Relevant Justifications (ongoing work)

Nenad Savić[1] and Thomas Studer[1]

Institute of Computer Science, University of Bern, Switzerland,
{savic,tstuder}@inf.unibe.ch

Justification logic replaces the $\Box$-operator of modal logic by explicit justifications [2, 5]. That is justification logic features formulas of the form $t : A$ meaning A *is believed for reason* t; hence we can reason with and about explicit justifications for an agent's belief. The framework of justification logic has been used to formalize and study a variety of epistemic situations [3, 6–8, 10].

However, traditional justification logic is based on classical logic, which can lead to the following paradoxical situation. Consider a person A visiting a foreign town, which she doesn't know well. In order to get to a certain restaurant, she asks two persons B and C for the way. Person B says that A can take path P to the restaurant whereas person C replies that P does not lead to the restaurant and A should take another way. Person A now has a reason s to believe P and a reason t to believe $\neg P$. We can formalize this in justification logic by saying that both

$$s : P \qquad \text{and} \qquad t : \neg P \tag{1}$$

hold. However, then there exists a justification $r(s, t)$ such that

$$r(s, t) : (P \wedge \neg P)$$

holds. Now this implies (under certain natural assumptions) that for any formula F, there is a justification u such that

$$u : F \tag{2}$$

holds. That means for any formula F, person A has a reason to believe F, which, of course, is an undesirable consequence.

It is the aim of this paper to introduce a justification logic, RJ, in which situations of this kind cannot occur, in particular, that means a logic in which (2) does not follow from (1). We achieve this by combining the relevant logic R with the justification logic J4.

Relevant logics are non-classical logics that avoid the paradoxes of material and strict implication and provide a more intuitive deductive inference. The central systems of relevant logic, according to Anderson and Belnap [1], are the system of relevant implication R, as well as the logic of entailment E. The choice of axioms for the relevant logic R can be varied in different ways, e.g., see [9]. We decided to use the first 12 axioms from [14].

Meyer [12] proposed the logic NR, which is the relevant logic R equipped with an $S4$-style theory of necessity, in order to investigate whether the resulting theory coincides with the theory of entailment provided by Anderson and

Belnap [1]. Adapting the semantics for the logic R [13], Routley and Meyer provided a complete semantics for the logic NR [14].

Our logic RJ is similar to NR but instead of the $\Box$-operator, we use explicit justifications and since we deal with beliefs, we do not include the truth principle $t : A \to A$ in the list of axioms.

Terms are built from countable sets of constants and variables as follows:

$$t ::= c \mid x \mid t \cdot t \mid t \tilde{\wedge} t \mid t + t \mid !t,$$

where c is a constant and x is a variable. Tm denotes the set of terms. Formulas are build from a countable set Prop of atomic propositions as follows:

$$A ::= p \mid A \to A \mid A \wedge A \mid A \vee A \mid A \circ A \mid t : A,$$

where $p \in \mathsf{Prop}$ and $t \in \mathsf{Tm}$. The set of formulas is called For.

There are two groups of axioms for RJ. The first group are the axioms of the logic R:

(A1) $A \to A$
(A2) $A \to ((A \to B) \to B)$
(A3) $(A \to B) \to ((B \to C) \to (A \to C))$
(A4) $(A \to (A \to B)) \to (A \to B)$
(A5) $A \wedge B \to A$
(A6) $A \wedge B \to B$
(A7) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
(A8) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
(A9) $\neg\neg A \to A$
(A10) $(A \to \neg B) \to (B \to \neg A)$
(A11) $A \vee B \leftrightarrow \neg(\neg A \wedge \neg B)$
(A12) $A \circ B \leftrightarrow \neg(A \to \neg B)$

The second group consists of the axioms of $\mathsf{J4}$:

(A13) $t : (A \to B) \to (s : A \to (t \cdot s) : B)$
(A14) $t : A \to !t : t : A$
(A15) $t : A \wedge s : B \to (t \tilde{\wedge} s)(A \wedge B)$
(A16) $t : A \to (t + s) : A$ and $t : A \to (s + t) : A$

To introduce the rules of our logic, we need the following notion: a *constant specification* is a set $\mathsf{CS} \subseteq \{(c, A) \mid c \text{ is a constant and } A \text{ is an axiom of } \mathsf{RJ}\}$.

Given a constant specification CS, the deductive system $\mathsf{RJ}_{\mathsf{CS}}$ is given by the axioms of RJ and the rules

$$\frac{F \qquad F \to G}{G} \qquad\qquad \frac{F \qquad G}{F \wedge G} \qquad\qquad \frac{(c, A) \in \mathsf{CS}}{c : A}$$

where the last rule is called *axiom necessitation.*

A constant specification CS is called *axiomatically appropriate* if for each axiom A there is a constant c such that $(c, A) \in \mathsf{CS}$. As usual in justification logics, we can show the following analogue of the necessitation rule.

Lemma 1 (Constructive necessitation). *Let* CS *be an axiomatically appropriate constant specification. For each formula* A,

$$\mathsf{RJ_{CS}} \vdash A \quad \textit{implies} \quad \mathsf{RJ_{CS}} \vdash t : A \textit{ for some term } t.$$

The semantics for RJ is based on a combination of possible world models for R and basic modular models for $\mathsf{J4}$. An $\mathsf{RJ_{CS}}$-model is a tuple of the form $(K, 0, R, *, \spadesuit, \nu)$ where

1. K is a set;
2. $0 \in K$;
3. R is a ternary relation on K;
4. $*$ is a function $* : K \to K$;
5. $\spadesuit$ is a function $\spadesuit : \mathsf{Tm} \times K \to \mathcal{P}(\mathsf{For})$;
6. ν is a function $\nu : K \to \mathcal{P}(\mathsf{Prop})$.

We often write $t_a^\spadesuit$ for $\spadesuit(t, a)$. Moreover we set $a \le b := R0ab$ and $R^2abcd := \exists x(Rabx \wedge Rxcd)$. For sets of formulas X and Y, we write

$$\begin{aligned} X \cdot Y &:= \{F \mid G \to F \in X \text{ and } G \in Y, \text{ for some formula } G\} \\ X \wedge Y &:= \{F \mid F = G \wedge H, \text{ for some } G \in X \text{ and } H \in Y\} \\ t : X &:= \{t : F \mid F \in X\}. \end{aligned}$$

An $\mathsf{RJ_{CS}}$-model $(K, 0, R, *, \spadesuit, \nu)$ must satisfy the following conditions:

$Raaa$	$R^2abcd \Rightarrow R^2acbd$	$Rabc \Rightarrow t_a^\spadesuit \cdot s_b^\spadesuit \subseteq (t \cdot s)_c^\spadesuit$
$a \le a$	$a \le b \wedge Rbcd \Rightarrow Racd$	$a \le b \Rightarrow t_a^\spadesuit \subseteq t_b^\spadesuit$
$Rabc \Leftrightarrow Rac^*b^*$	$a^{**} = a$	$s_a^\spadesuit \cdot t_a^\spadesuit \subseteq (s \cdot t)_a^\spadesuit$
$s_a^\spadesuit \cup t_a^\spadesuit \subseteq (s + t)_a^\spadesuit$	$A \in t_0^\spadesuit$ if $(t, A) \in \mathsf{CS}$	$t : (t_a^\spadesuit) \subseteq (!t)_a^\spadesuit$
$s_a^\spadesuit \wedge t_a^\spadesuit \subseteq (s \tilde{\wedge} t)_a^\spadesuit$	$a \le b \Rightarrow \nu(a) \subseteq \nu(b)$	

The elements of K are essentially basic modular models of justification logic (with an additional constraint for $\tilde{\wedge}$), see [4, 11]. However, $\mathsf{RJ_{CS}}$-models do not feature the *justification yields belief* principle of modular models. As in models for NR, we could add a binary relation S on K to $\mathsf{RJ_{CS}}$-model and require that justification yields belief in the sense of S. This construction would yield modular models for $\mathsf{RJ_{CS}}$.

Given a model $\mathcal{M} = (K, 0, R, *, \spadesuit, \nu)$ and $a \in K$ we define:

$$\begin{aligned} \mathcal{M}, a &\models p && \text{iff} \quad p \in \nu(a), \text{for } p \in Prop \\ \mathcal{M}, a &\models A \wedge B && \text{iff} \quad \mathcal{M}, a \models A \text{ and } \mathcal{M}, a \models B \\ \mathcal{M}, a &\models A \vee B && \text{iff} \quad \mathcal{M}, a \models A \text{ or } \mathcal{M}, a \models B \\ \mathcal{M}, a &\models A \circ B && \text{iff} \quad Rxya \text{ and } \mathcal{M}, x \models A \text{ and } \mathcal{M}, y \models B, \text{for some } x, y \in K \\ \mathcal{M}, a &\models A \to B && \text{iff} \quad Raxy \text{ and } \mathcal{M}, x \models A \text{ imply } \mathcal{M}, y \models B, \text{for all } x, y \in K \\ \mathcal{M}, a &\models \neg A && \text{iff} \quad \mathcal{M}, a* \not\models A \\ \mathcal{M}, a &\models t : A && \text{iff} \quad A \in t_a^\spadesuit \end{aligned}$$

We say that a formula A is *true* at a in $\mathcal{M}$ if $\mathcal{M}, a \models A$. Formula A is *verified* in M, iff $\mathcal{M}, 0 \models A$. Finally, formula A is CS-*valid* iff A is verified in every $\mathsf{RJ}_{\mathsf{CS}}$-model.

Conjecture 1 (Soundness and Completeness). Let CS be any constant specification. For each formula A we have

$$\mathsf{RJ}_{\mathsf{CS}} \vdash A \quad \text{iff} \quad A \text{ is } \mathsf{CS}\text{-valid.}$$

As mentioned in the introduction, there is a close relationship between NR and our logic of relevant justifications. Let RLP be the system RJ plus the axiom $t : A \to A$ based on the total constant specification, i.e., every constant justifies every axiom (including $t : A \to A$). A *realization* is a mapping from modal formulas to formulas of justification logic that replaces each $\Box$ with some expression t: (different occurrences of $\Box$ may be replaced with different terms).

Conjecture 2 (Realization). There is a realization r such that for each modal formula A

$$\mathsf{NR} \vdash A \quad \text{implies} \quad \mathsf{RLP} \vdash r(A).$$

References

1. A. R. Anderson and N. D. Belnap. *Entailment: The Logic of Relevance and Neccessity, Vol. I.* Princeton University Press, 1975.
2. S. N. Artemov. Explicit provability and constructive semantics. *BSL*, 7(1):1–36, Mar. 2001.
3. S. N. Artemov. The logic of justification. *RSL*, 1(4):477–513, Dec. 2008.
4. S. N. Artemov. The ontology of justifications in the logical setting. *Studia Logica*, 100(1–2):17–30, Apr. 2012. Published online February 2012.
5. S. N. Artemov and M. Fitting. Justification logic. In E. N. Zalta, editor, *The Stanford Encyclopedia of Philosophy.* Fall 2012 edition, 2012.
6. S. N. Artemov and R. Kuznets. Logical omniscience as infeasibility. *APAL*, 165(1):6–25, 2014.
7. S. Bucheli, R. Kuznets, and T. Studer. Justifications for common knowledge. *Applied Non-Classical Logics*, 21(1):35–60, Jan.–Mar. 2011.
8. S. Bucheli, R. Kuznets, and T. Studer. Realizing public announcements by justifications. *Journal of Computer and System Sciences*, 80(6):1046–1066, 2014.
9. M. Dunn and G. Restall. Relevance logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic.* Kluwer Academic Publishers, 2002.
10. I. Kokkinis, P. Maksimović, Z. Ognjanović, and T. Studer. First steps towards probabilistic justification logic. *Logic Journal of IGPL*, 23(4):662–687, 2015.
11. R. Kuznets and T. Studer. Justifications, ontology, and conservativity. In T. Bolander, T. Braüner, S. Ghilardi, and L. Moss, editors, *Advances in Modal Logic, Volume 9*, pages 437–458. College Publications, 2012.
12. R. K. Meyer. Entailment and relevant implication. *Logique et Analyse*, 11(44):472–479, 1968.
13. R. Routley and R. Meyer. The semantics of entailment. *Studies in Logic and the Foundations of Mathematics*, 68:199 – 243, 1973. Truth, Syntax and Modality.
14. R. Routley and R. K. Meyer. The semantics of entailment: II. *Journal of Philosophical Logic*, 1(1):53–73, 1972.