The strength of the SCT criterion

Emanuele Frittaion, Silvia Steila, and Keita Yokoyama

[1] Mathematical Institute, Tohoku University, Japan
emanuelefrittaion@gmail.com,
[2] Institute of Computer Science, University of Bern, Switzerland
steila@inf.unibe.ch,
[3] School of Information Science, Japan Advanced Institute of Science and Technology, Japan y-keita@jaist.ac.jp

Abstract. We undertake the study of size-change analysis in the context of Reverse Mathematics. In particular, we prove that the SCT criterion [9, Theorem 4] is equivalent to $\mathsf{I}\Sigma^0_2$ over RCA_0.

Keywords: Ramsey's theorem for pairs, Size-change termination, Reverse Mathematics, Σ^0_2-induction.

1 Introduction

Ramsey's theorem for pairs (RT^2) is one of the main characters in Reverse Mathematics. It states that for any natural number k and for any edge coloring of the complete graph with countably many nodes in k-many colors, there exists an infinite homogeneous set, i.e. there exists an infinite subset of nodes whose any two elements are connected with the same color [12].

As highlighted by Gasarch [4], Ramsey's theorem for pairs can be used to prove termination. For instance, Podelski and Rybalchenko characterized the termination of transition based programs as a property of well-founded relations by using Ramsey's theorem for pairs [11]. In [15] we started investigating the termination analysis from the point of view of Reverse Mathematics. We proved the equivalence between the termination theorem of Podelski and Rybalchenko and a corollary of Ramsey's theorem for pairs, which is weaker than Ramsey's theorem for pairs itself.

The termination theorem is not the only result which characterizes the termination of some class of programs. In [9] Lee, Jones and Ben-Amram introduced the notion of size-change termination (SCT) for first order functional programs. Size-change analysis is a general method for *automated termination proofs.* In fact, this method has been applied in the termination analysis of higher-order programs [8], logic programs [2], and term rewrite systems [16].

Informally, a program is size-change terminating (SCT) if every infinite state transition sequence would cause an infinite sequence of data values which is weakly decreasing and strictly decreasing infinitely many times. If the domain of data values is well-founded, such as the natural numbers, there cannot be such a sequence, thus SCT is a sufficient condition for termination [9, Theorem 1].

Size-change termination is based on the notion of size-change graph (see Subsection 2.2). Given a first order functional program P we associate to every call $f \to g$ a bipartite graph which describes the relation between source and target parameter values. These graphs are called size-change graphs.

In this paper we start the investigation of size-change termination in the framework of Reverse Mathematics. In particular, we analyse the following criterion for testing SCT [9, Theorem 4]:

Theorem 1 (SCT criterion). *Let $\mathcal{G}$ be a set of size-change graphs for a first order functional program P. Then $\mathcal{G}$ is SCT iff every idempotent $G \in \mathsf{cl}(\mathcal{G})$ has an arc $x \xrightarrow{\downarrow} x$.*

The original proof of the SCT criterion is based on Ramsey's theorem for pairs. In this paper we show that this is far from optimal and pinpoint the exact strength of the SCT criterion from the point of view of Reverse Mathematics. For our analysis we consider the following version, where we consider size-change graphs only.

Theorem 2 (SCT criterion for graphs). *Let $\mathcal{G}$ be a set of size-change graphs. Then $\mathcal{G}$ is SCT iff every idempotent $G \in \mathsf{cl}(\mathcal{G})$ has an arc $x \xrightarrow{\downarrow} x$.*

To the aim of studying the strength of the SCT criterion we introduce and study a corollary of Ramsey's theorem for pairs, called Triangle Ramsey's theorem (Triang). It states that for any natural number k and for any edge coloring of the complete graph with countably many nodes in k-many colors, there is some node which is, for some color $i \in k$, the first node of infinitely many triangles homogeneous in color i. As far as we know this corollary does not appear in the literature.

We show that Triang implies the SCT criterion and that the SCT criterion implies the Strong Pigeonhole Principle (SPP). From these (and some further) results we are able to conclude that both SCT criterion and Triang are equivalent to Σ^0_2-induction ($\mathsf{I}\Sigma^0_2$).

Theorem 3 (RCA_0). *The following are equivalent:*

1. $\mathsf{I}\Sigma^0_2$
2. Triang
3. *SCT criterion*

1.1 Notation

Given a set $X \subseteq \mathbb{N}$, let $[X]^2$ denote the set of 2-element subsets of X. As usual, we identify $[X]^2$ with the set $\{(x, y) \colon x, y \in X \wedge x < y\}$. We also identify a natural number k with the set $\{0, \ldots, k-1\}$. For $k \in \mathbb{N}$, we call a function $c : [\mathbb{N}]^2 \to k$ a *coloring* of $[\mathbb{N}]^2$ in k-many colors.

For a set $X \subseteq \mathbb{N}$, $X^{<\mathbb{N}}$ denotes the set of finite sequences of elements in X. Given a set X and a sequence $\sigma \in X^{<\mathbb{N}}$ we denote by $|\sigma|$ the length of the sequence, by $\mathrm{last}(\sigma)$ the last element of the sequence and by $\sigma(i)$ the i-th element of the sequence, if it exists. Note that $k^{<\mathbb{N}}$ is the set of finite sequences of natural numbers less than k.

1.2 Reverse Mathematics

Reverse Mathematics is a program in mathematical logic introduced by Harvey Friedman in [3], which stems from the following question. Given a theorem of ordinary mathematics, what is the weakest subsystem of second order arithmetic in which it is provable?

Amongst the several subsystems of second order arithmetic (see [13] for a detailed description), in this paper we consider only few extensions of RCA_0 (Recursive Comprehension Axiom). RCA_0 is the standard base system of Reverse Mathematics. It consists of the usual axioms of first order arithmetic for $0, 1, +, \times, <$, induction for Σ^0_1-formulas ($\mathsf{I}\Sigma^0_1$) and comprehension for Δ^0_1-formulas.

The infinite pigeonhole principle (RT^1) and Ramsey's theorem for pairs (RT^2) are defined as follows.

(RT^1_k) For any $c\colon \mathbb{N} \to k$ there exists $i < k$ such that $c(x) = i$ for infinitely many x.

(RT^1) $\forall k \in \mathbb{N}\ \mathrm{RT}^1_k$.

(RT^2_k) For any $c : [\mathbb{N}]^2 \to k$ there exists an infinite homogeneous set $X \subseteq \mathbb{N}$, that is $c \restriction [X]^2$ is constant.

(RT^2) $\forall k \in \mathbb{N}\ \mathrm{RT}^2_k$.

Let $\mathsf{I}\Sigma^0_2$ be induction for Σ^0_2-formulas. It is known that RT^2 implies the bounding principle for Σ^0_3-formulas ($\mathsf{B}\Sigma^0_3$) over RCA_0 [7], and so in particular $\mathsf{I}\Sigma^0_2$. As a side result here we provide a different proof of the fact that RT^2 implies $\mathsf{I}\Sigma^0_2$. Indeed we introduce an immediate consequence of RT^2, the Triangle Ramsey's theorem (Triang), which turns out to be equivalent to $\mathsf{I}\Sigma^0_2$.

(Triang_k) For any coloring $c : [\mathbb{N}]^2 \to k$ there exist $i \in k$ and $t \in \mathbb{N}$ such that $c(t, m) = c(t, l) = c(m, l) = i$ for infinitely many pairs $m < l$.

(Triang) $\forall k \in \mathbb{N}\ \mathrm{Triang}_k$

2 The SCT framework

In this section we describe the size-change method for first order functional programs as in [9]. All the definitions are made in RCA_0 except for the semantic notion of *safety*.

2.1 Syntax

We consider the following basic first order functional language:

$$
\begin{array}{rl}
x \in \mathrm{Par} & \text{parameter identifier} \\
f \in \mathrm{Fun} & \text{function identifier} \\
o \in \mathrm{Op} & \text{primitive operator} \\
a \in \mathrm{AExp} & \text{arithmetic expression} \\
 & ::= \ x \mid x+1 \mid x-1 \mid o(a,\dots,a) \mid f(a,\dots,a) \\
b \in \mathrm{BExp} & \text{boolean expression} \\
 & ::= \ x=0 \mid x=1 \mid x<y \mid x \leq y \mid b \wedge b \mid b \vee b \mid \neg b \\
e \in \mathrm{Exp} & \text{expression} \\
 & ::= \ a \mid \ \textbf{if} \ \ b \ \ \textbf{then} \ \ e \ \ \textbf{else} \ \ e \\
d \in \mathrm{Def} & \text{function definition} \\
 & ::= \ f(x_0,\dots,x_{n-1}) = e \\
P \in \mathrm{Prog} & \text{program} \\
 & ::= \ d_0,\dots,d_{m-1}
\end{array}
$$

Remark 1. This language is Turing complete.

A program P is a list of finitely many defining equations $f(x_0,\dots,x_{n-1}) = e^f$, where $f \in \mathrm{Fun}$ and e^f is an expression, called the *body* of f. Let $x_0,\dots,x_{n-1}$ be the *parameters* of f, denoted $\mathrm{Par}(f)$, and let n be the *arity* of f, denoted $\mathrm{ar}(f)$.

By $\mathrm{Fun}(P)$ we denote the set of functions of P. We also assume that a program P specifies an *initial* function $f \in \mathrm{Fun}(P)$. The idea is that P computes the (partial) function $f : \mathbb{N}^{\mathrm{ar}(f)} \to \mathbb{N}$.

In [9] the expression evaluation is based on a *left-to-right call-by-value* strategy given by *denotational semantics*. RCA_0 is not capable to formalize denotational semantics, and hence we need to consider other approaches if we want to study termination over RCA_0 (for instance, by *operational semantics*). Anyway we do not formally discuss semantics. For the sake of exposition, it is enough to say that one evaluates a program function f given an assignment of values $\mathbf{u}$ to its parameters (i.e. an element of $\mathbb{N}^{\mathrm{ar}(f)}$) by evaluating the body of f, that is $f(\mathbf{u}) = e^f(\mathbf{u})$.

Example 1 (Péter-Ackermann).

$$
\begin{array}{rl}
A(x,y) = & \textbf{if} \ \ x=0 \ \ \textbf{then} \ \ y+1 \ \ \textbf{else} \\
 & \textbf{if} \ \ y=0 \ \ \textbf{then} \ \ A(x-1,1) \\
 & \textbf{else} \ \ A(x-1,A(x,y-1))
\end{array}
$$

2.2 Size-change graphs

In order to express the notion of size-change termination, first of all we need the definition of size-change graph (see [9, Definition 3]).

Definition 1 (size-change graph). *Let P be a program and $f,\ g \in \mathrm{Fun}(P)$. A* size-change graph *$G : f \to g$ for P is a bipartite graph on $(\mathrm{Par}(f), \mathrm{Par}(g))$. The set of edges is a subset of $\mathrm{Par}(f) \times \{\downarrow, \Downarrow\} \times \mathrm{Par}(g)$ such that there is at most one edge for any $x \in \mathrm{Par}(f)$ and $y \in \mathrm{Par}(g)$.*

We say that f is the *source* function of G and g is the *target* function of G. We call $(x, \downarrow, y)$ the *decreasing edge* (strict arc), and we denote it by $x \xrightarrow{\downarrow} y$. We call $(x, \Downarrow, y)$ the *weakly-decreasing edge* (non-strict arc), and we denote it by $x \xrightarrow{\Downarrow} y$. We write $x \to y \in G$ as a shorthand for $x \xrightarrow{\downarrow} y \in G \lor x \xrightarrow{\Downarrow} y \in G$.

Note that the absence of edges between two parameters x and y in the size-change graph G indicates either an unknown or an increasing relation in the call $f \to g$.

Informally, a size-change graph is an approximation of the *state transition* relation induced by the program. A size-change graph $G : f \to g$ for a call $\tau : f \to g$ is *safe* if it reflects the relationship between the parameter values in the program call.

In more detail, a *state* of a program P is a pair $(f, \mathbf{u})$, where $f \in \mathrm{Fun}(P)$ and $\mathbf{u}$ is a tuple of length $\mathrm{ar}(f)$. If in the body of $f \in \mathrm{Fun}(P)$ there is a call

$$\ldots \tau : g(e_0, \ldots, e_{m-1})$$

we define a *state transition* $(f, \mathbf{u}) \xrightarrow{\tau} (g, \mathbf{v})$ to be a pair of states such that $\mathbf{v}$ is the sequence of values obtained by the expressions $(e_0, \ldots, e_{m-1})$ when f is evaluated with values $\mathbf{u}$.

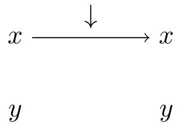
Let $\mathrm{Par}(f) = \{x_0, \ldots, x_{n-1}\}$ and $\mathrm{Par}(g) = \{y_0, \ldots y_{m-1}\}$. We say that a size-change graph $G : f \to g$ is *safe* for τ if every edge is safe, where an edge $x_i \xrightarrow{r} y_j$ is safe if for any $\mathbf{u} \in \mathbb{N}^n$ and $\mathbf{v} \in \mathbb{N}^m$ such that $(f, \mathbf{u}) \xrightarrow{\tau} (g, \mathbf{v})$, $r = \downarrow$ implies that $\mathbf{u}_i > \mathbf{v}_j$ and $r = \Downarrow$ implies that $\mathbf{u}_i \geq \mathbf{v}_j$.

Note for instance that the size-change graph without edges is always safe.

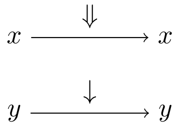
Example 2 (Péter-Ackermann).

$$\begin{aligned} A(x,y) = \ & \textbf{if} \ \ x = 0 \ \ \textbf{then} \ \ y + 1 \ \ \textbf{else} \\ & \textbf{if} \ \ y = 0 \ \ \textbf{then} \ \ \tau_0 : A(x-1, 1) \\ & \textbf{else} \ \ \tau_1 : A(x-1, \tau_2 : A(x, y-1)) \end{aligned}$$

There are three calls τ_i $(i < 3)$ safely described by the following size-change graphs:



$G_{0,1} : A \to A$



$G_2 : A \to A$

The size-change graph $G_{0,1}$ safely describes both calls $\tau_0 : A(x-1,1)$ and $\tau_1 : A(x-1, A(x, y-1))$. In particular, notice that in the call τ_1 the parameter value x decreases no matter what the value of the expression $A(x, y-1)$ is. Finally, the size-change graph G_2 safely describes the call $\tau_2 : A(x, y-1)$.

Note that we could have assumed that for any parameter in the target there is at most one edge, since in every call of the programs we consider any parameter value in the target depends at most from one parameter in the source. However this restriction is not essential. Note also that the SCT framework has been generalized in order to deal with other kinds of *monotonicity constraints* [1], where SCT only deals with two constraints $x > y$ (a strict arc) and $x \geq y$ (a non-strict arc).

Nonetheless we want to emphasize that the notion of size-change graph is clearly independent of that of a program and so we can define it directly. For simplicity we may assume that every function $f \in \mathrm{Fun}$ comes with a set of parameters $\mathrm{Par}(f)$ of size $\mathrm{ar}(f)$.

Definition 2 (size-change graph). *Let* $f, g \in \mathrm{Fun}$. *A* size-change graph $G : f \to g$ *is a bipartite graph on* $(\mathrm{Par}(f), \mathrm{Par}(g))$. *The set of edges is a subset of* $\mathrm{Par}(f) \times \{\downarrow, \Downarrow\} \times \mathrm{Par}(g)$ *such there is at most one edge for any* $x \in \mathrm{Par}(f)$ *and* $y \in \mathrm{Par}(g)$.

2.3 SCT criterion

Definition 3 (composition). *As in [6], given two size-change graphs* $G_0 : f \to g$ *and* $G_1 : g \to h$ *we define their* composition $G_0; G_1 : f \to h$. *The composition of two edges* $x \xrightarrow{\Downarrow} y$ *and* $y \xrightarrow{\Downarrow} z$ *is one edge* $x \xrightarrow{\Downarrow} z$. *In all other cases the composition of two edges from* x *to* y *and from* y *to* z *is the edge* $x \xrightarrow{\downarrow} z$. *Formally,* $G_0; G_1$ *is the size-change graph with the following set of edges:*

$$\begin{aligned}
E = \{x \xrightarrow{\downarrow} z : \exists y \in \mathrm{Par}(g)\ \exists r \in \{\downarrow, \Downarrow\}\ ((x \xrightarrow{\downarrow} y \in G_0 \wedge y \xrightarrow{r} z \in G_1) \\
\vee (x \xrightarrow{r} y \in G_0 \wedge y \xrightarrow{\downarrow} z \in G_1))\} \\
\cup \{x \xrightarrow{\Downarrow} z : \exists y \in \mathrm{Par}(g) (x \xrightarrow{\Downarrow} y \in G_0 \wedge y \xrightarrow{\Downarrow} z \in G_1) \wedge \forall y \in \mathrm{Par}(g) \\
\forall r, r' \in \{\downarrow, \Downarrow\}\ ((x \xrightarrow{r} y \in G_0 \wedge y \xrightarrow{r'} z \in G_1) \implies r = r' = \Downarrow)\}.
\end{aligned}$$

Observe that the composition operator ";" is associative. Moreover we say that the size-change graph G is *idempotent* if $G; G = G$.

Given a finite set of size-change graphs $\mathcal{G}$, $\mathsf{cl}(\mathcal{G})$ is the smallest set which contains $\mathcal{G}$ and is closed by composition. Formally $\mathsf{cl}(\mathcal{G})$ is the smallest set such that

- $\mathcal{G} \subseteq \mathsf{cl}(\mathcal{G})$;
- If $G_0 : f \to g$ and $G_1 : g \to h$ are in $\mathsf{cl}(\mathcal{G})$, then $G_0; G_1 \in \mathsf{cl}(\mathcal{G})$.

Definition 4 (multipath). *A* multipath $\mathcal{M}$ *is a sequence* $G_0, \ldots, G_n, \ldots$ *of graphs such that the target function of* G_i *is the source function of* G_{i+1} *for all* i*. A* thread *is a connected path of edges in* $\mathcal{M}$ *that starts at some* G_t*, where* $t \in \mathbb{N}$*. A multipath* $\mathcal{M}$ *has* infinite descent *if some thread in* $\mathcal{M}$ *contains infinitely many decreasing edges.*

Definition 5 (description). *A* description $\mathcal{G}$ *of* P *is a finite set of size-change graphs such that to every call* $\tau : f \to g$ *of* P *corresponds exactly one* $G_\tau \in \mathcal{G}$*.*

A description $\mathcal{G}$ of P is *safe* if each graph in $\mathcal{G}$ is safe. Note that there are finitely many descriptions, and in particular finitely many safe descriptions.

Definition 6 (SCT description). *We say that a description* $\mathcal{G}$ *of* P *is* size-change terminating *(SCT) if every infinite multipath* $\mathcal{M} = G_0, \ldots, G_n, \ldots$*, where every graph* $G_n \in \mathcal{G}$*, has an infinite descent.*

It is clear that a program P with a safe SCT description does not have infinite state transition sequences. Thus the existence of a safe SCT description is a sufficient condition for termination.

We now can state the SCT criterion.

Theorem 4 (SCT criterion). *Let* $\mathcal{G}$ *be a description of* P*. Then* $\mathcal{G}$ *is SCT iff every idempotent* $G \in \mathsf{cl}(\mathcal{G})$ *has an arc* $x \xrightarrow{\downarrow} x$*.*

To the aim of analysing in Reverse Mathematics it is convenient to state the SCT criterion for arbitrary sets of size-change graphs.

Definition 7 (SCT criterion for graphs). *Let* $\mathcal{G}$ *be a finite set of size-change graphs. Then* $\mathcal{G}$ *is SCT iff every idempotent* $G \in \mathsf{cl}(\mathcal{G})$ *has an arc* $x \xrightarrow{\downarrow} x$*.*

It is not difficult to see that the two formulations of the SCT criterion are equivalent. In fact, given a finite set $\mathcal{G}$ of size-change graphs, it is straightforward to define a program P such that $\mathcal{G}$ is a description of P. In more detail, let $f_0, \ldots, f_m$ be the finite set of source and target functions of $\mathcal{G}$. Without loss of generality, we may assume that all functions have the same arity $n \in \mathbb{N}$. For any i, let $f_{i_0}, \ldots, f_{i_{k-1}}$ be the functions (with repetition if there are more graphs with the same source and target functions) which correspond to the target of a graph whose source is f_i. Write the code:

$$\begin{aligned} f_i(x_0, \ldots x_{n-1}) &= \tau_0 : f_{i_0}(e_0^0, \ldots, e_{n-1}^0) && \text{if } x_0 = 0. \\ &= \ldots \\ &= \tau_{k-1} : f_{i_{k-1}}(e_0^{k-1}, \ldots, e_{n-1}^{k-1}) && \text{if } x_0 = k-1. \end{aligned}$$

where the expression e_j^h is determined by the source and the kind of the edge to x_j in the corresponding graph, if such an edge exists. Otherwise it is $x_j + 1$.

The union of these codes is a program $P_{\mathcal{G}}$. Of course, $\mathcal{G}$ is a description of $P_{\mathcal{G}}$. Therefore:

Proposition 1 (RCA_0)**.** *The following are equivalent:*

1. *SCT criterion*
2. *SCT criterion for graphs*

3 Proving the SCT criterion

The classical proof of the SCT criterion [6] uses Ramsey's theorem for pairs. Actually, what we really need is that there exist infinitely many monochromatic triangles which share a fixed vertex: we need the homogeneous cliques in order to prove that the graph is idempotent and that there are infinitely many strictly decreasing edges in the thread and we need that they share a fixed vertex in order to guarantee the continuity of the path. This is why we introduce the principle Triang.

(Triang$_k$) For any coloring $c : [\mathbb{N}]^2 \to k$ there exist $i \in k$ and $t \in \mathbb{N}$ such that $c(t, m) = c(t, l) = c(m, l) = i$ for infinitely many pairs $\{m, l\}$.
(Triang) $\forall k \in \mathbb{N}$ Triang$_k$.

We also introduce the following strengthening of the infinite pigeonhole principle:

(SPP$_k$) For any coloring $c\colon \mathbb{N} \to k$ there exists $I \subseteq k$ such that $i \in I$ iff $i < k$ and $c(x) = i$ for infinitely many x.
(SPP) $\forall k \in \mathbb{N}$ SPP$_k$.

For the reversal we use the fact that SPP is equivalent to Σ^0_2-induction.

Lemma 1 (RCA_0)**.** *The following are equivalent:*

1. $\mathsf{I}\Sigma^0_2$
2. SPP

Proof. It is well-known that $\mathsf{I}\Sigma^0_2$ is equivalent over RCA_0 to *bounded comprehension* for Π^0_2-formulas, that is the axiom schema

$$\forall k \ \exists X \ \forall i \ (i \in X \leftrightarrow i < k \wedge \varphi(i)),$$

where φ is Π^0_2. It immediately follows that $\mathsf{I}\Sigma^0_2$ implies SPP. Let us show that SPP implies bounded Π^0_2-comprehension. Let $\varphi(i) = \forall x \exists y \ \theta(i, x, y)$. We define $c\colon \mathbb{N} \to k + 1$ by primitive recursion as follows:

1. Let $s = 0$ and $x_i = 0$ for all $i < k$;
2. Suppose we have defined $c(x)$ for every $x < s$. For all $i < k$, if $\exists y < s \ \theta(i, x_i, y)$, let $c(s + i) = i$ and $x_i = x_i + 1$. Otherwise let $c(s + i) = k$;
3. Let $s = s + k$. Return to step 2.

By SPP, the set $I = \{i \leq k\colon \exists^\infty x \ c(x) = i\}$ exists. One can check that $I \setminus k = \{i < k\colon \forall x \exists y \ \theta(i, x, y)\}$.

The following shows that one direction of the SCT criterion is already provable in RCA_0.

Proposition 2 (RCA_0)**.** *Let $\mathcal{G}$ be a finite set of size-change graphs. If every multipath $M = G_0, \ldots, G_n, \ldots$ has an infinite descent, then every idempotent $G \in \mathsf{cl}(\mathcal{G})$ has an arc $x \xrightarrow{\downarrow} x$.*

Proof. Let G be idempotent. Then $M = G, G, \ldots, G, \ldots$ is a multipath. By hypothesis there exists an infinite descent. Since G is idempotent, one can define an infinite sequence $x_0, x_1, x_2, \ldots$ such that $x_i \xrightarrow{\downarrow} x_{i+1} \in G$. As there are finitely many parameters, by the finite pigeonhole principle, which is provable in RCA_0, there exist $i < j$ such that $x = x_i = x_j$. By idempotence of G, $x \xrightarrow{\downarrow} x \in G$.

Theorem 5 (RCA_0)**.** Triang *implies the SCT criterion.*

Proof. We prove the SCT criterion for graphs. Let $\mathcal{G}$ be a finite set of size-change graphs and assume that any idempotent graph in $\mathsf{cl}(\mathcal{G})$ has a strict arc $x \xrightarrow{\downarrow} x$ for some parameter x. Let

$$\mathcal{M}_\pi = G_0, \ldots, G_n, \ldots .$$

We aim to prove that $\mathcal{M}_\pi$ has an infinite descent. Define $c : [\mathbb{N}]^2 \to \mathsf{cl}(\mathcal{G})$ as follows:

$$c(i, j) = G_i; \ldots ; G_{j-1}.$$

By applying $\mathrm{Triang}_{|\mathsf{cl}(\mathcal{G})|}$ to the coloring c, we have:

$$\exists t \exists G \in \mathsf{cl}(\mathcal{G}) \forall n \exists m, l (n < m < l \wedge t < m \wedge c(t, m) = c(t, l) = c(m, l) = G).$$

Then G is idempotent, indeed

$$G; G = c(t, m); c(m, l) = c(t, l) = G.$$

By hypothesis, we have that there exists $x \xrightarrow{\downarrow} x \in G$. By Σ^0_0-comprehension, let $f : \mathbb{N}^3 \to \mathbb{N}$ be such that $f(n, m, l) = 0$ iff $n < m < l$ and $t < m$ and $c(t, m) = c(t, l) = c(m, l) = G$. By minimization (see Simpson [13, Theorem II.3.5]), there exists a function $h : \mathbb{N} \to \mathbb{N}^2$ such that for all n we have that $f(n, h_0(n), h_1(n)) = 0$, where $h(n) = (h_0(n), h_1(n))$. Now define by primitive recursion a Triang witness function $g : \mathbb{N} \to \mathbb{N}$ by letting $g(0) = h(0)$ and $g(n+1) = h(g_1(n))$, where $g(n) = (g_0(n), g_1(n))$. Therefore, for all n

- $t < g_0(n) < g_1(n) < g_0(n+1)$ and
- $c(t, g_0(n)) = c(t, g_1(n)) = c(g_0(n), g_1(n)) = G$.

We claim that there exists an infinite descent starting from x in G_t. Since $x \xrightarrow{\downarrow} x \in c(t, g_0(0))$, it is sufficient to show that $x \xrightarrow{\downarrow} x \in c(g_0(n), g_0(n+1))$ for any n. As $x \xrightarrow{\downarrow} x \in c(t, g_0(n+1))$, there exists y such that $x \to y \in c(t, g_1(n))$ and $y \to x \in c(g_1(n), g_0(n+1))$, and at least one of them is strict. Now $c(t, g_1(n)) = c(g_0(n), g_1(n))$, and so $x \to y \in c(g_0(n), g_1(n))$. Therefore we have $x \xrightarrow{\downarrow} x \in c(g_0(n), g_0(n+1))$, as desired.

Theorem 6 (RCA_0)**.** *The SCT criterion implies* SPP*.*

Proof. We show that the SCT criterion for graphs implies SPP.

We first prove the thesis for $k = 2$. This serves as an illustration of the general case. Note in fact that SPP_k is provable in RCA_0 for every standard $k \in \mathbb{N}$.

Given $c : \mathbb{N} \to 2$, we want to show that there exists $I \subseteq 2$ such that $i \in I$ iff $\exists^\infty x\ c(x) = i$. Let us define $\mathcal{G}$ as follows. The set $\mathcal{G}$ consists of three size-change graphs G_0, G_1, G_2 on parameters z_0, z_1, z_2. For $i < 3$, the graph G_i has only one strict arc $z_i \xrightarrow{\downarrow} z_i$ and non-strict arcs $z_j \xrightarrow{\Downarrow} z_j$ for $j > i$. Note that every $G \in \mathsf{cl}(\mathcal{G})$ contains a strict arc $z \xrightarrow{\downarrow} z$. Therefore, by the SCT criterion, every multipath of $\mathcal{G}$ has an infinite descent. Let

$$g(x) = \begin{cases} 0 & \text{if } c(x) = 0 \wedge c(x+1) = 0 \\ 1 & \text{if } c(x) = 1 \wedge c(x+1) = 1 \\ 2 & \text{otherwise} \end{cases}$$

Consider the multipath $M = G_{g(0)}, G_{g(1)}, \ldots$. Hence there exists an infinite descent in $\mathcal{M}$. This implies that there exists a parameter z_i that is strictly decreasing infinitely many times, that is $z_i \xrightarrow{\downarrow} z_i \in G_{g(x)}$, viz. $g(x) = i$, for infinitely many x. If $i < 2$, it means that from some point on $c(x) = i$ and so $I = \{i\}$. If $i = 2$, then the color changes infinitely many times and so $I = \{0, 1\}$.

General case. Let $c \colon \mathbb{N} \to k$ be a given coloring. We want to show that

$$I^\infty = \{i < k \colon \exists^\infty x\ c(x) = i\}$$

exists.

Let $\mathcal{I}$ be the set of nonempty subset of k and $\mathrm{Par}(\mathcal{I})$ consist of parameters z_I for every $I \in \mathcal{I}$. Define size-change graphs $G_\mathcal{A}$ on $(\mathrm{Par}(\mathcal{I}), \mathrm{Par}(\mathcal{I}))$ for any $\mathcal{A} \subseteq \mathcal{I}$ as follows. Let m be the maximum size of an element of $\mathcal{A}$. Then

- $z_I \xrightarrow{\downarrow} z_I \in G$ iff $I \in \mathcal{A}$ and $|I| = m$;
- $z_I \xrightarrow{\Downarrow} z_I \in G$ iff $I \notin \mathcal{A}$ and $|I| \geq m$.

Let $\mathcal{G} = \{G_\mathcal{A} \colon \mathcal{A} \subseteq \mathcal{I}\}$.

Claim. Every idempotent graph $G \in \mathsf{cl}(\mathcal{G})$ has an arc $z_I \xrightarrow{\downarrow} z_I$ for some I.

Proof. We show that every graph $G \in \mathsf{cl}(\mathcal{G})$ has a strict arc $z_I \xrightarrow{\downarrow} z_I$ for some I. Let $G = G_0; G_1; \ldots; G_{l-1}$ with $G_s \in \mathcal{G}$ for all $s < l$. Let $\mathcal{A}_s$ be the $\mathcal{A}$ corresponding to G_s. Choose $I \in \bigcup_{s<l} \mathcal{A}_s$ of maximum size. We claim that $z_I \xrightarrow{\downarrow} z_I \in G$. Let $p < l$ be such that $I \in \mathcal{A}_p$. By definition, $z_I \xrightarrow{\downarrow} z_I \in G_p$. By using the maximality of I it is easy to show that for every $s < l$ either $z_I \xrightarrow{\downarrow} z_I \in G_s$ or $z_I \xrightarrow{\Downarrow} z_I \in G_s$.

We now define a multipath $M = G_0, G_1, \ldots, G_x, \ldots$ as follows.

Let $\varGamma_I$ be a marker for $I \in \mathcal{I}$. At the beginning every marker $\varGamma_I$ points to the first color of I (in the standard ordering of the natural numbers). At stage x, if the marker $\varGamma_I$ points to the color i and $c(x)$ is the color right after i in I (in the standard ordering of the natural numbers), then move the marker to the color $c(x)$. If i is the last color of I and $c(x)$ is the first color of I, move the marker to the first color of I. It is not difficult to see within RCA_0 that every color in I appears infinitely often iff the marker points to the last color of I infinitely often.

Call I a guess at stage x if at the beginning of stage x the marker $\varGamma_I$ points to the last color of I and $c(x)$ equals the first color of I. The idea is that at stage x we are guessing that $I = I^\infty$. Note that we can have more guesses at the same stage and that I is a guess at infinitely many stages iff $I \subseteq I^\infty$.

Now let $G_x = G_{\mathcal{A}}$ where $\mathcal{A}$ is the set of guesses at stage x. By the SCT criterion for graphs, we have an infinite descent in M for some parameter z_I starting at some point t. We aim to show that I is the right guess, that is $I = I^\infty$. Now, there exist infinitely many x such that $z_I \overset{\downarrow}{\to} z_I \in G_x$, and in particular I is a guess at stage x for infinitely many x. It follows that $I \subseteq I^\infty$. It is sufficient to show that I is maximal. Suppose not and let $J \supset I$ be such that every color in J appears infinitely often. Therefore there exists $x > t$ such that J is a guess at stage x. By definition, in G_x there is no arc from z_I to z_I, a contradiction.

Therefore we can conclude that Triang $\geq$ SCT criterion $\geq \mathsf{I}\varSigma^0_2$. Actually we can prove that they are all equivalent.

Theorem 7. *Over* RCA_0 *the following are equivalent:*

1. $\mathsf{I}\varSigma^0_2$
2. Triang
3. *SCT criterion*
4. *SCT criterion for graphs*

Proof. We need only to show that $\mathsf{I}\varSigma^0_2$ implies Triang. As shown in [14] RT^2 is $\varPi^1_1$-conservative over $\mathsf{B}\varSigma^0_3$, the bounding principle for $\varSigma^0_3$-formulas. So, since RT^2 trivially implies Triang (which is a $\varPi^1_1$-statement), then also $\mathsf{B}\varSigma^0_3$ does. It is known that $\mathsf{B}\varSigma^0_3$ is $\widetilde{\varPi}^0_4$-conservative over $\mathsf{I}\varSigma^0_2$, where a statement is $\widetilde{\varPi}^0_4$ if it is of the form $\forall X \varphi(X)$ and $\varphi(X) \in \varPi^0_4$. This follows as a particular case from the analogue result in first order arithmetic that $\mathsf{B}\varSigma_{n+1}$ is $\varPi_{n+2}$-conservative over $\mathsf{I}\varSigma_n$ for all $n \geq 0$ (see [5, Chapter IV, Section 1(f)]). Finally, one can check that Triang is $\widetilde{\varPi}^0_4$, hence the thesis.

Remark 2. One can directly show that the Péter-Ackermann function is SCT in both senses. Indeed let $G_{0,1}, G_2$ be the size change graphs of the Péter-Ackermann function as in Example 2. Let $\mathcal{M} = G'_0, \ldots, G'_n, \ldots$ be an infinite multipath. We have

$$\forall n \, \exists m \geq n \; G'_n = G_{0,1} \vee \exists n \, \forall m \geq n \; G'_n = G_2.$$

In the first case we have an infinite descent for x starting in G'_0. The second case yields an infinite descent for y starting in some G_n, since all graphs in the multipath from n on are G_2. Note that this proof is in classical logic, since it requires the Law of Excluded Middle.

In general, if $\mathcal{G}$ has size k for some standard $k \in \omega$, then $\mathsf{RCA_0}$ proves the SCT criterion for $\mathcal{G}$. This follows from the following:

Proposition 3. *For any standard $k \in \omega$,*

$$\mathsf{RCA_0} \vdash \mathrm{Triang}_k.$$

Proof. Note that $\mathsf{RCA_0} \vdash \mathrm{RT}^1_k$ for all standard $k \in \mathbb{N}$. We prove Triang_k by (external) induction on k.

Given a coloring $c : [\mathbb{N}]^2 \to k$, let $c_0 : \mathbb{N} \to k$ such that $c_0(x) = c(0, x)$ and let X be the infinite homogeneous set given by RT^1_k. Let $\{x_n : n \in \mathbb{N}\}$ be the increasing enumeration of X. Suppose $i = c(0, x_0)$. By the law of excluded middle, we have:

$$\forall n \exists m, l(l > m > n \wedge c(x_m, x_l) = i) \ \vee \ \exists n \forall m, l(l > m > n \implies c(x_m, x_l) \neq i).$$

In the first case we are done. In the second case let $Y = \{x \in X : x > x_n\}$. Then Y is an infinite homogeneous set in $(k-1)$-many colors. By the induction hypothesis (on $d : [\mathbb{N}]^2 \to k-1$ such that $d(a, b) = c(x_{n+a}, x_{n+d})$) we are done again.

4 Conclusion and further works

In this paper we addressed the study of size-change analysis in the context of Reverse Mathematics. We determined the exact strength of the SCT criterion by proving that it is equivalent to a weak version of Ramsey's theorem for pairs, which turns out to be equivalent to Σ^0_2-induction over $\mathsf{RCA_0}$. In particular the proof of the SCT criterion does not require full Ramsey's theorem for pairs.

One of the motivations for studying size-change termination in the framework of Reverse Mathematics is that the Péter-Ackermann function is size-change-terminating. Actually, this can be proved in $\mathsf{RCA_0}$, whereas it is well known that the totality of the Péter-Ackermann function is not provable in $\mathsf{RCA_0}$. This arises the question of what is needed in order to show the *soundness* of size-change termination (SCT soundness), that is the statement that every SCT program terminates.

The classical proof is based on the fact that "if a program does not terminate then there exists an infinite state transition sequence". This statement seems to require König's lemma, which is equivalent to Arithmetical Comprehension Axiom ($\mathsf{ACA_0}$) over the base system $\mathsf{RCA_0}$. Roughly, $\mathsf{ACA_0}$ asserts the existence of the jump of every set of natural numbers.

We suspect that a direct proof of the SCT soundness does not require any comprehension (set existence) axiom. In fact, it is known that SCT programs compute exactly the *multiply recursive* functions [1]. On the other hand, the class of multiply recursive functions coincides with the class $\mathcal{M} = \bigcup_{\alpha<\omega^\omega} \mathcal{F}_\alpha$, where $(\mathcal{F}_\alpha)_\alpha$ is the fast growing hierarchy [10]. Since well-foundedness of ω^{ω^ω} implies the totality of every function in $\mathcal{M}$, we thus conjecture that SCT soundness is provable in RCA_0 plus well-foundedness of ω^{ω^ω}.

References

1. Ben-Amram, A.M.: General Size-Change Termination and Lexicographic Descent. In: Mogensen, T., Schmidt, D., Sudborough, I.H. (eds.) The Essence of Computation: Complexity, Analysis, Transformation. Essays Dedicated to Neil D. Jones, Lecture Notes in Computer Science, vol. 2566, pp. 3–17. Springer-Verlag (2002)
2. Codish, M., Genaim, S.: Proving Termination One Loop at a Time. In: Mesnard, F., Serebrenik, A. (eds.) Proceedings of the 13th International Workshop on Logic Programming Environments, Tata Institute of Fundamental Research, Mumbai, India, December 8, 2003. Report, vol. CW371, pp. 48–59. Katholieke Universiteit Leuven, Department of Computer Science, Celestijnenlaan 200A, B-3001 Heverlee (Belgium) (2003)
3. Friedman, H.: Some systems of second order arithmetic and their use. Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1 pp. 235–242 (1975)
4. Gasarch, W.: Chapter Four - Proving Programs Terminate Using Well-Founded Orderings, Ramsey's Theorem, and Matrices. Advances in Computers, vol. 97, pp. 147 – 200. Elsevier (2015)
5. Hájek, P., Pudlák, P.: Metamathematics of First-Order Arithmetic, Perspectives in Mathematical Logic, vol. 3. Springer-Verlag, Berlin (1998)
6. Heizmann, M., Jones, N.D., Podelski, A.: Size-Change Termination and Transition Invariants. In: SAS 2010, Perpignan, France, September 14-16, 2010. Proceedings. pp. 22–50 (2010)
7. Hirst, J.L.: Combinatorics in Subsystems of Second Order Arithmetic. Phd thesis, The Pennsylvania State University (1987)
8. Jones, N.D., Bohr, N.: Termination Analysis of the Untyped lamba-Calculus. In: van Oostrom, V. (ed.) Rewriting Techniques and Applications, 15th International Conference, RTA 2004, Aachen, Germany, June 3-5, 2004, Proceedings. Lecture Notes in Computer Science, vol. 3091, pp. 1–23. Springer (2004)
9. Lee, C.S., Jones, N.D., Ben-Amram, A.M.: The size-change principle for program termination. In: Conference Record of POPL 2001: The 28th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, London, UK, January 17-19, 2001. pp. 81–92 (2001)
10. Löb, M.H., Wainer, S.S.: Hierarchies of number-theoretic functions. I. Arch. Math. Logic 13(1-2), 39–51 (1970)
11. Podelski, A., Rybalchenko, A.: Transition invariants. In: Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, LICS 2004, July 13-17, Turku, Finland. pp. 32–41 (2004)
12. Ramsey, F.P.: On a problem in formal logic. Proc. London Math. Soc. 30, 264–286 (1930)
13. Simpson, S.G.: Subsystems of Second Order Arithmetic. Perspectives in Mathematical Logic, Springer-Verlag (1999)

14. Slaman, T.A., Yokoyama, K.: The strength of Ramsey's Theorem for pairs and arbitrary many colors (2016), in preparation
15. Steila, S., Yokoyama, K.: Reverse mathematical bounds for the Termination Theorem. Annals of Pure Applied Logic 167(12), 1213–1241 (2016)
16. Thiemann, R., Giesl, J.: The size-change principle and dependency pairs for termination of term rewriting. Applicable Algebra in Engineering, Communication and Computing 16(4), 229–270 (2005)