TRUNCATION AND SEMI-DECIDABILITY NOTIONS IN APPLICATIVE THEORIES

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Abstract. BON^+ is an applicative theory and closely related to the first order parts of the standard systems of explicit mathematics. As such it is also a natural framework for abstract computations. In this article we analyze this aspect of BON^+ more closely. First a point is made for introducing a new operation $\boldsymbol{\tau}_{\mathsf{N}}$, called truncation, to obtain a natural formalization of partial recursive functions in our applicative framework. Then we introduce the operational versions of a series of notions that are all equivalent to semi-decidability in ordinary recursion theory on the natural numbers, and study their mutual relationships over BON^+ with $\boldsymbol{\tau}_{\mathsf{N}}$.

§1. Introduction. Starting point of the following considerations is the applicative theory BON^+ whose universe consists of so-called operations; self-application is possible though not necessarily defined. This basic theory of operations and numbers BON^+ comprises the axioms of partial combinatory algebra, some natural axioms for the data type of the natural numbers, and the schema of complete induction on the natural numbers for all first order formulae (hence the symbol "+" in its name).

Moreover, BON^+ is (closely related to) the first order parts of the standard systems of explicit mathematics introduced in Feferman [3, 4]. Since the notion of a partial combinatory algebra is an interesting generalization of and an abstract framework for computations, this applicative part of explicit mathematics is sometimes called its "computational engine".

In this article we analyze this aspect of BON^+ more closely. First a point is made for introducing a new operation $\boldsymbol{\tau}_{\mathsf{N}}$, called truncation, to obtain a natural formalization of partial recursive functions in our applicative framework. Then we introduce the operational versions of a series of notions that are all equivalent to semi-decidability in ordinary recursion theory on the natural numbers, and study their mutual relationships over BON^+ with $\boldsymbol{\tau}_{\mathsf{N}}$. As it turns out, not all these equivalences can be transferred to their operational variants, and interesting mutual relationships can be discovered.

This paper is organized as follows. In the next section we present the basic theory BON^+ as well as two notions of representing partial number-theoretic functions and state some central properties of BON^+. In Section 3 we first discuss a few shortcomings of BON^+ with respect to a natural treatment of partial recursive functions and then introduce the truncation operator $\boldsymbol{\tau}_{\mathsf{N}}$ as

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a possibility to compensate for these deficits. Section 4 gives two proofs of the undefinability of $\boldsymbol{\tau}_{\mathsf{N}}$ in BON^+. Section 5 is about models of $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$. These models serve several purposes: (i) they underline that the operator $\boldsymbol{\tau}_{\mathsf{N}}$ reflects a very natural principle in our operational context, (ii) they give us the consistency of $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$ with the assertion that application is total and a kind of existence property, and (iii) they provide some complexity results that we use in Section 6, where the relationships between our operational versions of classical semi-decidability notions are studied.

§2. The theory BON^+. In this section we introduce the basic theory BON^+ of operations and numbers, which is the point of departure for our considerations. BON^+ axiomatizes the basic operational behavior of the first order objects of explicit mathematics. It is closely related to the theory BON introduced in, for example, Feferman and Jäger [5] and Feferman, Jäger, and Strahm [6], to the theory EON of Beeson [2, VI.2.4], and to the theory APP of Troelstra and van Dalen [19, 9.3.3]. In Section 3 we extend BON^+ to the theory $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$.

The language L of BON^+ and $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$ is a first order language with countably many individual variables $a, b, c, u, v, w, x, y, z, f, g, h, \dots$ (possibly with subscripts) and the individual constants 0, $\mathbf{k}$, $\mathbf{s}$, $\mathbf{p}$, $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{s}_{\mathsf{N}}$, $\mathbf{p}_{\mathsf{N}}$, $\mathbf{d}_{\mathsf{N}}$, $\boldsymbol{\tau}_{\mathsf{N}}$, the meaning of which will be explained later. In addition, there is a binary function symbol $\cdot$ for application. The basic relation symbols are countably many unary relation variables $U, V, W, \dots$ plus the specific unary relation symbols $\downarrow$, N, and the binary relation symbol $=$.

The principal term formation operation is term application and thus the *terms* ($r, s, t, r_1, s_1, t_1, \dots$) are generated as follows:

(1) Each individual variable is a term.
(2) Each individual constant is a term.
(3) If s and t are terms, then so also is $(s \cdot t)$.

We write $(s \cdot t)$ often just as (st) or st. In this simplified form we adopt the convention of association to the left such that, e.g., $s_1 s_2 \dots s_n$ stands for $(\dots(s_1 \cdot s_2) \dots \cdot s_n)$. We also use the notation $s(t_1, \dots, t_n)$ for $st_1 \dots t_n$. If n is a natural number, we write $\overline{n}$ for the corresponding numeral, i.e., for the closed term given recursively by $\overline{0} := 0$ and $\overline{n{+}1} := \mathbf{s}_{\mathsf{N}}\overline{n}$.

The *formulae* ($\varphi, \chi, \psi, \varphi_1, \chi_1, \psi_1, \dots$) of L are generated from the atomic formulae $t\downarrow$, $(s = t)$, $\mathsf{N}(t)$, and $U(t)$ by closing them under the usual propositional connectives and quantification over individuals. We will often omit parentheses if there is no danger of confusion.

The logic of BON^+ is the classical version of Beeson's *logic of partial terms* (see [2, VI.1]). It corresponds to the E^+-logic with equality and strictness of Troelstra and van Dalen [18, 2.2.4], where $E(t)$ is written instead of $t\downarrow$. Here $t\downarrow$ is read "t is defined" or "t has a value". The partial equality $\simeq$ is introduced by

$$(s \simeq t) \ := \ ((s\downarrow \vee \ t\downarrow) \rightarrow \ s = t).$$

Further we write $t \in Z$ instead of $Z(t)$ in case Z is a relation variable or the relation constant N. As usual, $t \notin Z$ and $s \neq t$ stand for $\neg(t \in Z)$ and $\neg(s = t)$, respectively. As additional abbreviations we will use:

$$
\begin{aligned}
t \in (\mathsf{N} \to \mathsf{N}) &:= (\forall x \in \mathsf{N})(tx \in \mathsf{N}),\\
t \in (\mathsf{N}^m \to \mathsf{N}) &:= (\forall x_0, \dots, x_{m-1} \in \mathsf{N})(t(x_0, \dots, x_{m-1}) \in \mathsf{N}),\\
f \in \mathit{Char} &:= (\forall x \in \mathsf{N})\big(fx = \overline{0} \vee fx = \overline{1}\big),\\
f \in \mathit{Char}_2 &:= (\forall x, y \in \mathsf{N})\big(f(x,y) = \overline{0} \vee f(x,y) = \overline{1}\big),\\
t \in \mathsf{N}\backslash U &:= t \in \mathsf{N} \wedge t \notin U,\\
U = \emptyset &:= \neg\exists x\ (x \in U).
\end{aligned}
$$

The so-called *strictness axioms* of the logic of partial terms are all formulae of the following form where $\varphi[u]$ is an atomic formula with an occurrence of u:

$$\varphi[s] \to s\downarrow.$$

Keep in mind that in general $t \notin Z$ does not imply $t\downarrow$ and that we cannot deduce $s\downarrow$ or $t\downarrow$ from $s \neq t$.

The non-logical axioms of BON^+ can be divided into the following four groups:

I. Partial combinatory algebra:

(1) $\mathbf{k}(x,y) = x$,

(2) $\mathbf{s}(x,y)\downarrow \wedge \mathbf{s}(x,y,z) \simeq x(z,yz)$.

II. Pairing and projection:

(3) $\mathbf{p}_0(\mathbf{p}(x,y)) = x \wedge \mathbf{p}_1(\mathbf{p}(x,y)) = y$.

III. Natural numbers:

(4) $0 \in \mathsf{N} \wedge \mathbf{s}_{\mathsf{N}} \in (\mathsf{N} \to \mathsf{N})$,

(5) $\mathbf{s}_{\mathsf{N}}x \neq 0 \wedge \mathbf{p}_{\mathsf{N}}0 = 0 \wedge (\forall x \in \mathsf{N})(\mathbf{p}_{\mathsf{N}}(\mathbf{s}_{\mathsf{N}}x) = x)$,

(6) $x \in U \to x \in \mathsf{N}$,

(7) $\varphi[0] \wedge (\forall x \in \mathsf{N})(\varphi[x] \to \varphi[\mathbf{s}_{\mathsf{N}}x]) \to (\forall x \in \mathsf{N})\varphi[x]$ for all L formulae $\varphi[x]$.

IV. Definition by cases on N:

(8) $x \in \mathsf{N} \wedge y \in \mathsf{N} \wedge x = y \to \mathbf{d}_{\mathsf{N}}(a,b,x,y) = a$,

(9) $x \in \mathsf{N} \wedge y \in \mathsf{N} \wedge x \neq y \to \mathbf{d}_{\mathsf{N}}(a,b,x,y) = b$.

$\mathbf{k}$ and $\mathbf{s}$ are the partial versions of the well-known combinators of Curry's combinatory logic. $\mathbf{p}$ provides an injective pairing of the universe with the inverse operations $\mathbf{p}_0$ and $\mathbf{p}_1$. $\mathbf{s}_{\mathsf{N}}$ represents the successor function on the natural numbers and $\mathbf{p}_{\mathsf{N}}$ the predecessor function. Axioms (4) and (5) formulate some basic properties of the natural numbers, axiom (6) simply states that the relation variables range over subsets of the natural numbers, and (7) is the schema of complete induction. $\mathbf{d}_{\mathsf{N}}$ gives definition by integer cases. Since BON^+ comprises the axioms (1)–(2) of a partial combinatory algebra, we clearly have λ abstraction and the usual fixed point theorem; this is mentioned already in Feferman [3] and proved in detail in, e.g., Beeson [2, VI.2.2], Troelstra and van Dalen [19, 9.3.5], and Feferman, Jäger, and Strahm [6].

LEMMA 1 (λ abstraction). *For each variable x and term t we can construct a term $\lambda x.t$ whose free variables are those of t, excluding x, such that BON^+ proves*

$$\lambda x.t\downarrow \wedge (\lambda x.t)x \simeq t.$$

LEMMA 2 (Fixed point). *There exists a closed term* $\mathbf{fix}$ *such that* BON^+ *proves*

$$\mathbf{fix}(f)\downarrow \wedge (g = \mathbf{fix}(f) \to \forall x\ (gx \simeq f(g,x))).$$

The generalization of λ abstraction to several variables is by simply iterating abstraction for one argument, and we usually write $\lambda x_1 \dots x_n.t$ for the corresponding term.

BON^+ is proof-theoretically equivalent to the theory BON (see Feferman and Jäger [5] and Feferman, Jäger, and Strahm [6]) extended by the schema of complete induction for arbitrary formulae. In particular, it can be shown that all primitive recursive functions can be represented in BON^+ as explained below.

We write ω for the set of natural numbers. Given a (possibly partial) function $\mathcal{F}$ from ω^k to ω we say that a closed term t *numeralwise represents* $\mathcal{F}$ in an L theory T iff

$$\mathcal{F}(m_1,\dots,m_k) = n \iff T \vdash t(\overline{m_1},\dots,\overline{m_k}) = \overline{n}$$

for all $m_1,\dots,m_k,n \in \omega$. However, this does not guarantee the expected behavior of t on nonstandard natural numbers. In order to impose such a condition, we have to assume that it is described by a formula, e.g., by equations. For example, let us consider an unary function $\mathcal{G}$ that is defined by primitive recursion from a natural number n and a binary function $\mathcal{F}$ as

$$\mathcal{G}(0) = n \text{ and } \mathcal{G}(m+1) = \mathcal{F}(m, g(m))$$

for all natural numbers m. Then, if the terms s and t represent the functions $\mathcal{F}$ and $\mathcal{G}$, respectively, we want the conditional equations

$$t0 = \overline{n} \text{ and } (\forall x \in \mathsf{N})(t(\mathbf{s_N}x) = s(x, tx))$$

to be provable in T. If the defining formula of a function $\mathcal{F}$ is provable for a term t in T, we say that t *definitionally represents* $\mathcal{F}$ *in* T. The following is immediate from Troelstra and van Dalen [19, 9.3].

THEOREM 3 (Prim. rec. func.). *For any (definition of a) k-ary primitive recursive function $\mathcal{F}$, there exists a closed term* $\mathbf{prim}_{\mathcal{F}}$ *that numeralwise and definitionally represents $\mathcal{F}$ in* BON^+ *and for which* BON^+ *proves* $\mathbf{prim}_{\mathcal{F}} \in (\mathsf{N}^k \to \mathsf{N})$.

Observe, however, that this theorem does not imply that BON^+ proves

$$\mathbf{prim}_{\mathcal{F}}(a_1,\dots,a_k) \in \mathsf{N} \to a_1,\dots,a_k \in \mathsf{N}$$

for the representation $\mathbf{prim}_{\mathcal{F}}$ of a k-ary primitive recursive $\mathcal{F}$; this implication is not provable in BON^+ in general.

According to the last theorem there are closed terms $\mathbf{pair}$, $\mathbf{proj}_0$, $\mathbf{proj}_1$, numeralwise and definitionally representing a primitive recursive bijective pairing function with its corresponding projections, respectively, such that BON^+ proves:

(1) $s,t \in \mathsf{N} \to (\mathbf{pair}(s,t) \in \mathsf{N} \wedge \mathbf{proj}_0(s) \in \mathsf{N} \wedge \mathbf{proj}_1(s) \in \mathsf{N})$,
(2) $s,t \in \mathsf{N} \to (\mathbf{proj}_0(\mathbf{pair}(s,t)) = s \wedge \mathbf{proj}_1(\mathbf{pair}(s,t)) = t)$,
(3) $s \in \mathsf{N} \to \mathbf{pair}(\mathbf{proj}_0(s), \mathbf{proj}_1(s)) = s$.

There is also a closed term **less** for the characteristic function of the primitive recursive less relation, and we often write $a < b$ for $(a, b \in \mathsf{N} \wedge \mathbf{less}(a, b) = 0)$.

In Troelstra and van Dalen [19, 9.3.10], a specific minimum operator is considered. Later, we need the following part of this result.

THEOREM 4 ($\mathbf{min}_0$). *There exists a closed term* $\mathbf{min}_0$ *such that* BON^+ *proves*

$$(\exists x \in \mathsf{N})(fx = 0 \wedge (\forall y < x)(fy \in \mathsf{N}))$$
$$\rightarrow \mathbf{min}_0(f) \in \mathsf{N} \wedge f(\mathbf{min}_0(f)) = 0 \wedge (\forall y < \mathbf{min}_0(f))(0 < fy).$$

PROOF. Let $t := \lambda fhx.\mathbf{d}_{\mathsf{N}}(\lambda u.x, \lambda u.h(\mathbf{s}_{\mathsf{N}}x), fx, 0)0$. Then, as far as $fx \in \mathsf{N}$,

$$\mathbf{fix}(tf, x) \simeq t(f, \mathbf{fix}(tf), x) \simeq \mathbf{d}_{\mathsf{N}}(\lambda u.x, \lambda u.\mathbf{fix}(tf, \mathbf{s}_{\mathsf{N}}x), fx, 0)0$$
$$\simeq \begin{cases} x & \text{if } fx = 0, \\ \mathbf{fix}(tf, \mathbf{s}_{\mathsf{N}}x) & \text{otherwise.} \end{cases}$$

Define $\mathbf{min}_0 := \lambda f.\mathbf{fix}(tf, 0)$. Now we assume

$$(\exists x \in \mathsf{N})(fx = 0 \wedge (\forall y < x)(fy \in \mathsf{N})).$$

By induction there exists $a \in \mathsf{N}$ with $fa = 0$ and $(\forall y < a)(0 < fy)$. If $a = 0$, then $\mathbf{min}_0(f) \simeq \mathbf{fix}(tf, a) = a$ is provable. If $0 < a$, a further induction yields

$$y < a \rightarrow \mathbf{min}_0(f) \simeq \mathbf{fix}(tf, \mathbf{s}_{\mathsf{N}}y).$$

Therefore, we have $\mathbf{min}_0(f) \simeq \mathbf{fix}(tf, a) = a$ as well. This proves our claim. ⊣

Making use of this minimum operator and following [19] it is routine work to show that every total recursive function can be represented numeralwise (but not definitionally in general) in BON^+ by a closed term. Having primitive recursion and $\mathbf{min}_0$, it is easy to see that even Kleene's enumeration $\{e\}$ of the partial recursive number-theoretic functions can be obtained in BON^+.

§3. Truncation to N. In this section we discuss some deficiencies of BON^+ with respect to a "natural treatment" of partial recursive number-theoretic functions within BON^+ and propose the introduction of a new truncation operator to compensate for them.

There are two interesting additional axioms, the totality assertion (TOT-AP) and the assertion (TOT-N) that every object is a natural number,

(TOT-AP) $\forall x\, \forall y\, (xy\downarrow)$,

(TOT-N) $\forall x\, (x \in \mathsf{N})$.

Folk results tell us that both are unprovable in BON^+, but BON^+ is consistent with (TOT-AP) and with (TOT-N). However, BON^+ + (TOT-AP) + (TOT-N) is inconsistent. Hence, if we want to be compatible with both possible extensions of BON^+, the only way to formally express undefinedness of a partial number-theoretic function $\mathcal{F}$ at input x is to state $t_{\mathcal{F}}(x) \notin \mathsf{N}$ for the associated term $t_{\mathcal{F}}$.

Now suppose that the unary partial number-theoretic function $\mathcal{F}$ is the composition of the unary partial number-theoretic functions $\mathcal{G}$ and $\mathcal{H}$,

$$\mathcal{F}(n) \simeq \mathcal{H}(\mathcal{G}(n))$$

for all natural numbers n. Also, if $\mathcal{G}(n)$ is undefined, then $\mathcal{F}(n)$ is undefined as well. If the terms s and t represent $\mathcal{G}$ and $\mathcal{H}$, respectively, we would expect that $r := \lambda x.t(s(x))$ represents $\mathcal{F}$ and

$$a \in \mathsf{N} \wedge sa \notin \mathsf{N} \rightarrow ra \notin \mathsf{N}$$

within BON^+. However, if $\mathcal{H}$ is the function constant 0 and $t := \lambda x.0$ its canonical representation, then

$$a \in \mathsf{N} \wedge sa \notin \mathsf{N} \wedge ra = 0$$

is possible in BON^+. Simply assume that sa has a value outside N.

In ordinary computation theory on the natural numbers and many of its generalizations there exist

(i) a closed term r such that

$$(\forall x \in \mathsf{N})(rx = 0) \wedge \forall x\ (rx \in \mathsf{N} \rightarrow x \in \mathsf{N}),$$

(ii) an operator **op** that maps any partial computable function f to a partial computable function $g = \mathbf{op}(f)$ such that

$$(\forall x \in \mathsf{N})(fx \in \mathsf{N} \leftrightarrow gx = 0).$$

In the following section we will show that both such terms do not exist in our present environment BON^+.

To overcome these problems and similar difficulties, we now make use of the constant $\boldsymbol{\tau}_\mathsf{N}$, which did not play a role thus far. Consider the following two $\boldsymbol{\tau}_\mathsf{N}$-axioms.

VI. Truncation to N:

($\boldsymbol{\tau}_\mathsf{N}$.1) $x \in \mathsf{N} \rightarrow \boldsymbol{\tau}_\mathsf{N}(f, x) \simeq fx$,

($\boldsymbol{\tau}_\mathsf{N}$.2) $x \notin \mathsf{N} \rightarrow \boldsymbol{\tau}_\mathsf{N}(f, x) \notin \mathsf{N}$.

These two axioms state that on N any operation f behaves exactly as its truncation $\boldsymbol{\tau}_\mathsf{N} f$. Moreover, if an object that does not belong to N is fed to $\boldsymbol{\tau}_\mathsf{N} f$, then $\boldsymbol{\tau}_\mathsf{N}(f, x)$ is undefined or an object not belonging to N. In this sense, $\boldsymbol{\tau}_\mathsf{N}$ truncates every operation f to the natural numbers N.

$\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ is defined to be the extension of BON^+ by the axioms ($\boldsymbol{\tau}_\mathsf{N}$.1) and ($\boldsymbol{\tau}_\mathsf{N}$.2). In Section 4, we will show that $\boldsymbol{\tau}_\mathsf{N}$ cannot be defined in BON^+. Hence $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ is a proper extension of BON^+. And it is easy to check that by means of $\boldsymbol{\tau}_\mathsf{N}$ the problems described above can be healed. There is a close relationship between our truncation operator $\boldsymbol{\tau}_\mathsf{N}$ and Kahle's notion of N-strictness, introduced in Kahle [8, 9]; for details see Rosebrock [17].

Before turning to the undefinability of $\boldsymbol{\tau}_\mathsf{N}$ in BON^+ we want to illustrate that $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ is a natural framework for explicitly dealing with partial recursive functions and their defining equations. We leave it to the reader to verify that without $\boldsymbol{\tau}_\mathsf{N}$ and the $\boldsymbol{\tau}_\mathsf{N}$-axioms this approach would not have been possible.

It turns out to be important to have a minimum operator that is stronger than $\mathbf{min}_0$ of Theorem 4. To establish its existence we start with a preparatory lemma that asserts the existence of a term for deciding admissibility in the sense of Troelstra and van Dalen [19, 9.3.9] up to a natural number.

LEMMA 5. *There exists a closed term* **adm** *such that* $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ *proves:*

(1) $(\forall x \in \mathsf{N})(fx = 0 \wedge (\forall y < x)(fy \in \mathsf{N}) \to \mathbf{adm}(f, x) = 0)$,
(2) $(\forall x \in \mathsf{N})(\mathbf{adm}(f, x) \in \mathsf{N} \to \mathbf{adm}(f, x) = fx \wedge (\forall y < x)(fy \in \mathsf{N}))$.

PROOF. We work within $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ and define

$$\mathbf{adm} := \lambda f.\mathbf{fix}(\lambda hx.\mathbf{d}_\mathsf{N}(f, \lambda u.\boldsymbol{\tau}_\mathsf{N}(\lambda z.fx, h(\mathbf{p}_\mathsf{N}x)), x, 0)0).$$

Then we have for all $y \in \mathsf{N}$,

$$\begin{aligned} \mathbf{adm}(f, y) &\simeq \mathbf{d}_\mathsf{N}(f, \lambda u.\boldsymbol{\tau}_\mathsf{N}(\lambda z.fy, \mathbf{adm}(f, \mathbf{p}_\mathsf{N}y)), y, 0)0 \\ &\simeq \begin{cases} f0 & \text{if } y = 0, \\ \boldsymbol{\tau}_\mathsf{N}(\lambda z.fy, \mathbf{adm}(f, \mathbf{p}_\mathsf{N}y)) & \text{otherwise.} \end{cases} \end{aligned}$$

To show (1), pick $x \in \mathsf{N}$ with $fx = 0$ and $(\forall y < x)(fy \in \mathsf{N})$. We prove $y < x \to \mathbf{adm}(f, y) \in \mathsf{N}$ by induction on y and continue with

$$\mathbf{adm}(f, x) \simeq \left.\begin{cases} f0 & \text{if } x = 0 \\ \boldsymbol{\tau}_\mathsf{N}(\lambda z.fx, \mathbf{adm}(f, \mathbf{p}_\mathsf{N}x)) & \text{otherwise} \end{cases}\right\} \simeq fx = 0.$$

For establishing (2), we prove

$$\mathbf{adm}(f, x) \in \mathsf{N} \to \mathbf{adm}(f, x) = fx \wedge (\forall y < x)(fy \in \mathsf{N}).$$

by induction on x. This is obvious for $x = 0$. Assume $\mathbf{adm}(f, \mathbf{s}_\mathsf{N}x) \in \mathsf{N}$. This means $\boldsymbol{\tau}_\mathsf{N}(\lambda z.f(\mathbf{s}_\mathsf{N}x), \mathbf{adm}(f, x)) \in \mathsf{N}$. Hence ($\boldsymbol{\tau}_\mathsf{N}$.2) implies $\mathbf{adm}(f, x) \in \mathsf{N}$, and so ($\boldsymbol{\tau}_\mathsf{N}$.1) yields $\mathbf{adm}(f, \mathbf{s}_\mathsf{N}x) = f(\mathbf{s}_\mathsf{N}x)$. By induction hypothesis we also have $fx = \mathbf{adm}(f, x) \in \mathsf{N}$ and $(\forall y < x)(fy \in \mathsf{N})$. Thus we can conclude $(\forall y < \mathbf{s}_\mathsf{N}x)(fy \in \mathsf{N})$. ⊣

THEOREM 6 (**min**). *There exists a closed term* **min** *such that* $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ *proves:*

(1) $(\exists x \in \mathsf{N})(fx = 0 \wedge (\forall y < x)(fy \in \mathsf{N})) \to \mathbf{min}(f) \in \mathsf{N}$,
(2) $\mathbf{min}(f) \in \mathsf{N} \to f(\mathbf{min}(f)) = 0 \wedge (\forall y < \mathbf{min}(f))(0 < fy)$.

PROOF. Because of Lemma 2 we know that there exists a closed term $\mathbf{nt}_\mathsf{N}$ such that BON^+ proves $\mathbf{nt}_\mathsf{N}\downarrow \wedge \forall x\ (\mathbf{nt}_\mathsf{N}(x) \simeq \mathbf{s}_\mathsf{N}(\mathbf{nt}_\mathsf{N}(x)))$. We define

$$\mathbf{min} := \lambda f.\boldsymbol{\tau}_\mathsf{N}(\lambda u.\mathbf{d}_\mathsf{N}(\lambda v.\mathbf{min}_0(f), \mathbf{nt}_\mathsf{N}, u, 0)0, \mathbf{adm}(f, \mathbf{min}_0(f))).$$

In order to prove (1), assume $(\exists x \in \mathsf{N})(fx = 0 \wedge (\forall y < x)(fy \in \mathsf{N}))$. Theorem 4 implies $\mathbf{min}_0(f) \in \mathsf{N}$, $f(\mathbf{min}_0(f)) = 0$, and $(\forall y < \mathbf{min}_0(f))(fy \in \mathsf{N})$. Therefore, $\mathbf{adm}(f, \mathbf{min}_0(f)) = 0$ in view of the previous lemma. By ($\boldsymbol{\tau}_\mathsf{N}$.1) we have

$$\mathbf{min}(f) \simeq \mathbf{d}_\mathsf{N}(\lambda v.\mathbf{min}_0(f), \mathbf{nt}_\mathsf{N}, 0, 0)0 = \mathbf{min}_0(f).$$

Now we turn to (2) and assume $\mathbf{min}(f) \in \mathsf{N}$. Then $\mathbf{adm}(f, \mathbf{min}_0(f)) \in \mathsf{N}$ by ($\boldsymbol{\tau}_\mathsf{N}$.2). Hence,

$$\begin{aligned} \mathbf{min}(f) &\simeq \mathbf{d}_\mathsf{N}(\lambda v.\mathbf{min}_0(f), \mathbf{nt}_\mathsf{N}, \mathbf{adm}(f, \mathbf{min}_0(f)), 0)0 \\ &\simeq \begin{cases} \mathbf{min}_0(f) & \text{if } \mathbf{adm}(f, \mathbf{min}_0(f)) = 0, \\ \mathbf{nt}_\mathsf{N}(0) & \text{otherwise.} \end{cases} \end{aligned}$$

Since $\mathbf{min}(f) \in \mathsf{N}$, the second case is ruled out. Thus $\mathbf{adm}(f, \mathbf{min}_0(f)) = 0$ and $\mathbf{min}(f) = \mathbf{min}_0(f)$. According to the previous lemma, we thus have

$$(\forall y < \mathbf{min}(f))(fy \in \mathsf{N}) \wedge f(\mathbf{min}(f)) = \mathbf{adm}(f, \mathbf{min}(f)) = 0.$$

Now we apply Theorem 4 and obtain $(\forall y < \mathbf{min}(f))(0 < fy)$. ⊣

Now we are ready to turn to the definitional representation of all partial recursive (number-theoretic) functions. We start off from the definition of the partial recursive functions as the least class of number-theoretic functions that (i) contains the function constant zero, the successor function, the projections and (ii) is closed under composition and the minimum operator.

THEOREM 7 (Part. rec. func.: definit. repr.). *For any (definition of a) partial recursive number-theoretic function $\mathcal{F}$, there is a closed term $\mathbf{g}_\mathcal{F}$ such that $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ proves the defining formulae for both the domain and the values of $\mathbf{g}_\mathcal{F}$.*

PROOF. We prove this by induction on the definition of the class of the partial recursive functions.

(i) Initial functions. Clearly, the term $\mathbf{s}_\mathsf{N}$ represents the unary successor function and the corresponding defining equations are provable in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$. The term $\mathbf{zero}^k := \lambda x_0 \ldots x_{k-1}.0$ and the term $\mathbf{proj}_i^k := \lambda x_0 \ldots x_{k-1}.x_i$ represent the k-ary zero function and the k-ary projection function (for $0 \le i < k$), respectively, with the equations

(1) $\mathbf{zero}^k(x_0, \ldots, x_{k-1}) \in \mathsf{N} \leftrightarrow \top$,

(2) $\mathbf{zero}^k(x_0, \ldots, x_{k-1}) = 0$,

(3) $\mathbf{proj}_i^k(x_0, \ldots, x_{k-1}) \in \mathsf{N} \leftrightarrow \top$,

(4) $\mathbf{proj}_i^k(x_0, \ldots, x_{k-1}) = x_i$

provable in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ for all $x_0, \ldots, x_{k-1} \in \mathsf{N}$.

(ii) Composition. For notational simplicity we restrict ourselves to the case of the composition of a binary with two unary functions,

$$\mathcal{F}(n) \simeq \mathcal{I}(\mathcal{G}(n), \mathcal{H}(n));$$

the generalization to the general case is obvious. By induction hypothesis we have the terms $\mathbf{g}_\mathcal{G}$, $\mathbf{g}_\mathcal{H}$, and $\mathbf{g}_\mathcal{I}$. Then define

$$\mathbf{g}_\mathcal{F} := \lambda x.\boldsymbol{\tau}_\mathsf{N}(\lambda y_0.\boldsymbol{\tau}_\mathsf{N}(\lambda y_1.\mathbf{g}_\mathcal{I}(y_0, y_1), \mathbf{g}_\mathcal{H} x), \mathbf{g}_\mathcal{G} x)$$

and check that $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ proves, for all $x \in \mathsf{N}$,

(5) $\mathbf{g}_\mathcal{F} x \in \mathsf{N} \leftrightarrow (\mathbf{g}_\mathcal{G} x \in \mathsf{N} \wedge \mathbf{g}_\mathcal{H} x \in \mathsf{N} \wedge \mathbf{g}_\mathcal{I}(\mathbf{g}_\mathcal{G} x, \mathbf{g}_\mathcal{H} x) \in \mathsf{N})$,

(6) $\mathbf{g}_\mathcal{F} x \in \mathsf{N} \to \mathbf{g}_\mathcal{F} x = \mathbf{g}_\mathcal{I}(\mathbf{g}_\mathcal{G} x, \mathbf{g}_\mathcal{H} x)$.

(iii) Minimization. For notational simplicity we restrict ourselves to the case that the unary $\mathcal{F}$ is defined from the binary $\mathcal{G}$ by minimization, i.e.,

(♯) $\mathcal{F}(n)$ is the least m with $\mathcal{G}(n, m) = 0$ and for all $k < m$, $\mathcal{G}(n, k)$ is defined,

if such m exists; otherwise $\mathcal{F}(n)$ is undefined. By the induction hypothesis we have a term $\mathbf{g}_\mathcal{G}$ and define

$$\mathbf{g}_\mathcal{F} := \lambda x.\mathbf{min}(\lambda y.\mathbf{g}_\mathcal{G}(x, y)).$$

In view of Theorem 6 it is clear that $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ proves, for all $x \in \mathsf{N}$,

(7) $\mathbf{g}_{\mathcal{F}} x \in \mathsf{N} \leftrightarrow (\exists y \in \mathsf{N})(\mathbf{g}_{\mathcal{G}}(x,y) = 0 \wedge (\forall z < y)(\mathbf{g}_{\mathcal{G}}(x,z) \in \mathsf{N}))$,

(8) $\mathbf{g}_{\mathcal{F}} x \in \mathsf{N} \to (\mathbf{g}_{\mathcal{G}}(x, \mathbf{g}_{\mathcal{F}} x) = 0 \wedge (\forall z < \mathbf{g}_{\mathcal{F}} x)(0 < \mathbf{g}_{\mathcal{G}}(x,z)))$.

This finishes the proof of the definitional representation theorem for all partial recursive number-theoretic functions. ⊣

Then it is natural to ask for the numeralwise representation of the partial recursive functions. For this purpose, we need the following lemma.

Lemma 8. *For any closed term t, if $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ proves $t \in \mathsf{N}$ then there exists a natural number n such that $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ proves $t = \overline{n}$.*

This lemma is proved in full detail in Rosebrock [17]. The underlying idea of its proof is also sketched in Section 5.1.

Theorem 9 (Part. rec. func.: numeralwise repr.). *For any (definition of a) partial recursive number-theoretic function $\mathcal{F}$, there is a closed term $\mathbf{g}_{\mathcal{F}}$ which numeralwise represents $\mathcal{F}$ in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$.*

Proof. We can use the same closed term as in Theorem 7. For the case of the initial functions, the claim is trivial.

Let us consider the case of composition, namely $\mathcal{F}(n) \simeq \mathcal{I}(\mathcal{G}_0(n), \mathcal{G}_1(n))$. If $\mathcal{F}(n) = m$, then let $l_i := \mathcal{G}_i(n)$ for $i < 2$. By the induction hypothesis, $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ proves $\mathbf{g}_{\mathcal{G}_i}\overline{n} = \overline{l_i}$ for $i < 2$ and $\mathbf{g}_{\mathcal{I}}(\overline{l_0}, \overline{l_1}) = \overline{m}$. Therefore $\mathbf{g}_{\mathcal{F}}\overline{n} = \boldsymbol{\tau}_\mathsf{N}(\lambda y_0.\boldsymbol{\tau}_\mathsf{N}(\lambda y_1.\mathbf{g}_{\mathcal{I}}(y_0,y_1), \mathbf{g}_{\mathcal{G}_1}\overline{n}), \mathbf{g}_{\mathcal{G}_0}\overline{n}) = \overline{m}$ is provable in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$.

Conversely, if $\mathbf{g}_{\mathcal{F}}\overline{n} = \overline{m}$ is provable in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$, then by the axioms ($\boldsymbol{\tau}_\mathsf{N}$.2) and ($\boldsymbol{\tau}_\mathsf{N}$.1), $\mathbf{g}_{\mathcal{G}_i}\overline{n} \in \mathsf{N}$ is provable for $i < 2$. By the last lemma, there exist l_i for $i < 2$ such that $\mathbf{g}_{\mathcal{G}_i}\overline{n} = \overline{l_i}$ is provable. Then, by the induction hypothesis, $\mathcal{F}(n) = \mathcal{I}(l_0, l_1) = m$.

Next, we look at the case of minimization, namely (♯). If $\mathcal{F}(n) = m$, then $\mathcal{G}(n,m) = 0$ and $\mathcal{G}(n,k) > 0$ for $k < m$. By the induction hypothesis, $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ proves $\mathbf{g}_{\mathcal{G}}(\overline{n}, \overline{m}) = 0$ and $0 < \mathbf{g}_{\mathcal{G}}(\overline{n}, \overline{k})$ for any $k < m$. By Theorem 6(1), $\mathbf{min}(\lambda y.\mathbf{g}_{\mathcal{G}}(\overline{n}, y)) \in \mathsf{N}$ is provable. By the last lemma, there exists l such that $\mathbf{min}(\lambda y.\mathbf{g}_{\mathcal{G}}(\overline{n}, y)) = \overline{l}$ is provable. By Theorem 6(2),

$$\mathbf{g}_{\mathcal{G}}(\overline{n}, \overline{l}) = 0 \wedge (\forall y < \overline{l})(0 < \mathbf{g}_{\mathcal{G}}(\overline{n}, y))$$

is provable. Again by the induction hypothesis, $\mathcal{G}(n,l) = 0$. Therefore $m \leq l$. If $m < l$, then $0 < \mathbf{g}_{\mathcal{G}}(\overline{n}, \overline{m})$ is provable and by the induction hypothesis, $\mathcal{G}(n,m) > 0$, a contradiction. Thus $m = l$ and $\mathbf{g}_{\mathcal{F}}\overline{n} = \overline{m}$ is provable.

Conversely, if $\mathbf{g}_{\mathcal{F}}\overline{n} = \overline{m}$ is provable, $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ proves

$$\mathbf{g}_{\mathcal{G}}(\overline{n}, \overline{m}) = 0 \wedge (\forall y < \overline{m})(0 < \mathbf{g}_{\mathcal{G}}(\overline{n}, y))$$

in view of Theorem 6(2). By the induction hypothesis, $\mathcal{F}(n) = m$. ⊣

We close this section with the following easy lemma, which will be useful later. It asserts that subclasses of N that are represented as ranges of operations are *exactly* the projections of those represented as preimages of 0 under operations (cf. Definition 27).

Lemma 10. *In $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ we can prove:*

(1) $\forall g\ \exists f\ (\forall x \in \mathsf{N})((\exists z \in \mathsf{N})(fz = x) \leftrightarrow (\exists y \in \mathsf{N})(g(\mathbf{pair}(x,y)) = 0))$,

(2) $\forall f\ \exists g\ (\forall x \in \mathsf{N})((\exists y \in \mathsf{N})(fy = x) \leftrightarrow (\exists y \in \mathsf{N})(g(\mathbf{pair}(x,y)) = 0))$.

PROOF. For (1), given g, take $f := \lambda z.\boldsymbol{\tau}_{\mathsf{N}}(\lambda v.\mathbf{d}_{\mathsf{N}}(\mathbf{proj}_0(z), \mathbf{nt}_{\mathsf{N}}, v, 0), gz)$. It is easy to see $fz = x$ iff $gz = 0 \wedge \mathbf{proj}_0(z) = x$ for $x, z \in \mathsf{N}$.

For (2), given f, set $g := \lambda z.\boldsymbol{\tau}_{\mathsf{N}}(\lambda u.\mathbf{d}_{\mathsf{N}}(0, \mathbf{nt}_{\mathsf{N}}, u, \mathbf{proj}_0(z)), f(\mathbf{proj}_1(z)))$. For $x, y \in \mathsf{N}$, it is easy to see $fy = x$ iff $g(\mathbf{pair}(x, y)) = 0$. ⊣

§4. Undefinability of $\boldsymbol{\tau}_{\mathsf{N}}$ in BON^+.

We have seen how the truncation operator $\boldsymbol{\tau}_{\mathsf{N}}$ is used for a representation of the partial recursive functions within our operational framework. In this section, we prove that $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$ is not a definable extension of BON^+.

Our strategy is to show that there is no closed term s such that BON^+ proves

$$(\forall x \in \mathsf{N})(sx = 0) \wedge \forall x\ (sx \in \mathsf{N} \rightarrow x \in \mathsf{N}).$$

On the other hand such s is easily definable from $\boldsymbol{\tau}_{\mathsf{N}}$ by $\lambda x.\boldsymbol{\tau}_{\mathsf{N}}(\lambda y.0, x)$.

To show this fact and for further unprovability results in Section 6 we make use of semantic considerations, and thus begin with introducing some basic notions.

DEFINITION 11. An *operational structure* is a 4-tuple

$$\mathfrak{M} = (M, App, Nat, I)$$

with the following properties:

(1) M is a nonempty set, the so-called universe of $\mathfrak{M}$, App is a subset of M^3, unique in its last argument, and Nat is a subset of M.
(2) I is a mapping that assigns an element $I(r)$ of M to any constant r of the language L.

Furthermore, a *valuation* over this structure is a mapping J that assigns an element $J(u)$ of M to any individual variable u and a subset $J(U)$ of Nat to any relation variable U.

Given the operational structure $\mathfrak{M} = (M, App, Nat, I)$ and the valuation J over $\mathfrak{M}$, the value $\|r\|^J_{\mathfrak{M}}$ of a term r is inductively defined as follows. If r is an individual constant, then $\|r\|^J_{\mathfrak{M}} := I(r)$; if r is an individual variable, then $\|r\|^J_{\mathfrak{M}} := J(r)$. If r is the compound term st we have to distinguish a few cases:

(1) If $\|s\|^J_{\mathfrak{M}}$ and $\|t\|^J_{\mathfrak{M}}$ are elements of M and if there exists $m \in M$ such that $(\|s\|^J_{\mathfrak{M}}, \|t\|^J_{\mathfrak{M}}, m) \in App$, then this element m is uniquely determined, and we set $\|r\|^J_{\mathfrak{M}} := m$;
(2) If $\|s\|^J_{\mathfrak{M}}$ and $\|t\|^J_{\mathfrak{M}}$ are elements of M and if there exists no $m \in M$ such that $(\|s\|^J_{\mathfrak{M}}, \|t\|^J_{\mathfrak{M}}, m) \in App$, then $\|r\|^J_{\mathfrak{M}}$ is the value **undef**;
(3) If $\|s\|^J_{\mathfrak{M}}$ or $\|t\|^J_{\mathfrak{M}}$ is the value **undef**, then $\|r\|^J_{\mathfrak{M}}$ is the value **undef**.

Clearly, the value of a closed term does not depend on the valuation J and, therefore, we simply write $\|r\|_{\mathfrak{M}}$ for the value of the closed term r with respect to the operational structure $\mathfrak{M}$.

Similarly, the value $\|\varphi\|^J_{\mathfrak{M}}$ of an L formula φ with respect to the operational structure $\mathfrak{M} = (M, App, Nat, I)$ and the valuation J over $\mathfrak{M}$ is either $\mathbf{T}$ or $\mathbf{F}$. For atomic formulae we set:

(1) $\|t{\downarrow}\|^J_{\mathfrak{M}} := \mathbf{T}$ if $\|t\|^J_{\mathfrak{M}} \in M$, and $\|t{\downarrow}\|^J_{\mathfrak{M}} := \mathbf{F}$ if $\|t\|^J_{\mathfrak{M}}$ is the value **undef**;
(2) $\|s = t\|^J_{\mathfrak{M}} := \mathbf{T}$ if $\|s\|^J_{\mathfrak{M}} = \|t\|^J_{\mathfrak{M}} \in M$, and $\|s = t\|^J_{\mathfrak{M}} := \mathbf{F}$ if (at least) one of $\|s\|^J_{\mathfrak{M}}$ or $\|t\|^J_{\mathfrak{M}}$ is the value **undef** or if they are both in M but different;

(3) $\|\mathsf{N}(t)\|^J_{\mathfrak{M}} := \mathbf{T}$ if $\|t\|^J_{\mathfrak{M}} \in \mathit{Nat}$, and $\|\mathsf{N}(t)\|^J_{\mathfrak{M}} := \mathbf{F}$ if $\|t\|^J_{\mathfrak{M}}$ is the value **undef** or an element of $M \setminus \mathit{Nat}$;

(4) $\|U(t)\|^J_{\mathfrak{M}} := \mathbf{T}$ if $\|t\|^J_{\mathfrak{M}} \in J(U)$, and $\|U(t)\|^J_{\mathfrak{M}} := \mathbf{F}$ if $\|t\|^J_{\mathfrak{M}}$ is the value **undef** or an element of $M \setminus J(U)$.

Starting off from this treatment of the atomic formulae, the values of the compound formulae are introduced as usual. We say that an L formula φ is *valid* in the operational structure $\mathfrak{M}$, in symbols $\mathfrak{M} \models \varphi$, iff $\|\varphi\|^J_{\mathfrak{M}} = \mathbf{T}$ for all valuations J over this structure. Let T be the theory BON^+ or $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$. Then we call an operational structure $\mathfrak{M}$ a *model of* T iff all axioms of T are valid in $\mathfrak{M}$. Moreover we call it an ω-*model* if additionally $\mathit{Nat} = \{\|\overline{n}\|_{\mathfrak{M}} : n \in \omega\}$.

Recall that ω is the set of the standard natural numbers and in the following we write $\mathbb{N} = (\omega, \dots)$ for the standard model of Peano arithmetic PA. We may assume without loss of generality that any model $\mathcal{M} = (M, \dots)$ of PA is an extension of $\mathbb{N}$ and that ω is an initial segment of M.

Models of Peano arithmetic PA can be easily extended to operational structures. Let $\{e\}$ be an indexing of the partial recursive functions, keeping in mind that there exists a Σ_1 formula *Kleene* of the language of PA that defines $\{e\}(x) \simeq y$ in PA by $\mathit{Kleene}[e, x, y]$. If N is either the set ω or the set M, the N-*extension of* $\mathcal{M}$ is defined to be the operational structure

$$[\mathcal{M}, N] := (M, \mathit{App}_{\mathcal{M}}, N, I_\omega),$$

where $\mathit{App}_{\mathcal{M}}$ is defined to be the set

$$\{(e, x, y) \in M^3 : \mathcal{M} \models \mathit{Kleene}[e, x, y]\}$$

and I_ω is an arbitrary but fixed assignment of standard natural numbers to the constants of L such that the axioms of BON^+ are satisfied and any numeral $\overline{n}$ is interpreted as the natural number n; this is possible by ordinary recursion theory. Hence for any model $\mathcal{M} = (M, \dots)$ of PA, the structures $[\mathcal{M}, \omega]$ and $[\mathcal{M}, M]$ are models of BON^+. By the upward Σ_1 persistency, we have the following.

REMARK 12. If t is a closed term that is defined in $[\mathbb{N}, \omega]$, i.e., $\|t\|_{[\mathbb{N},\omega]} \in \omega$, then for any model $\mathcal{M} = (M, \dots)$ of PA,

$$\|t\|_{[\mathbb{N},\omega]} = \|t\|_{[\mathcal{M},\omega]} = \|t\|_{[\mathcal{M},M]}.$$

THEOREM 13. *There exists no closed term s such that* BON^+ *proves*

$$(\forall x \in \mathsf{N})(sx = 0) \land \forall x\, (sx \in \mathsf{N} \to x \in \mathsf{N}).$$

PROOF. For contradiction, let s be a closed term such that BON^+ proves

(i) $(\forall x \in \mathsf{N})(sx = 0)$,

(ii) $\forall x\, (sx \in \mathsf{N} \to x \in \mathsf{N})$.

Then we take any non-standard model $\mathcal{M} = (M, \dots)$ of Peano arithmetic and arbitrary $n \in M \setminus \omega$. In view of (i) and (ii) we thus have

(iii) $[\mathcal{M}, M] \models (\forall x \in \mathsf{N})(sx = 0)$,

(iv) $[\mathcal{M}, \omega] \models \forall x\, (sx \in \mathsf{N} \to x \in \mathsf{N})$.

From (iii) we conclude that $\mathit{Kleene}[m, n, 0]$ holds in $\mathcal{M}$ if m is the value of s in $[\mathcal{M}, M]$. Together with (iv) we thus obtain $n \in \omega$, a contradiction. ⊣

COROLLARY 14. *The operator $\boldsymbol{\tau}_{\mathsf{N}}$ is not definable in* BON^+.

This corollary can also be obtained by showing that another term cannot exist in BON^+, see Theorem 17 below. This result, or better the strategy to show it, is interesting in its own and proceeds as follows.

Given a model $\mathcal{M}$ of PA, we write $\mathbb{N} \prec_1 \mathcal{M}$ iff for every Σ_1 formula $\varphi[u]$ of the language of PA with at most u free and all $n \in \omega$,

$$\mathbb{N} \models \varphi[\widehat{n}] \iff \mathcal{M} \models \varphi[\widehat{n}].$$

Here, $\widehat{n}$ is the numeral in the sense of the language of PA corresponding to $n \in \omega$. We use this different notation in order to avoid confusing the numerals in the sense of BON^+ with those in the sense of PA. The following observation is logical folklore and will play a central role in the proof of Theorem 17.

LEMMA 15. *There exists a model $\mathcal{M}$ of Peano arithmetic* PA *with* $\mathbb{N} \not\prec_1 \mathcal{M}$.

PROOF. Assume that $\mathbb{N} \prec_1 \mathcal{M}$ for all models $\mathcal{M}$ of PA, and let φ be a Σ_1 sentence logically equivalent to $\neg \mathit{Con}(\mathsf{PA})$. From $\mathbb{N} \not\models \varphi$ we thus obtain that $\mathcal{M} \models \mathit{Con}(\mathsf{PA})$ for all models $\mathcal{M}$ of PA. By Gödel-Henkin's completeness this yields $\mathsf{PA} \vdash \mathit{Con}(\mathsf{PA})$, a contradiction. ⊣

There is a further well-known property of PA, dealing with formalized recursion theory, that will be used in the proof of Theorem 17.

LEMMA 16. *Let $\varphi[u,v]$ be a Δ_0 formula of the language of* PA *with at most u and v free. Then there exists a natural number e_φ such that* PA *proves*

$$\forall x\,\big(\exists y\;\varphi[x,y] \leftrightarrow \{\widehat{e_\varphi}\}(x)\!\downarrow\big) \wedge \forall x\,\big(\{\widehat{e_\varphi}\}(x)\!\downarrow \to \varphi[x,\{\widehat{e_\varphi}\}(x)]\big).$$

The proof of this lemma is by a straightforward formalization of a "search from below" argument.

THEOREM 17. *There exists no closed term t such that* BON^+ *proves*

$$(\flat) \qquad \forall f\;(tf\!\downarrow \wedge\, (\forall x \in \mathsf{N})(fx \in \mathsf{N} \leftrightarrow t(f,x) = 0)).$$

PROOF. We proceed indirectly and assume that BON^+ proves $(\flat)$ for a closed term t. By Lemma 15 take a model $\mathcal{M} = (M, \dots)$ of PA for which $\mathbb{N} \not\prec_1 \mathcal{M}$.

Now we pick an arbitrary Δ_0 formula $\varphi[u,v]$ of the language of PA with at most u, v free and choose $e_\varphi \in \omega$ according to the previous lemma such that

$$(1) \qquad \mathsf{PA} \vdash \forall x\,\big(\exists y\;\varphi[x,y] \leftrightarrow \{\widehat{e_\varphi}\}(x)\!\downarrow\big) \wedge \forall x\,\big(\{\widehat{e_\varphi}\}(x)\!\downarrow \to \varphi[x,\{\widehat{e_\varphi}\}(x)]\big).$$

In BON^+, $(\flat)$ implies $t\,\overline{e_\varphi}\!\downarrow$, hence also $t\!\downarrow$ by strictness. This implies that the value of t in $[\mathbb{N},\omega]$ is a natural number and that

$$\|t\|_{[\mathbb{N},\omega]} = \|t\|_{[\mathcal{M},\omega]} = \|t\|_{[\mathcal{M},M]}$$

according to Remark 12. From $\mathsf{BON}^+ \vdash t\,\overline{e_\varphi}\!\downarrow$ we also obtain that there exists a natural number m such that

$$[\mathbb{N},\omega] \models t\,\overline{e_\varphi} = \overline{m} \qquad \text{and} \qquad [\mathcal{M},M] \models t\,\overline{e_\varphi} = \overline{m}.$$

Since we assume the provability of $(\flat)$ in BON^+, this implies

$$(2) \qquad [\mathcal{M},\omega] \models (\forall x \in \mathsf{N})(\overline{e_\varphi}\,x \in \mathsf{N} \leftrightarrow \overline{m}\,x = 0),$$

$$(3) \qquad [\mathcal{M},M] \models (\forall x \in \mathsf{N})(\overline{e_\varphi}\,x \in \mathsf{N} \leftrightarrow \overline{m}\,x = 0).$$

For any $n \in \omega$ we have the following equivalences. The first ones are consequences of (1) and the interpretation of N in $[\mathcal{M}, M]$,

$$\mathcal{M} \models \exists y\ \varphi[\widehat{n}, y] \iff \mathcal{M} \models \{\widehat{e_\varphi}\}(\widehat{n})\downarrow \iff [\mathcal{M}, M] \models \overline{e_\varphi}\,\overline{n} \in \mathsf{N}. \tag{4}$$

Because of (3) we continue with

$$[\mathcal{M}, M] \models \overline{e_\varphi}\,\overline{n} \in \mathsf{N} \iff [\mathcal{M}, M] \models \overline{m}\,\overline{n} = 0. \tag{5}$$

Then the interpretation of the application in $[\mathcal{M}, M]$ and $[\mathcal{M}, \omega]$ yields

$$[\mathcal{M}, M] \models \overline{m}\,\overline{n} = 0 \iff \mathcal{M} \models \{\widehat{m}\}(\widehat{n}) = 0 \iff [\mathcal{M}, \omega] \models \overline{m}\,\overline{n} = 0. \tag{6}$$

Now we apply (2) and obtain

$$[\mathcal{M}, \omega] \models \overline{m}\,\overline{n} = 0 \iff [\mathcal{M}, \omega] \models \overline{e_\varphi}\,\overline{n} \in \mathsf{N}. \tag{7}$$

By the interpretations of N and the application in $[\mathcal{M}, \omega]$ we have

$$[\mathcal{M}, \omega] \models \overline{e_\varphi}\,\overline{n} \in \mathsf{N} \iff \mathcal{M} \models \{\widehat{e_\varphi}\}(\widehat{n}) = \widehat{k} \text{ for some } k \in \omega. \tag{8}$$

By (1) and the absoluteness of φ we further have

$$\mathcal{M} \models \{\widehat{e_\varphi}\}(\widehat{n}) = \widehat{k} \iff \mathcal{M} \models \varphi\Big[\widehat{n}, \widehat{k}\Big] \iff \mathbb{N} \models \varphi\Big[\widehat{n}, \widehat{k}\Big]$$

for any $k \in \omega$. Therefore, together with (8),

$$[\mathcal{M}, \omega] \models \overline{e_\varphi}\,\overline{n} \in \mathsf{N} \iff \mathbb{N} \models \exists y\ \varphi[\widehat{n}, y]. \tag{9}$$

Equivalences (4)–(9) thus give us

$$\mathcal{M} \models \exists y\ \varphi[\widehat{n}, y] \iff \mathbb{N} \models \exists y\ \varphi[\widehat{n}, y]$$

for all $n \in \omega$. Since $\varphi[u, v]$ has been an arbitrary Δ_0 formula of the language of PA, this is a contradiction to $\mathbb{N} \not\prec_1 \mathcal{M}$. ⊣

This also shows the undefinability of $\boldsymbol{\tau}_\mathsf{N}$ in BON^+ since $t := \lambda f x.r_2(fx)$, where r_2 is from Lemma 31 below, cannot exist in BON^+ according to this theorem.

§5. ω-models of $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$. For the standard recursion-theoretic operational structure $[\mathbb{N}, \omega]$ with ω as universe and application $ab \simeq c$ interpreted as $\{a\}(b) \simeq c$ we can easily validate the $\boldsymbol{\tau}_\mathsf{N}$-axioms: Simply interpret $\boldsymbol{\tau}_\mathsf{N}$ as the identity operation. Hence $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\textsc{Tot}\text{-}\mathsf{N})$ is clearly consistent, and thus the $\boldsymbol{\tau}_\mathsf{N}$-axioms are justified with respect to this standard model of BON^+, even under the additional assumption that all individuals are natural numbers.

In order to make a point that $\boldsymbol{\tau}_\mathsf{N}$ is a natural operator, we also look at further typical operational ω-models: the canonical term model as well as two variants of Kleene's second model and of the graph model, respectively. In this article we confine ourselves to some basic definitions and results. A detailed analysis of these structures will be given in Rosebrock's forthcoming dissertation [17].

Before seeing the particular ones, we summarize general results for ω-models.

THEOREM 18. *Let $\mathfrak{M} = (M, \circ, Nat, I)$ be an ω-model of $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ and $S \subseteq \omega$.*

(1) *If S is Σ^0_1, there is $f \in M$ with $\mathfrak{M} \models f \in Char_2$ such that for all $m \in \omega$,*

$$m \in S \iff (f \circ \|\overline{m}\|_\mathfrak{M}) \circ \|\overline{n}\|_\mathfrak{M} = \big\|\overline{0}\big\|_\mathfrak{M} \text{ for some } n \in \omega.$$

(2) *The following are equivalent:*

- *There is* $f \in M$ *such that for all* $m \in \omega$,

$$m \in S \iff f \circ (\|\mathbf{pair}(\overline{m}, \overline{n})\|_{\mathfrak{M}}) = \|\overline{0}\|_{\mathfrak{M}} \text{ for some } n \in \omega.$$

- *There is* $g \in M$ *such that for all* $m \in \omega$,

$$m \in S \iff g \circ \|\overline{n}\|_{\mathfrak{M}} = \|\overline{m}\|_{\mathfrak{M}} \text{ for some } n \in \omega.$$

PROOF. For (1), let S be Σ^0_1. There is a total recursive function $\mathcal{F}$ such that $S = \{m \in \omega : (\exists n \in \omega)(\mathcal{F}(m,n) = 0)\}$ and $\mathcal{F}(m,n) \in \{0,1\}$ for any $m, n \in \omega$. Then $f := \mathbf{g}_{\mathcal{F}}$ from Theorem 9 is what is required, since $\mathfrak{M}$ is an ω-model.

(2) follows immediately from Lemma 10. ⊣

5.1. The canonical term model. We begin with the canonical term model. On the closed terms a binary relation *conv* is introduced such that we have $conv(r,s)$ if and only if there exist closed terms t_0, t_1, t_2 as well as different natural numbers m and n for which one of the cases in Table 1 holds. If we have $conv(r,s)$ then r is called a redex and s the contractum of r.

Let $\approx$ be the smallest congruence relation, with respect to application, on the collection of all closed terms that contains *conv*. Thus, given any closed term r there exists the equivalence class $[r]$ of r modulo $\approx$.

Now we write $|\mathfrak{CT}|$ for the collection of all equivalence classes of the closed terms and define an application relation $\cdot^{\mathfrak{CT}}$ on $|\mathfrak{CT}|$ by setting, for all closed terms r and s,

$$[r] \cdot^{\mathfrak{CT}} [s] := [rs].$$

DEFINITION 19. The *operational term structure* is the 4-tuple

$$\mathfrak{CT} = \big(|\mathfrak{CT}|, \cdot^{\mathfrak{CT}}, \{[\overline{n}] : n \in \omega\}, I^{\mathfrak{CT}}\big),$$

where $I^{\mathfrak{CT}}(r) = [r]$ for every constant r of L.

Essentially by exploiting the confluence property it is shown in Rosebrock [17] that $\mathfrak{CT}$ is a model of $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}}) + (\text{TOT-AP})$. We obtain also the second part of the following theorem, where the essence of its proof is

$$conv(r,s) \implies \mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}}) \vdash r{\downarrow} \to r = s,$$

but $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}}) \vdash s{\downarrow} \to r = s$ does not follow from $conv(r,s)$ in general.

r is of the form	s is of the form
$\mathbf{k}(t_0, t_1)$	t_0
$\mathbf{s}(t_0, t_1, t_2)$	$t_0(t_2, t_1 t_2)$
$\mathbf{p}_0(\mathbf{p}(t_0, t_1))$	t_0
$\mathbf{p}_1(\mathbf{p}(t_0, t_1))$	t_1
$\mathbf{p}_{\mathsf{N}} 0$	0
$\mathbf{p}_{\mathsf{N}}(\mathbf{s}_{\mathsf{N}} t_0)$	t_0
$\mathbf{d}_{\mathsf{N}}(t_0, t_1, \overline{m}, \overline{m})$	t_0
$\mathbf{d}_{\mathsf{N}}(t_0, t_1, \overline{m}, \overline{n})$ with $m \neq n$	t_1
$\boldsymbol{\tau}_{\mathsf{N}}(t_0, \overline{n})$	$t_0 \overline{n}$

TABLE 1. The relation $conv(r,s)$

THEOREM 20. *$\mathfrak{CT}$ is an ω-model of* $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\text{TOT-AP})$. *In addition, for all closed terms r and s,*

$$r \approx s \implies \mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) \vdash (r\downarrow \wedge\ s\downarrow) \to r = s.$$

Hence, if $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ proves $t \in \mathsf{N}$ for some closed term t, then there exists an $n \in \omega$ such that $t \approx \overline{n}$. Therefore Lemma 8 immediately follows.

5.2. Two variants of Kleene's second model. Kleene's second model provides a further interesting approach to constructing a model of a partial combinatory algebra; see, e.g., Beeson [2, VI.7.4] and Troelstra and van Dalen [19, 9.9.2]. First we have to introduce some notations.

In the following we will make use of the standard primitive recursive coding machinery: $\langle m_0, \dots, m_{n-1}\rangle$ stands for the primitive recursively formed n-tuple of the natural numbers $m_0, \dots, m_{n-1}$ and $*$ is the primitive recursive concatenation of the finite sequences, i.e.,

$$\langle m_0, \dots, m_{i-1}\rangle * \langle n_0, \dots, n_{j-1}\rangle = \langle m_0, \dots, m_{i-1}, n_0, \dots, n_{j-1}\rangle.$$

If $\mathcal{F}$ is a function from ω to ω and n a natural number, then we write $\mathcal{F}\restriction n$ for the code of the initial segment of $\mathcal{F}$ up to $n-1$, i.e.,

$$\mathcal{F}\restriction n := \langle \mathcal{F}(0), \dots, \mathcal{F}(n-1)\rangle.$$

Finally, if $\mathcal{F}$ and $\mathcal{G}$ are functions from ω to ω then $\mathcal{F} \bullet \mathcal{G}$ is the possibly partial function from ω to ω that is defined as follows:

$$(\mathcal{F} \bullet \mathcal{G})(n) := \begin{cases} \mathcal{F}(\langle n\rangle * \mathcal{G}\restriction m) - 1 & \text{if } m \text{ is minimal with } \mathcal{F}(\langle n\rangle * \mathcal{G}\restriction m) > 0, \\ \text{undefined} & \text{if there is no such } m. \end{cases}$$

On functions from ω to ω we define a partial application relation by

$$\mathcal{F}|\mathcal{G} := \begin{cases} \mathcal{F} \bullet \mathcal{G} & \text{if } \mathcal{F} \bullet \mathcal{G} \text{ is a total function from } \omega \text{ to } \omega, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Note that $\mathcal{F} \bullet \mathcal{G}$ is Σ^0_1 definable relative to $\mathcal{F}$ and $\mathcal{G}$, and that the definedness of $\mathcal{F}|\mathcal{G}$ is a Π^0_2 statement on $\mathcal{F}$ and $\mathcal{G}$.

In the following we denote the collection of total recursive functions by *TRec*. Because of the Σ^0_1 definability, if $\mathcal{F}, \mathcal{G} \in \mathit{TRec}$ and $\mathcal{F}|\mathcal{G}$ exists then $\mathcal{F}|\mathcal{G} \in \mathit{TRec}$.

The following lemma is easily proved by Kleene's normal form theorem. Together with the definition of $|$, it characterizes the functionals on Baire space ω^ω described by this operation. It is not difficult to generalize it to the characterization of multi-argument functionals.

LEMMA 21. *We have the following results about the existence of specific functions:*

(1) *Any partial continuous functional on Baire space ω^ω whose domain is a G_δ set can be expressed as $\mathcal{G} \mapsto \mathcal{F}|\mathcal{G}$ for some total function $\mathcal{F}$.*
(2) *For Σ^0_1 formulae $\varphi[n, \mathcal{G}]$ and $\psi[n, m, \mathcal{G}]$ without other parameters such that,*
for any $\mathcal{G} \in \omega^\omega$ if $(\forall n \in \omega)\varphi[n, \mathcal{G}]$ then $(\forall n \in \omega)(\exists m \in \omega)\psi[n, m, \mathcal{G}]$,
there exists $\mathcal{F} \in \mathit{TRec}$ such that for any total function $\mathcal{G}$ from ω to ω,

$$\begin{aligned} \mathcal{F}|\mathcal{G} \text{ is defined } &\iff \varphi[n, \mathcal{G}] \text{ for any } n \in \omega \\ &\implies \psi[n, (\mathcal{F}|\mathcal{G})(n), \mathcal{G}] \text{ for any } n \in \omega. \end{aligned}$$

Note that (1) follows from the relativized version of (2). Thus (1) is a boldface version of (2). While the *lightface* version (2) is proved in Nemoto and Sato [15, 3.22(1)] (applied to $\varphi[n,\mathcal{G}]\wedge\psi[n,m,\mathcal{G}]$), the boldface one seems more popular in the literature; see, e.g., Rin and Walsh [16, 3.3] and Longley and Normann [11, 12.2.2]. This explains why the operations based on $|$ are sometimes called *partial continuous*, e.g., in Troelstra and van Dalen [19, 9.4.1].

C_n is written for the constant function with value n and

$$Con := \{C_n : n \in \omega\}.$$

In the structures below this is the interpretation of the predicate N. It is known that the structures are models of BON^+ (see, e.g., Beeson [2, VI.7.4.1, 7.5.1]) and we can easily extend them to those of $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ by using Lemma 21(2). The details will be shown in Rosebrock [17].

Theorem 22 (Bold- and Lightface Kleene's Second Model). *There exists an interpretation I of the constants of L in TRec such that the operational structures*

$$\mathfrak{B}\mathfrak{K}_2 = (\omega^\omega, |, Con, I) \qquad \text{and} \qquad \mathfrak{L}\mathfrak{K}_2 = (TRec, |, Con, I)$$

are ω-models of $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ and that $\|\overline{n}\|_{\mathfrak{L}\mathfrak{K}_2} = C_n$ for any $n \in \omega$.

Despite the popularity of the boldface $\mathfrak{B}\mathfrak{K}_2$, later we will need the following result, which is specific to the lightface $\mathfrak{L}\mathfrak{K}_2$.

Theorem 23. *For every subset S of ω we have the following equivalences:*

(1) *S is Σ^0_1 iff there is $\mathcal{F} \in TRec$ with $\mathfrak{L}\mathfrak{K}_2 \models \mathcal{F} \in Char_2$ such that for all $m \in \omega$,*

$$m \in S \iff (\mathcal{F}|C_m)|C_n = C_0 \text{ for some } n \in \omega.$$

(2) *S is Π^0_2 iff there exists $\mathcal{F} \in TRec$ such that for all $m \in \omega$,*

$$m \in S \iff \mathcal{F}|C_m = C_0.$$

(3) *S is Σ^0_3 iff there exists $\mathcal{F} \in TRec$ such that for all $m \in \omega$,*

$$m \in S \iff \mathcal{F}|C_n = C_m \text{ for some } n \in \omega.$$

Proof. (1) The "only-if" part is by Theorem 18(1). For the "if" part, note that $(\mathcal{F}|C_m)|C_n = C_0$ iff $((\mathcal{F} \bullet C_m) \bullet C_n)(0) = 0$ by $\mathfrak{L}\mathfrak{K}_2 \models \mathcal{F} \in Char_2$.
(2) The "if" part is obvious. Let $S = \{n \in \omega : (\forall m \in \omega)\theta[m,n]\}$ with θ being Σ^0_1. Lemma 21(2) with $\varphi[n,\mathcal{G}] \equiv \theta[n,\mathcal{G}(0)]$ and $\psi[n,m,\mathcal{G}] \equiv m = 0$ yields the required $\mathcal{F} \in TRec$.
(3) Theorem 18(2) asserts that S satisfies the latter condition iff S is a projection of a set satisfying the latter condition of (2). Hence (2) yields the statement. ⊣

5.3. Two variants of graph model. The so-called graph model for the untyped lambda calculus was discovered independently by Engeler, Plotkin, and Scott.

The universes of our variants are included in the power set $\mathrm{Pow}(\omega)$ of ω. To define the application relation, we let $(e_n : n \in \omega)$ be a standard enumeration of all finite sets of natural numbers. For arbitrary $P, Q \subseteq \omega$ we then set

$$P \cdot^{\mathfrak{G}} Q := \{m \in \omega : \langle n, m\rangle \in P \text{ and } e_n \subseteq Q \text{ for some } n \in \omega\}.$$

Clearly, this application is total on $\mathrm{Pow}(\omega)$ and the class Σ^0_1 is closed under it.

The next lemma is analogous to Lemma 21. Now $\mathrm{Pow}(\omega)$ is equipped with the so-called Scott topology, and must not be confused with Cantor space 2^ω.

LEMMA 24. *We have the following results about the existence of specific sets:*

(1) *Any continuous functional from the Scott domain* $\mathrm{Pow}(\omega)$ *to* $\mathrm{Pow}(\omega)$ *can be expressed as* $Q \mapsto P \cdot^{\mathfrak{G}} Q$ *for some* $P \subseteq \omega$.

(2) *For any* Σ^0_1 *formula* $\theta[n, Q]$ *in which* Q *occurs only positively, there exists a* Σ^0_1 *subset* P *of* ω *such that, for any* $Q \subseteq \omega$,

$$P \cdot^{\mathfrak{G}} Q = \{n \in \omega : \theta[n, Q]\}.$$

Similarly to Lemma 21, (1) follows from the relativization of (2). As the Scott continuity is equivalent to the positive Σ^0_1 definability with set parameters, we can see the contrast between boldface and lightface again. While the boldface version (1) seems more popular (e.g., Barendregt [1, 18.1.8.(ii)], and Rin and Walsh [16, 3.6]), we can prove (2) easily by the Σ^0_1 normal form theorem in second order arithmetic with a modification for the positiveness of the set variables.

The natural numbers are represented by the singletons of ω, and we set

$$\mathit{Sing} := \{\{m\} : m \in \omega\}.$$

It is shown in, e.g., Beeson [2, VI.7.2.4, 7.5.2] that the following structures are models of BON^+ and we can easily extend them to those of $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ by using Lemma 24(2). The details of this result will also be shown in Rosebrock [17].

THEOREM 25 (Bold- and Lightface Graph Model). *There exists an interpretation* I *of the constants of* L *in* $\mathrm{Pow}(\omega) \cap \Sigma^0_1$ *such that the operational structures*

$$\mathfrak{BG} = \big(\mathrm{Pow}(\omega), \cdot^{\mathfrak{G}}, \mathit{Sing}, I\big) \qquad \text{and} \qquad \mathfrak{LG} = \big(\mathrm{Pow}(\omega) \cap \Sigma^0_1, \cdot^{\mathfrak{G}}, \mathit{Sing}, I\big)$$

are ω*-models of* $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\textsc{Tot-Ap})$ *and that* $\|\overline{n}\|_{\mathfrak{LG}} = \{n\}$ *for any* $n \in \omega$.

We can also have an analogue of Theorem 23 as follows. $\Sigma^0_1 \wedge \Pi^0_1$ denotes the class consisting of intersections of Σ^0_1 sets and Π^0_1 sets. This class must not be confused with $\Delta^0_1 = \Sigma^0_1 \cap \Pi^0_1$, the intersection of the classes Σ^0_1 and Π^0_1.

THEOREM 26. *For every subset* S *of* ω *we have the following equivalences.*

(1) S *is* Σ^0_1 *iff there is* $P \in \Sigma^0_1$ *with* $\mathfrak{LG} \models P \in \mathit{Char}_2$ *such that for all* $m \in \omega$,

$$m \in S \iff \big(P \cdot^{\mathfrak{G}} \{m\}\big) \cdot^{\mathfrak{G}} \{n\} = \{0\} \text{ for some } n \in \omega.$$

(2) S *is* $\Sigma^0_1 \wedge \Pi^0_1$ *iff there exists* $P \in \Sigma^0_1$ *such that for all* $m \in \omega$,

$$m \in S \iff P \cdot^{\mathfrak{G}} \{m\} = \{0\}.$$

(3) S *is* Σ^0_2 *iff there exists* $P \in \Sigma^0_1$ *such that for all* $m \in \omega$,

$$m \in S \iff P \cdot^{\mathfrak{G}} \{n\} = \{m\} \text{ for some } n \in \omega.$$

PROOF. (1) The "only-if" part is by Theorem 18(1). For the "if" part, note that $\big(P \cdot^{\mathfrak{G}} \{m\}\big) \cdot^{\mathfrak{G}} \{n\} = \{0\}$ iff $\big(P \cdot^{\mathfrak{G}} \{m\}\big) \cdot^{\mathfrak{G}} \{n\} \ni 0$ by $\mathfrak{LG} \models P \in \mathit{Char}_2$.
(2) For the 'if' part, the latter condition is equivalent to

$$\big(0 \in P \cdot^{\mathfrak{G}} \{m\}\big) \wedge (\forall k \in \omega)\big(k \in P \cdot^{\mathfrak{G}} \{m\} \to k = 0\big).$$

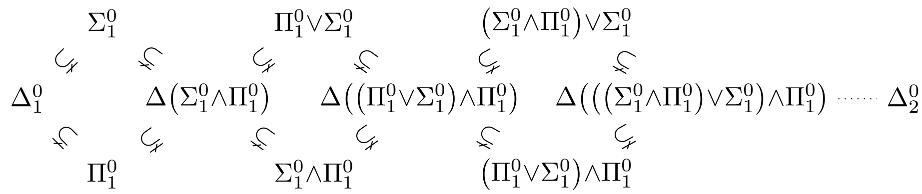

FIGURE 1. Semi-Recursive Difference Hierarchy

For the converse, let $S = \{m \in \omega : \varphi[m] \wedge (\forall k \in \omega)\psi[m,k]\}$ with φ and ψ being Σ^0_1 and Δ^0_0 respectively. Lemma 24(2) yields P such that

$$P \cdot^{\mathfrak{G}} \{m\} = \{0 : \varphi[m]\} \cup \{k+1 : \neg\psi[m,k]\} \text{ for any } m \in \omega$$

by $\theta[n, Q] \;\equiv\; (\exists m \in Q)((n = 0 \wedge \varphi[m]) \vee (n > 0 \wedge \neg\psi[m, n-1]))$.
(3) Similar to Lemma 23(3), for projections of $\Sigma^0_1{\wedge}\Pi^0_1$ sets are exactly Σ^0_2 sets. ⊣

As the class $\Sigma^0_1{\wedge}\Pi^0_1$ is not so popular as the classes Σ^0_n, a short remark seems to be justified. Since the elements are of the form $R \setminus S$ with R and S being Σ^0_1, it is the second level of the lightface analogue of *Hausdorff-Kuratowski difference hierarchy*. The corresponding boldface class is denoted by $D_2(\boldsymbol{\Sigma}^0_1)$ in Louveau [12, 1.1], $(\boldsymbol{\Sigma}^0_1)_2$ in Nemoto [14], and would be by 2-$\boldsymbol{\Sigma}^0_1$ in the notation of Kanamori [10, §31] and $\boldsymbol{\Sigma}^0_{1,2}$ in that of Montalbán and Shore [13, 2.4]. Note however that they consider classes of subsets of Baire space ω^ω or Cantor space 2^ω, whereas we consider classes of subsets of ω. Even so, we can define a similar hierarchy by defining $\Pi^0_1{\vee}\Sigma^0_1$, $(\Sigma^0_1{\wedge}\Pi^0_1){\vee}\Sigma^0_1$ and so on in the obvious way. From a universal Σ^0_1 set, we can define universal sets for these classes. This yields the strictness of the hierarchy similarly to the arithmetical hierarchy as in Figure 1, where $\Delta(\Sigma^0_1{\wedge}\Pi^0_1)$ denotes $(\Sigma^0_1{\wedge}\Pi^0_1) \cap (\Pi^0_1{\vee}\Sigma^0_1)$ and so on. In particular, $\Sigma^0_1{\wedge}\Pi^0_1$ is properly between Σ^0_1 and Δ^0_2.

§6. Operational semi-decidability and the like. Section 2 explains the role of $\boldsymbol{\tau}_{\mathsf{N}}$ for formalizing the basic parts of recursion theory within $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$. According to this, any partial recursive function is represented as a partial operator on N and moreover the basic closure properties of the structure of all partial recursive functions are also formalized as those of partial operators on N. Therefore we could say that it also formalizes the structure of the partial recursive functions relative to some class of functions. In this sense, we could consider partial operations on N as "generalized" partial recursive functions.

Now let us go further with this paradigm, to the recursion-theoretic notions for sets of natural numbers. It is natural in our paradigm to call U *operationally decidable* iff there exists an operation f with

$$f \in \mathit{Char} \;\wedge\; (\forall x \in \mathsf{N})(x \in U \;\leftrightarrow\; fx = 0).$$

Correspondingly, we call U *operationally semi-decidable* iff there is f with

$$(\forall x \in \mathsf{N})(x \in U \;\leftrightarrow\; fx = 0).$$

So the totality requirement is dropped in the case of semi-decidability.

In ordinary recursion theory a subset of ω is decidable iff the set itself and its complement are semi-decidable. As we will see below, this is not the case in our paradigm. Moreover, in ordinary recursion theory there are many (equivalent) ways how semi-decidability can be defined, but operationally the situation is more complex. The second part of the following definition lists some of the possible "standard" definitions of semi-decidability, tailored for our present context. Afterwards we will say more about their relationships.

Definition 27. Given any U, we use the following abbreviations to express that U is operationally decidable, semi-decidable, a projection of an operationally decidable set, a domain of an operation, a range of an operation or operationally enumerable:

$$\begin{aligned}
OD[U] &:= (\exists f \in \mathit{Char})(\forall x \in \mathsf{N})(x \in U \leftrightarrow fx = 0),\\
OSD[U] &:= \exists f\ (\forall x \in \mathsf{N})(x \in U \leftrightarrow fx = 0),\\
Pr[U] &:= (\exists f \in \mathit{Char}_2)(\forall x \in \mathsf{N})(x \in U \leftrightarrow (\exists y \in \mathsf{N})(f(x,y) = 0)),\\
Dom[U] &:= \exists f\ (\forall x \in \mathsf{N})(x \in U \leftrightarrow fx \in \mathsf{N}),\\
Rng[U] &:= \exists f\ (\forall x \in \mathsf{N})(x \in U \leftrightarrow (\exists y \in \mathsf{N})(x = fy)),\\
OE[U] &:= U = \emptyset \vee (\exists f \in (\mathsf{N} \to \mathsf{N}))(\forall x \in \mathsf{N})(x \in U \leftrightarrow (\exists y \in \mathsf{N})(x = fy)).
\end{aligned}$$

The notions $OD[\mathsf{N}\backslash U]$, $OSD[\mathsf{N}\backslash U]$, $Pr[\mathsf{N}\backslash U]$, ... are defined accordingly.

We begin with the more or less obvious relationship between these notions.

Theorem 28. *In* BON^+ *we can prove:*

(1) $OD[U] \to OD[\mathsf{N}\backslash U]$,

(2) $OD[U] \to Pr[U]$.

Proof. We assume $f \in \mathit{Char}$ and $(\forall x \in \mathsf{N})(x \in U \leftrightarrow fx = 0)$, and we set $r := \lambda u.\mathbf{d}_{\mathsf{N}}(\overline{1}, 0, fu, 0)$. Then $r \in \mathit{Char}$ and, for any $x \in \mathsf{N}$, $x \in \mathsf{N}\backslash U$ iff $rx = 0$. Hence we have (1). Furthermore, for $s := \lambda uv.fu$ we have $s \in \mathit{Char}_2$ and, for any $x \in \mathsf{N}$, $x \in U$ iff $(\exists y \in \mathsf{N})(s(x,y) = 0)$; thus we also have (2). ⊣

Theorem 29. *In* BON^+ *we can prove*

$$OD[U] \leftrightarrow (Pr[U] \wedge Pr[\mathsf{N}\backslash U]).$$

Proof. By the previous theorem, the direction from left to right is obvious. For the converse, let $f, g \in \mathit{Char}_2$ be such that

$$x \in U \leftrightarrow (\exists y \in \mathsf{N})(f(x,y) = 0) \quad \text{and} \quad x \in \mathsf{N}\backslash U \leftrightarrow (\exists y \in \mathsf{N})(g(x,y) = 0)$$

for all $x \in \mathsf{N}$. Now we define

$$r := \lambda uv.\mathbf{d}_{\mathsf{N}}(0, g(u,v), f(u,v), 0) \qquad \text{and} \qquad s := \lambda u.f(u, \mathbf{min}_0(ru)).$$

Clearly, $rx \in \mathit{Char}$ and $(\exists y \in \mathsf{N})(r(x,y) = 0)$ for all $x \in \mathsf{N}$. Applying Theorem 4, we see $\mathbf{min}_0(rx) \in \mathsf{N}$ if $x \in \mathsf{N}$. This implies $s \in \mathit{Char}$. Assume now $x \in \mathsf{N}$ and $sx = 0$. Then $f(x, \mathbf{min}_0(rx)) = 0$. Thus, $x \in U$. Conversely, if $x \in \mathsf{N}$ with $sx = \overline{1}$, we conclude $0 = r(x, \mathbf{min}_0(rx)) = g(x, \mathbf{min}_0(rx))$ by Theorem 4 again. Hence, $x \in \mathsf{N}\backslash U$. We have shown $(\forall x \in \mathsf{N})(x \in U \leftrightarrow sx = 0)$. ⊣

THEOREM 30. *In* BON^+ *we can prove*

$$Pr[U] \leftrightarrow OE[U].$$

PROOF. The equivalence is clearly provable for $U = \emptyset$. So let us assume $a \in U$. To show the direction from left to right we assume

$$(\natural) \qquad (\forall x \in \mathsf{N})(x \in U \leftrightarrow (\exists y \in \mathsf{N})(f(x,y) = 0))$$

for some $f \in Char_2$ and set

$$r := \lambda u.\mathbf{d}_{\mathsf{N}}(\mathbf{proj}_0(u), a, f(\mathbf{proj}_0(u), \mathbf{proj}_1(u)), 0).$$

$r \in (\mathsf{N} \to \mathsf{N})$ is clear, and it remains to show that, for all $x \in \mathsf{N}$,

$$x \in U \leftrightarrow (\exists y \in \mathsf{N})(x = ry).$$

Given $x \in U$, the equivalence $(\natural)$ yields $f(x,y) = 0$ for some $y \in \mathsf{N}$. Hence $r(\mathbf{pair}(x,y)) = x$, and thus $(\exists z \in \mathsf{N})(x = rz)$. Conversely, if $x = rz$ for some $z \in \mathsf{N}$, then $x = a$ or $x = \mathbf{proj}_0(z) \wedge f(\mathbf{proj}_0(z), \mathbf{proj}_1(z)) = 0$. In both cases we have $x \in U$.

Turning to the direction from right to left of our theorem, assume $OE[U]$, say $g \in (\mathsf{N} \to \mathsf{N})$ with

$$(\forall x \in \mathsf{N})(x \in U \leftrightarrow (\exists y \in \mathsf{N})(x = gy)).$$

Set $s := \lambda uv.\mathbf{d}_{\mathsf{N}}\big(0, \overline{1}, u, gv\big)$. Clearly, $s \in Char_2$ and

$$(\forall x \in \mathsf{N})(x \in U \leftrightarrow (\exists y \in \mathsf{N})(s(x,y) = 0)).$$

But this implies $Pr[U]$, as we had to show. ⊣

LEMMA 31. *There exist closed terms* r_1, r_2, r_3 *such that:*

(1) $\mathsf{BON}^+ \vdash r_1 0 = 0 \wedge (\forall x \in \mathsf{N})(x \neq 0 \to r_1 x \notin \mathsf{N})$,
(2) $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}}) \vdash \forall x\,(r_2 x = 0 \leftrightarrow x \in \mathsf{N})$,
(3) $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}}) \vdash \forall x\,(r_3 x \in \mathsf{N} \leftrightarrow x = 0)$.

PROOF. Let $\mathbf{nt}_{\mathsf{N}}$ be the closed term introduced in the proof of Theorem 6. Then BON^+ proves $\mathbf{nt}_{\mathsf{N}}\!\downarrow$ and $\mathbf{nt}_{\mathsf{N}}(0) \notin \mathsf{N}$. For

$$r_1 := \lambda x.\mathbf{d}_{\mathsf{N}}(\lambda u.0, \mathbf{nt}_{\mathsf{N}}, x, 0)0 \qquad \text{and} \qquad r_2 := \lambda x.\boldsymbol{\tau}_{\mathsf{N}}(\lambda u.0, x),$$

(1) and (2) are immediately proved. For (3) consider

$$r_3 := \lambda x.\boldsymbol{\tau}_{\mathsf{N}}(r_1, x).$$

Then $x = 0$ implies $r_3 x = 0 \in \mathsf{N}$. Conversely $r_3 x \in \mathsf{N}$ yields $x \in \mathsf{N}$. Hence $r_1 x = r_3 x \in \mathsf{N}$ and thus $x = 0$. ⊣

The existence of the two closed terms r_2 and r_3 according to the previous lemma is also the core of the proof of (1) in the following theorem. As shown in Theorem 33, the converse directions of (2) and (3) do not hold in $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$.

THEOREM 32. *In* $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$ *we can prove:*

(1) $OSD[U] \leftrightarrow Dom[U]$,
(2) $Dom[U] \to Rng[U]$,
(3) $Pr[U] \to Dom[U]$.

Proof. (1) is easy by r_2 and r_3 from Lemma 31. For (2), assume $Dom[U]$. Then (1) tells us $OSD[U]$, i.e., there exists f with

$$(\forall x \in \mathsf{N})(x \in U \leftrightarrow fx = 0).$$

Now we make use of Lemma 10. Set $t := \lambda u.f(\mathbf{proj}_0 u)$. Then obviously we have $(\forall x \in \mathsf{N})(x \in U \leftrightarrow (\exists y \in \mathsf{N})(t(\mathbf{pair}(x, y)) = 0))$. By Lemma 10(1), there exists g such that $(\forall x \in \mathsf{N})(x \in U \leftrightarrow (\exists z \in \mathsf{N})(gz = x))$, implying $Rng[U]$.

Turning to (3), if $Pr[U]$ then there exists $f \in Char_2$ with

$$(\forall x \in \mathsf{N})(x \in U \leftrightarrow (\exists y \in \mathsf{N})(f(x, y) = 0)).$$

We first observe $fx \in (\mathsf{N} \to \mathsf{N})$ for all $x \in \mathsf{N}$ and set $t := \lambda u.\mathbf{min}(fu)$. Hence Theorem 6 gives us, for all $x \in \mathsf{N}$, that $tx \in \mathsf{N}$ iff $(\exists y \in \mathsf{N})(f(x, y) = 0)$. Consequently, U is the domain of t. ⊣

Now we turn to some unprovability results that we directly obtain from the complexity results in connection with our lightface models; see Theorems 23 and 26. In what follows, given a set S of natural numbers and an operational structure $\mathfrak{M}$, we write $S^{\mathfrak{M}}$ for $\{\|\overline{n}\|_{\mathfrak{M}} : n \in S\}$.

Theorem 33. *The following are* <u>*not*</u> *provable in* $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\text{Tot-Ap})$*:*

(1) $Dom[U] \to Pr[U]$,
(2) $Rng[U] \to Dom[U]$.

Proof. To show the unprovability of (1), choose a universal Π^0_1 set R of the natural numbers. R is $\Sigma^0_1 \wedge \Pi^0_1$ but not Σ^0_1. In view of Theorems 26 and 32(1), the lightface graph model $\mathfrak{LG}$ satisfies $Dom\big[R^{\mathfrak{LG}}\big]$ but not $\mathrm{Pr}\big[R^{\mathfrak{LG}}\big]$.

For establishing the unprovability of (2), we pick a universal Σ^0_2 set S of the natural numbers. S is Σ^0_2 but not $\Sigma^0_1 \wedge \Pi^0_1$. Then Theorem 26 tells us that $\mathfrak{LG}$ satisfies $Rng\big[S^{\mathfrak{LG}}\big]$ but not $Dom\big[S^{\mathfrak{LG}}\big]$. ⊣

Theorem 23 also gives us similar unprovability results but on $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ or on $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + \neg(\text{Tot-N}) + \neg(\text{Tot-Ap})$.

We summarize the results of Theorems 28, 30, 32, and 33 (together with an obvious observation) in Figure 2, where a black arrow means the provability of the corresponding implication in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ while a crossed arrow represents the unprovability in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\text{Tot-Ap})$. It depicts the interdependencies of our (semi-)decidability notions, relative to the theory $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$.

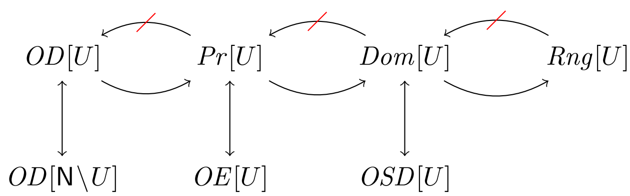

Figure 2. Summary of our main results

Theorem 29 naturally leads us to be interested also in the "decidability notions", $Pr[U] \wedge Pr[\mathsf{N}\backslash U]$, $Dom[U] \wedge Dom[\mathsf{N}\backslash U]$ and $Rng[U] \wedge Rng[\mathsf{N}\backslash U]$. By Theorem 26 with help of Figure 1, we can similarly obtain the unprovability of the respective implications between them corresponding to Theorem 33, and moreover the following unprovability of implications from the "decidabilities" to the "semi-decidabilities" (while the others of this type, e.g., $Pr[U] \wedge Pr[\mathsf{N}\backslash U] \to Dom[U]$, are obviously provable in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$, because of, e.g., Theorem 32(3)). In particular, in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ or even in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\text{Tot-Ap})$ we do not have that a relation on the natural numbers is operationally decidable iff the relation and its complement in the natural numbers are operationally semi-decidable.

THEOREM 34. *The following are* <u>*not*</u> *provable in* $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\text{Tot-Ap})$*:*

(1) $Dom[U] \wedge Dom[\mathsf{N}\backslash U] \to Pr[U]$,
(2) $Rng[U] \wedge Rng[\mathsf{N}\backslash U] \to Dom[U]$,
(3) $Rng[U] \wedge Rng[\mathsf{N}\backslash U] \to Pr[U]$.

How about the converses of the implications in Theorem 34? Trivially, they are all false in the ordinary recursion-theoretic operational structure $[\mathbb{N}, \omega]$ or Kleene's first model, and hence not provable in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$. However this does not show the unprovabilities on $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\text{Tot-Ap})$. Theorems 23 and 26 do not show these unprovabilities, either. For this, we need the analogous results for the canonical term model $\mathfrak{CT}$, as follows.

LEMMA 35. *For any $S \subseteq \omega$, the following equivalences hold:*

$$S \text{ is } \Sigma^0_1 \iff \mathfrak{CT} \models Pr\big[S^{\mathfrak{CT}}\big] \iff \mathfrak{CT} \models Dom\big[S^{\mathfrak{CT}}\big] \iff \mathfrak{CT} \models Rng\big[S^{\mathfrak{CT}}\big].$$

PROOF. Theorem 18(1) yields the first $\Longrightarrow$ and Theorem 32(3)(2) yield the second and the third. It remains to imply that S is Σ^0_1 from $\mathfrak{CT} \models Rng\big[S^{\mathfrak{CT}}\big]$.

Code the closed terms by Gödel numbering. Then the relation *conv* is Δ^0_0 and its congruent closure $\approx$ is Σ^0_1. Thus $\{m \in \omega : (\exists n \in \omega)(t\,\overline{n} \approx \overline{m})\}$ is Σ^0_1. ⊣

THEOREM 36. *The following are* <u>*not*</u> *provable in* $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\text{Tot-Ap})$*:*

(1) $Pr[U] \to Dom[U] \wedge Dom[\mathsf{N}\backslash U]$,
(2) $Dom[U] \to Rng[U] \wedge Rng[\mathsf{N}\backslash U]$,
(3) $Pr[U] \to Rng[U] \wedge Rng[\mathsf{N}\backslash U]$.

Table 2 summarizes our complexity results, namely Theorems 23 and 26 and Lemma 35. It characterizes the subsets S of ω for which $S^{\mathfrak{M}}$ satisfies the semi-decidability notions in our four ω-models $\mathfrak{M}$ of $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$.

model / notion	Kleene's first $[\mathbb{N}, \omega]$	canonical term $\mathfrak{CT}$	graph $\mathfrak{LG}$	Kleene's second $\mathfrak{LK}_2$
Pr, OE	Σ^0_1	Σ^0_1	Σ^0_1	Σ^0_1
Dom, OSD	Σ^0_1	Σ^0_1	$\Sigma^0_1 \wedge \Pi^0_1$	Π^0_2
Rng	Σ^0_1	Σ^0_1	Σ^0_2	Σ^0_3

TABLE 2. Corresponding complexities to our semi-decidability notions

The complexity results yield further interesting consequences. First, $\Sigma^0_1 \wedge \Pi^0_1$ is not closed under projections nor under binary unions (for otherwise $(\Sigma^0_1 \wedge \Pi^0_1) \vee \Sigma^0_1$ would be included in $\Sigma^0_1 \wedge \Pi^0_1$). Therefore, for example, $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\text{TOT-AP})$ cannot prove the closure of *OSD* under binary unions, formalized as

$$(\forall x \in \mathsf{N})(x \in W \leftrightarrow x \in U \vee x \in V) \wedge \mathit{OSD}[U] \wedge \mathit{OSD}[V] \rightarrow \mathit{OSD}[W].$$

Second, since Π^0_2 is known not to have the reduction property (see, e.g., Hinman [7, III.1.10(ii)]), $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ cannot prove the operational form of the reduction property for *OSD*. (Also, since Σ^0_1 does not have the separation property, $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\text{TOT-AP})$ cannot prove that of the separation property for *OSD*.) These results suggest that, despite its simple definition, the notion *OSD* as we defined is not a right operational formalization of semi-decidability, since it does not satisfy the basic properties which we expect from the word “semi-decidable”.

We conclude this article with some open questions.

QUESTION 37. It might be interesting to ask

- if $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ proves the operational form of the reduction property of *Rng*;
- how to characterize a many-one degree Γ for which there is an ω-model $\mathfrak{M}$ of $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ such that $S \in \Gamma$ iff $\mathfrak{M} \models \mathit{OSD}\big[S^{\mathfrak{M}}\big]$ for any $S \subseteq \omega$;
- if $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\text{TOT-}\mathsf{N})$ can prove the implications in Theorems 33 and 34.

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