots, t_m$,*
– *for $a \in \Sigma$ and $w \in W$, $LUP(w,a) = \langle W(w,a), H(w,a), P(w,a) \rangle$ is defined:*
  - *$W(w,a) = W$,*
  - *$H(w,a) = \{\{u \mid u \in W(w,a), \alpha \in u\} \mid \alpha \in For_{\mathcal{L}_{lu}}\}$,*
  - *$P(w,a)$ is any set of probability measures such that $P^\star(w,a)(\{u \mid u \in W(w,a), \alpha \in u\}) = \sup\{s \mid U^a_{\geq s}\alpha \in w\}$.*

Lemma 1 *For every formula α and every $w \in W$, $\alpha \in w$ iff $w \models \alpha$.*

Theorem 3 (Strong completeness) *. Every consistent set of formulas T is $\mathcal{L}_{lu_{Meas}} - satisfiable$.*

Sketch of the proof. Let T be a consistent set of formulas and let $\mathcal{M}_{Can} = \langle W, D, I, LUP \rangle$ be a canonical model. It can be shown that $\mathcal{M}_{Can}$ is a well defined measurable structure. Furthermore, from Lemma 1we obtain that for every formula α, and every $w \in W$, $w \models \alpha$ iff $\alpha \in w$. Finally, using Theorem 2, we can extend T to a saturated set T^*, and since $T^* \in W$, we obtain $\mathcal{M}_{Can}, T^* \models T$. □

5 Conclusion

In this paper we present the proof-theoretical analysis of a logic which allows making statements about upper and lower probabilities of formulas according to some agent. We combine the approaches from [12, 20] and [6] and generalize them to an expressive modal language $\mathcal{L}_{lu}$ which extend first-order logic with the unary operators $U^a_{\geq r}$ and $L^a_{\geq r}$, where r ranges over the unit interval of rational numbers. The corresponding semantics $\mathcal{L}_{lu_{Meas}}$ consists of the measurable Kripke models with a set of finitely additive probability measures attached to each possible world. For a given world of a model, every probability form the corresponding set of probabilities is defined on the same algebra of a chosen sets of worlds. We prove that the proposed axiomatic system $Ax_{\mathcal{L}_{lu}}$ is strongly complete with respect to the class of $\mathcal{L}_{lu_{Meas}}$-models. Since the logic is not compact, the axiomatization contains infinitary rules of inference.

Finally, upper and lower probabilities are just one approach in development of imprecise probability models [2, 5, 18, 23, 24]. In the future work, we also wish to logically formalize different approaches to imprecise probabilities.

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