Intuitionistic common knowledge or belief

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Abstract

Starting off from the usual language of modal logic for multi-agent systems dealing with the agents' knowledge/belief and common knowledge/belief we define so-called epistemic Kripke structures for intuitionistic (common) knowledge/belief. Then we introduce corresponding deductive systems and show that they are sound and complete with respect to these semantics.
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1 Introduction

Common knowledge (and in particular the modal logic approach to common knowledge) has received a lot of attention in recent years; see, e.g., the textbooks Fagin, Halpern, Moses, and Vardi [4] and Meyer and van der Hoek [11] and the article [16] on common knowledge in the Stanford Encyclopedia of Philosophy by Sillari and Vanderschraaf. In these texts the general landscape of common knowledge is described and sound as well as complete formalizations of common knowledge in a multi-agent scenario are presented. Examples of proof-theoretic work on modal systems for common knowledge are Alberucci and Jäger [1], Brünnler and Studer [3], Jäger, Kretz, and Studer [7], Kretz and Studer [8], and Lescanne [9].

However, all these approaches are embedded in a framework of classical (multi-)modal logic. On the other hand, there is also the interesting – though not so popular – world of intuitionistic modal logics. First important results, including the completeness proof for the logic **IK**, are presented in Fischer Servi [5], and Simpson [15] provides an excellent survey of intuitionistic modal logics. It also leads to present research in this area.

In this article we start off from the traditional approach to common knowledge, but couched into an intuitionistic base logic. We present the system **ICK** for intuitionistic common knowledge and show that it is sound and complete. Hence this work is a technical contribution concerning an, as we think, natural system for dealing with common knowledge from an intuitionistic perspective.

We do not enter into the discussion what the "right" intuitionistic epistemic logic is. There has been an interesting recent proposal by Artemov and Protopopescu in [2], but there are also alternative approaches by Williamson [17], Hirai [6], Proietti [14], and several others. This indicates that intuitionistic epistemic logic with and without common knowledge is an interesting area of ongoing research. But more work and a deeper conceptual analysis is necessary, and we hope that we will come back to this topic in a future publication.

Our formalism starts off from a framework for intuitionistic modal logic presented in Fischer Servi [5] and Plotkin and Stirling [13] and discussed from a broader perspective in Simpson [15]. We extend its $\Box$-fragment to several agents and treat common knowledge as a greatest fixed point, as it is common in epistemic logic; see, for example, Fagin, Halpern, Moses, and Vardi [4] or Meyer and van der Hoek [11]. More details about the relationship between our semantics and standard approaches in the literature are given at the end of Section 2.

The corresponding deductive systems are presented as sequent calculi, simply taking the intuitionistic variants of those in Alberucci and Jäger [1]. Their soundness with respect to our semantics will be obvious and their completeness will be shown in Section 4. There is nothing specific about this choice, the use of sequent calculi is a matter of personal taste rather than logical necessity. Equally well we could have adapted the Hilbert calculi of [5, 13, 15] to intuitionistic common knowledge.

2 The language $\mathcal{L}_{CK}$ and its semantics

In this section we introduce our language $\mathcal{L}_{CK}$ for intuitionistic common knowledge/belief and interpret its formulas over so-called epistemic Kripke structures, thus also providing a semantic approach to intuitionistic common knowledge/belief. The next section is dedicated to corresponding deductive systems.

The general assumption is that we want to deal with ℓ agents $a_1, \ldots, a_\ell$. To formally express that agent a_i knows or believes α, we will write $\mathsf{K}_i(\alpha)$, and $\mathsf{C}(\alpha)$ says that α is common knowledge or common belief. Hence the language $\mathcal{L}_{CK}$ comprises the following primitive symbols:

PS.1 Countably many atomic propositions p, q, r (possibly with subscripts); the collection of all atomic propositions is called $PROP$.

PS.2 The logical constant $\bot$ and the logical connectives $\vee$, $\wedge$, and $\to$.

PS.3 The modal operators $\mathsf{K}_1, \ldots, \mathsf{K}_\ell, \mathsf{C}$.

The *formulas* $\alpha, \beta, \gamma, \delta$ (possibly with subscripts) of $\mathcal{L}_{CK}$ are generated by the following BNF:

$$\alpha \ ::\equiv \ p \ | \ \bot \ | \ (\alpha \vee \alpha) \ | \ (\alpha \wedge \alpha) \ | \ (\alpha \to \alpha) \ | \ \mathsf{K}_i(\alpha) \ | \ \mathsf{C}(\alpha)$$

We make use of the standard syntactic abbreviations, for example, $\neg\alpha := (\alpha \to \bot)$ and $(\alpha \leftrightarrow \beta) := ((\alpha \to \beta) \wedge (\beta \to \alpha))$ and often omit parentheses and brackets if there is no danger of confusion. In addition, we abbreviate

$$\mathsf{E}(\alpha) \ := \ \mathsf{K}_1(\alpha) \wedge \ldots \wedge \mathsf{K}_\ell(\alpha)$$

in order to express that "everybody knows α" or "everybody believes α", depending on what the $\mathsf{K}_i(\alpha)$ are supposed to formalize.

In the classical setting we have for all operators K_i the necessitation rule

$$\frac{\alpha}{\mathsf{K}_i(\alpha)}$$

and the normality axiom

$$\mathsf{K}_i(\alpha \to \beta) \ \to \ (\mathsf{K}_i(\alpha) \to \mathsf{K}_i(\beta)).$$

Depending on what we want to formalize further axioms can be added, for example,

truth property: $\mathsf{K}_i(\alpha) \to \alpha$,

positive introspection: $\mathsf{K}_i(\alpha) \to \mathsf{K}_i(\mathsf{K}_i(\alpha))$,

negative introspection: $\neg\mathsf{K}_i(\alpha) \to \mathsf{K}_i(\neg\mathsf{K}_i(\alpha))$.

Sometimes it is argued that interpreting $\mathsf{K}_i(\alpha)$ as "agent a_i knows α" at least requires the presence of the truth axioms and possibly positive as well as negative introspection; otherwise $\mathsf{K}_i(\alpha)$ should be seen as stating that agent a_i only believes α. In this case often the

consistency property: $\mathsf{K}_i(\bot) \to \bot$

is considered; it is also known under the name of axiom (D). However, since we are primarily interested in technical questions, we do not make this distinction and speak of knowledge and common knowledge to simplify matters.

In the traditional approach, put forward, for example, in Fagin, Halpern, Moses, and Vardi [4], common knowledge $\mathsf{C}(\alpha)$ of α is interpreted as the infinite conjunction $\bigwedge\{\mathsf{E}^n(\alpha) : n \geq 1\}$ of the iterations of everybody knows α, where $\mathsf{E}^0(\alpha) := \alpha$ and $\mathsf{E}^{n+1}(\alpha) := \mathsf{E}(\mathsf{E}^n(\alpha))$. This is a perfect semantic characterization of common knowledge, but there is a problem: Since our language is finite, we cannot have an axiom of the form

$$\mathsf{C}(\alpha) \;\leftrightarrow\; \bigwedge_{n\geq 1} \mathsf{E}^n(\alpha).$$

To overcome this complication, $\mathsf{C}(\alpha)$ is typically axiomatized via the fixed point characterization

$$\mathsf{C}(\alpha) \;\leftrightarrow\; \mathsf{E}(\alpha \wedge \mathsf{C}(\alpha)),$$

more precisely as the greatest fixed point of this equivalence. It is easy to see that semantically both approaches agree. Again we refer to Fagin, Halpern, Moses, and Vardi [4], Meyer and van der Hoek [11], or Sillari and Vanderschraaf [16] for further reading.

In the intuitionistic case we can proceed similarly. But first we have to fix the adequate structures over which the formulas of $\mathcal{L}_{CK}$ will be interpreted.

Definition 1. *An* epistemic Kripke structure (EK-structure for short) *is an $(\ell+3)$-tuple $\mathfrak{M} = (W, \preceq, f_1, \ldots, f_\ell, V)$ such that*

(EK.1) W is a nonempty set and $\preceq$ is a preorder on W.

(EK.2) Every f_i for $1 \leq i \leq \ell$ is a function from W to the power set of W such that for any $v, w \in W$,

$$v \preceq w \quad\Longrightarrow\quad f_i(w) \subseteq f_i(v).$$

(EK.3) V is a function from W to the power set of PROP such that for any $v, w \in W$,

$$v \preceq w \quad\Longrightarrow\quad V(v) \subseteq V(w).$$

$\mathfrak{M}$ is called a reflexive EK-structure *iff $v \in f_i(v)$ for all $v \in W$ and all i with $1 \leq i \leq \ell$.*

(EK.1) and (EK.3) are the usual properties of a Kripke structure for intuitionistic propositional logic. Given the EK-structure

$$\mathfrak{M} = (W, \preceq, f_1, \ldots, f_\ell, V),$$

the functions f_i assign to any world w the collection $f_i(w)$ of worlds that are accessible for agent a_i from world w and take care of the modal operators

K_i (see below) in the following sense: Agent a_i "knows" α in world w – i.e. $\mathsf{K}_i(\alpha)$ holds in world w – iff α holds in all worlds v that are accessible from w via f_i. Then (EK.2) guarantees monotonicity; if agent a_i "knows" α at world w, then also in all worlds v that satisfy $w \preceq v$. For the treatment of common knowledge it is convenient to introduce the notion of reachability within $\mathfrak{M}$.

Definition 2. *Assume that $\mathfrak{M} = (W, \preceq, f_1, \ldots, f_\ell, V)$ is an EK-structure and n a natural number.*

1. *We say that there exists an $\mathfrak{M}$-path of length n from world $v \in W$ to world $w \in W$ – written $Path_{\mathfrak{M}}(v, w, n)$ – iff there exist $u_0, \ldots, u_n \in W$ such that $v = u_0$, $w = u_n$, and $u_{i+1} \in \bigcup\{f_j(u_i) : 1 \leq j \leq \ell\}$ for $i = 0, \ldots, n-1$.*

2. *$Reach_{\mathfrak{M}}(v, n)$ and $Reach_{\mathfrak{M}}(v)$ are defined to be the collections of all elements of W that are reachable from $v \in W$ by an $\mathfrak{M}$-path of length n and any $\mathfrak{M}$-path, respectively,*

$$Reach_{\mathfrak{M}}(v, n) := \{u \in W : Path_{\mathfrak{M}}(v, u, n)\},$$

$$Reach_{\mathfrak{M}}(v) := \bigcup_{m \geq 1} Reach_{\mathfrak{M}}(v, m).$$

Given an EK-structure $\mathfrak{M} = (W, \preceq, f_1, \ldots, f_\ell, V)$, the set $\|\alpha\|_{\mathfrak{M}}$ of worlds satisfying an $\mathcal{L}_{CK}$ formula α is now defined as follows:

$$\|\bot\|_{\mathfrak{M}} := \varnothing,$$

$$\|p\|_{\mathfrak{M}} := \{v \in W : p \in V(v)\} \text{ for any } p \in PROP,$$

$$\|\alpha \vee \beta\|_{\mathfrak{M}} := \|\alpha\|_{\mathfrak{M}} \cup \|\beta\|_{\mathfrak{M}},$$

$$\|\alpha \wedge \beta\|_{\mathfrak{M}} := \|\alpha\|_{\mathfrak{M}} \cap \|\beta\|_{\mathfrak{M}},$$

$$\|\alpha \to \beta\|_{\mathfrak{M}} := \{v \in W : \{w \in W : v \preceq w\} \cap \|\alpha\|_{\mathfrak{M}} \subseteq \|\beta\|_{\mathfrak{M}}\},$$

$$\|\mathsf{K}_i(\alpha)\|_{\mathfrak{M}} := \{v \in W : f_i(v) \subseteq \|\alpha\|_{\mathfrak{M}}\},$$

$$\|\mathsf{C}(\alpha)\|_{\mathfrak{M}} := \{v \in W : Reach_{\mathfrak{M}}(v) \subseteq \|\alpha\|_{\mathfrak{M}}\}.$$

A simple proof by induction on the structure of α shows that the sets $\|\alpha\|_{\mathfrak{M}}$ satisfy the usual monotonicity condition of intuitionistic logic.

Lemma 3. *For all EK-structures $\mathfrak{M} = (W, \preceq, f_1, \ldots, f_\ell, V)$, all elements $v, w \in W$, and all $\mathcal{L}_{CK}$ formulas α we have that*

$$v \preceq w \text{ and } v \in \|\alpha\|_{\mathfrak{M}} \implies w \in \|\alpha\|_{\mathfrak{M}}.$$

The following lemma tells us that also over our (intuitionistic) EK-structures common knowledge is handled as in the case of classical logic. The proof of this lemma is as in the classical case and left to the reader.

Lemma 4. *For all EK-structures $\mathfrak{M} = (W, \preceq, f_1, \ldots, f_\ell, V)$, all $v \in W$, and all natural numbers n we have:*

1. $v \in \|\mathsf{E}^n(\alpha)\|_{\mathfrak{M}} \quad \Longleftrightarrow \quad \mathit{Reach}_{\mathfrak{M}}(v, n) \subseteq \|\alpha\|_{\mathfrak{M}}$.

2. $\|\mathsf{C}(\alpha)\|_{\mathfrak{M}} \; = \; \bigcap_{m \geq 1} \|\mathsf{E}^m(\alpha)\|_{\mathfrak{M}}$.

3. *Consider the operator $\mathcal{O}_\alpha$ from the power set of W to the power set of W defined by*

$$\mathcal{O}_\alpha(X) \; := \; \|\mathsf{E}(\alpha)\|_{\mathfrak{M}} \cap \{w \in W : \bigcup_{i=1}^{\ell} f_i(w) \subseteq X\}$$

for all $X \subseteq W$. Then $\|\mathsf{C}(\alpha)\|_{\mathfrak{M}}$ is the greatest fixed point of $\mathcal{O}_\alpha$.

As usual, we call an $\mathcal{L}_{CK}$ formula α *valid in the EK-structure* $\mathfrak{M}$ iff $W \subseteq \|\alpha\|_{\mathfrak{M}}$ for the universe W of $\mathfrak{M}$. Accordingly, the *EK-valid* formulas are those $\mathcal{L}_{CK}$ formulas that are valid in all EK–structures. The question now is whether there exists a deductive system that proves exactly the EK-valid formulas.

We end this section with comparing our semantics to some common approaches in the literature, in particular that of Fischer Servi, Plotkin and Stirling, and Simpson. Since common knowledge is not treated there, we confine us to C-free $\mathcal{L}_{CK}$ formulas for this comparison.

A first though not important difference is that we use functions $f_1, \ldots, f_\ell$ to encode accessibility of worlds in Kripke models. Often this is done by using accessibility relations $R_1, \ldots, R_\ell$, and then $f_i(w)$ corresponds to

$$R_i[w] := \{v \in W : (w, v) \in R_i\}.$$

Furthermore, these authors impose certain restrictions on their frames to deal with the interplay between $\Box$- and $\Diamond$-formulas. Since we work in multi-agent versions of the $\Box$-fragment, we do not need these frame conditions.

Intuitionistic logic requires monotonicity, and in Fischer Servi [5], Plotkin and Stirling [13], and Simpson [15] this is done by building it into the truth definition. To make this distinction precise, we call an $(\ell + 3)$-tuple

$$\mathfrak{M} \; = \; (W, \preceq, f_1, \ldots, f_\ell, V)$$

a *pseudo EK-structure* iff it satisfies the properties (EK.1) and (EK.3) of an EK-structure; i.e. the monotonicity condition (EK.2) is dropped. Hence pseudo EK-structures correspond to the structures considered in [5, 13, 15].

For a pseudo EK-structure $\mathfrak{M} = (W, \preceq, f_1, \ldots, f_\ell, V)$, the set of worlds $\|\alpha\|^*_{\mathfrak{M}}$ satisfying a C-free $\mathcal{L}_{CK}$ formula α in the sense of Fischer Servi, Plotkin and Stirling, and Simpson is inductively defined in analogy to $\|\alpha\|_{\mathfrak{M}}$ for EK-structures on page 5 with the only difference that now

$$\|\mathsf{K}_i(\alpha)\|^*_{\mathfrak{M}} \;:=\; \{v \in W : \bigcup_{v \preceq w} f_i(w) \subseteq \|\alpha\|^*_{\mathfrak{M}}\}.$$

The following observation, which is proved by straightforward induction on α, shows that the definitions of $\|\alpha\|_{\mathfrak{M}}$ and $\|\alpha\|^*_{\mathfrak{M}}$ agree in the case of EK-structures.

Lemma 5. *For all EK-structures $\mathfrak{M}$ and all C-free $\mathcal{L}_{CK}$ formulas α we have that $\|\alpha\|_{\mathfrak{M}} = \|\alpha\|^*_{\mathfrak{M}}$.*

Clearly, there are pseudo EK-structures that are no EK-structures. However, it easy to transform them into EK-structures that validate the same formulas.

Lemma 6. *Given the pseudo EK-structure $\mathfrak{M} = (W, \preceq, f_1, \ldots, f_\ell, V)$, its* completion *is the structure*

$$\overline{\mathfrak{M}} \;:=\; (W, \preceq, \overline{f}_1, \ldots, \overline{f}_\ell, V) \quad \textit{with} \quad \overline{f}_i(w) \;:=\; \bigcup_{w \preceq v} f_i(v)$$

for all $i = 1, \ldots, \ell$ and $w \in W$. Then $\overline{\mathfrak{M}}$ is an EK-structure and we have $\|\alpha\|_{\overline{\mathfrak{M}}} = \|\alpha\|^*_{\mathfrak{M}}$ for all C-free $\mathcal{L}_{CK}$ formulas α.

That $\overline{\mathfrak{M}}$ is an EK-structure is obvious; the second assertion is established by induction on α.

Let us call an $\mathcal{L}_{CK}$ formulas α *valid in the pseudo EK-structure* $\mathfrak{M}$ iff $W \subseteq \|\alpha\|^*_{\mathfrak{M}}$ for the universe W of $\mathfrak{M}$. Also, we say that α is *pseudo-EK-valid* iff α is valid in all pseudo EK-structures. It is an immediate consequence of the previous two lemmas that EK-validity and pseudo-EK-validity agree.

Corollary 7. *Any C-free $\mathcal{L}_{CK}$ formula α is EK-valid if and only if it is pseudo-EK-valid.*

Since pseudo-EK-validity is the same as validity à la Fischer Servi, Plotkin and Stirling, and Simpson for $\mathcal{L}_{CK}$ formulas without the operator C for common knowledge we see that our semantics for intuitionistic common knowledge builds on established semantic concepts.

Finally, a word about Ono's semantics studied in [12]. He only considers models where the accessibility relation is reflexive and transitive. His so-called I models of type 0 correspond to reflexive and transitive EK-structures (with only one agent).

3 The deductive systems ICK and ICKT

There exist numerous formalisms for intuitionistic modal logic, ranging from Hilbert-style systems to sequent calculi and frameworks dealing with nested sequents. A series of those is presented in, e.g., Simpson [15] and Marin and Straßburger [10].

In the following we choose the general framework of a sequent calculus for intuitionistic propositional logic, equipped with rules for the knowledge operators and the operator for common knowledge. However, to use a sequent-style approach is not important; Hilbert-style formalizations would have been equally suitable. It is more a matter of taste than necessity. Also, it allows us to simply take the intuitionistic version of Alberucci and Jäger [1].

The capital Greek letters Σ, Φ, Ψ (possibly with subscripts) denote finite sets of $\mathcal{L}_{CK}$ formulas. We often write (for example) $\Sigma, \Phi, \alpha, \beta$ for the union $\Sigma \cup \Phi \cup \{\alpha, \beta\}$. Expressions of the form $\Sigma \supset \alpha$ are called sequents. In addition, if Σ is the set $\{\alpha_1, \ldots, \alpha_n\}$, we use the following abbreviations:

$$\begin{aligned} \mathsf{K}_i(\Sigma) &:= \{\mathsf{K}_i(\alpha_1), \ldots, \mathsf{K}_i(\alpha_n)\}, \\ \mathsf{C}(\Sigma) &:= \{\mathsf{C}(\alpha_1), \ldots, \mathsf{C}(\alpha_n)\}, \\ d(\Sigma) &:= (\ldots(\alpha_1 \vee \alpha_2) \vee \ldots \alpha_n), \\ c(\Sigma) &:= (\ldots(\alpha_1 \wedge \alpha_2) \wedge \ldots \alpha_n). \end{aligned}$$

Clearly, if Σ is the empty set, then $\mathsf{K}_i(\Sigma)$ and $\mathsf{C}(\Sigma)$ are empty as well and $d(\Sigma) := \bot$ and $c(\Sigma) := \neg\bot$. The deductive system **ICK** proves sequents and comprises the following axioms and rules of inference.

Axioms of ICK.

$$\Sigma, \bot \supset \bot \qquad \Sigma, \bot \supset p \qquad \Sigma, p \supset p$$

Cut rules of ICK.

$$\frac{\Sigma \supset \alpha \qquad \Sigma, \alpha \supset \beta}{\Sigma \supset \beta}$$

α is called the *cut formula* of this cut.

Propositional rules of ICK.

$$\frac{\Sigma, \alpha \supset \gamma \qquad \Sigma, \beta \supset \gamma}{\Sigma, \alpha \vee \beta \supset \gamma} \qquad \frac{\Sigma \supset \alpha}{\Sigma \supset \alpha \vee \beta} \qquad \frac{\Sigma \supset \beta}{\Sigma \supset \alpha \vee \beta}$$

$$\frac{\Sigma, \alpha, \beta \supset \gamma}{\Sigma, \alpha \wedge \beta \supset \gamma} \qquad \frac{\Sigma \supset \alpha \qquad \Sigma \supset \beta}{\Sigma \supset \alpha \wedge \beta}$$

$$\frac{\Sigma \supset \alpha \qquad \Sigma, \beta \supset \gamma}{\Sigma, \alpha \rightarrow \beta \supset \gamma} \qquad \frac{\Sigma, \alpha \supset \beta}{\Sigma \supset \alpha \rightarrow \beta}$$

K **rules of ICK.**

$$\frac{\Sigma, \mathsf{C}(\Phi) \supset \alpha}{\mathsf{K}_i(\Sigma), \mathsf{C}(\Phi), \Psi \supset \mathsf{K}_i(\alpha)}$$

C **rules of ICK.**

$$\frac{\Sigma, \mathsf{E}(\alpha) \supset \beta}{\Sigma, \mathsf{C}(\alpha) \supset \beta} \qquad \frac{\mathsf{C}(\Sigma) \supset \mathsf{E}(\alpha)}{\mathsf{C}(\Sigma), \Psi \supset \mathsf{C}(\alpha)}$$

Induction rules of ICK.

$$\frac{\mathsf{C}(\Sigma), \beta \supset \mathsf{E}(\alpha) \qquad \mathsf{C}(\Sigma), \beta \supset \mathsf{E}(\beta)}{\mathsf{C}(\Sigma), \Psi, \beta \supset \mathsf{C}(\alpha)}$$

In the K rules, the second C rule, and in the induction rules the set Ψ is added in the conclusion to have closure under weakening. In case of the axioms and the other rules, weakening is clear.

ICKT is the extension of **ICK** by the truth rules

$$\frac{\Sigma, \alpha \supset \beta}{\Sigma, \mathsf{K}_i(\alpha) \supset \beta}$$

that are the sequent-style versions of the truth property $\mathsf{K}_i(\alpha) \rightarrow \alpha$ mentioned above.

We write **ICK** $\vdash \Sigma \supset \alpha$ iff the sequent $\Sigma \supset \alpha$ is derivable in **ICK** in the usual sense; then we also say that **ICK** proves $\Sigma \supset \alpha$. **ICK** $\vdash \alpha$ is short for **ICK** $\vdash \varnothing \supset \alpha$. Moreover, if M is any set of $\mathcal{L}_{CK}$ formulas, then $M \vdash_{\mathbf{ICK}} \alpha$ says that there exists a finite subset Σ of M such that **ICK** $\vdash \Sigma \supset \alpha$. **ICKT** $\vdash \Sigma \supset \alpha$, **ICKT** $\vdash \alpha$, and $M \vdash_{\mathbf{ICKT}} \alpha$ are defined accordingly.

In the following we do not present detailed derivations within **ICK** or **ICKT** and work freely in these systems; it is assumed that the reader has some familiarity with such calculi. The following theorem list a series of important properties of **ICK** and **ICKT**; their proofs are completely standard.

Theorem 8. *Let* $\mathbf{ICK}^\bullet$ *be the system* **ICK** *or the system* **ICKT**. *For all* $\mathcal{L}_{CK}$ *formulas* α, β *and all sets* M, N *of* $\mathcal{L}_{CK}$ *formulas we have:*

1. $\mathbf{ICK}^\bullet \vdash \alpha \supset \alpha$ *and* $\mathbf{ICK}^\bullet \vdash \bot \supset \alpha$.
2. $\mathbf{ICK}^\bullet \vdash \mathsf{K}_i(\alpha), \mathsf{K}_i(\alpha \to \beta) \supset \mathsf{K}_i(\beta)$.
3. $\mathbf{ICK}^\bullet \vdash \alpha \implies \mathbf{ICK}^\bullet \vdash \mathsf{K}_i(\alpha) \wedge \mathsf{C}(\alpha)$.
4. $\mathbf{ICK}^\bullet \vdash \mathsf{E}(\alpha) \wedge \mathsf{E}(\mathsf{C}(\alpha)) \supset \mathsf{C}(\alpha)$.
5. $\mathbf{ICK}^\bullet \vdash \beta \supset \mathsf{E}(\alpha) \wedge \mathsf{E}(\beta) \implies \mathbf{ICK}^\bullet \vdash \beta \supset \mathsf{C}(\alpha)$.
6. $\mathbf{ICK}^\bullet \vdash (\mathsf{E}(\alpha) \wedge \mathsf{E}(\mathsf{C}(\alpha))) \leftrightarrow \mathsf{C}(\alpha)$.
7. $\mathbf{ICK}^\bullet \vdash \beta \to (\mathsf{E}(\alpha) \wedge \mathsf{E}(\beta)) \implies \mathbf{ICK}^\bullet \vdash \beta \to \mathsf{C}(\alpha)$.
8. $M \subseteq N$ *and* $M \vdash_{\mathbf{ICK}^\bullet} \alpha \implies N \vdash_{\mathbf{ICK}^\bullet} \alpha$.
9. $\mathbf{ICKT} \vdash \mathsf{K}_i(\alpha) \supset \alpha$.

ICK does in general not prove the truth property, positive, or negative introspection. However, whenever wanted or needed a simple modification of our calculus can accommodate for those.

The soundness of **ICK** with respect to EK-structures is easy to check. To do so, pick an arbitrary EK-structure $\mathfrak{M}$ and convince yourself by simple calculations that

(i) all axioms of **ICK** are valid in $\mathfrak{M}$,

(ii) the $\mathcal{L}_{CK}$ formulas valid in $\mathfrak{M}$ are closed under the rules of inference of **ICK**.

Then induction on the length of the derivations in **ICK** yields the following result. Analogously for **ICKT** and reflexive EK-structures.

Theorem 9 (Soundness of **ICK** and **ICKT**). *Let* $\mathfrak{M} = (W, \preceq, f_1, \ldots, f_\ell, V)$ *be an EK-structure.*

1. *If* **ICK** *proves* $\Sigma \supset \alpha$, *then* $c(\Sigma) \to \alpha$ *is valid in* $\mathfrak{M}$.
2. *If* **ICKT** *proves* $\Sigma \supset \alpha$ *and* $\mathfrak{M}$ *is reflexive, then* $c(\Sigma) \to \alpha$ *is valid in* $\mathfrak{M}$.

4 Completeness of ICK and ICKT

In this section we show that **ICK** is complete with respect to EK-structures and **ICKT** is complete with respect to reflexive EK-structures. Our approach is an adaptation of the completeness proof presented in Fagin, Halpern, Moses, and Vardi [4] for a system of classical common knowledge.

Until the end of this section we fix an $\mathcal{L}_{CK}$ formula α and build the so-called canonical model with respect to α.

Definition 10. *The fragment $M(\alpha)$ is the collection of all $\mathcal{L}_{CK}$ formulas that is inductively generated as follows:*

(M.1) $\alpha, \bot \in M(\alpha)$.

(M.2) If $\beta \in M(\alpha)$, then all subformulas of β belong to $M(\alpha)$.

(M.3) If $\mathsf{C}(\beta) \in M(\alpha)$, then $\mathsf{E}(\beta)$ and $\mathsf{E}(\mathsf{C}(\beta))$ belong to $M(\alpha)$.

It is checked immediately that $M(\alpha)$ is a finite set. The most important ingredients of our canonical model with respect to α are the α-prime sets of formulas.

Definition 11 (α-prime)**.** *A set N of $\mathcal{L}_{CK}$ formulas is called* α-prime *iff it satisfies the following conditions:*

(P.1) $N \subseteq M(\alpha)$.

(P.2) $\beta \in M(\alpha)$ *and* $N \vdash_{\mathbf{ICK}} \beta \quad \Longrightarrow \quad \beta \in N$.

(P.3) $\beta \vee \gamma \in N \quad \Longrightarrow \quad \beta \in N$ *or* $\gamma \in N$.

(P.4) $\bot \notin N$.

The α-prime sets of $\mathcal{L}_{CK}$ formulas will form the worlds of the canonical model depending on α. Crucial for this model construction is the following property of α-prime sets.

Lemma 12 (Prime lemma)**.** *Suppose that $N \subseteq M(\alpha)$ and $N \nvdash_{\mathbf{ICK}} \beta$ for some $\mathcal{L}_{CK}$ formula β; observe that it is not assumed that $\beta \in M(\alpha)$. Then there exists an α-prime set N^* such that $N \subseteq N^*$ and $N^* \nvdash_{\mathbf{ICK}} \beta$.*

Proof. Let $\gamma_0, \ldots, \gamma_k$ be an enumeration of the elements of $M(\alpha)$. Now we define by induction, for $n = 0, \ldots, k$,

$$N_0 := N,$$

$$N_{n+1} := \begin{cases} N_n \cup \{\gamma_n\} & \text{if } N_n \cup \{\gamma_n\} \nvdash_{\mathbf{ICK}} \beta, \\ N_n & \text{if } N_n \cup \{\gamma_n\} \vdash_{\mathbf{ICK}} \beta. \end{cases}$$

Clearly, we have $N \subseteq N_n \subseteq N_{n+1}$ and $N_{n+1} \nvdash_{\mathbf{ICK}} \beta$ for $n = 0, \ldots, k$. We set $N^* := N_{k+1}$ and show that N^* is α-prime. The conditions (P.1) and (P.4) are trivially satisfied.

To show (P.2) assume that $\delta \in M(\alpha)$ and $N^* \vdash_{\mathbf{ICK}} \delta$. Then δ is a formula γ_n for some $n = 0, \ldots, k$. If $N_n \cup \{\gamma_n\} \nvdash_{\mathbf{ICK}} \beta$, then $\gamma_n \in N_{n+1} \subseteq N^*$, and we have what we want. Otherwise,

$$N_n \cup \{\gamma_n\} \vdash_{\mathbf{ICK}} \beta$$

and $N_{n+1} = N_n$. Since γ_n is the formula δ and $N^* \vdash_{\mathbf{ICK}} \delta$, a cut gives us $N^* \vdash_{\mathbf{ICK}} \beta$; a contradiction.

It remains (P.3). So assume that $\delta_0 \vee \delta_1 \in N^*$ and both, δ_0 and δ_1, do not belong to N^*. Since δ_0 is a formula γ_{i_0} and δ_1 a formula γ_{i_1}, we have $\gamma_{i_0} \notin N_{i_0+1}$ and $\gamma_{i_1} \notin N_{i_1+1}$ and thus

$$N_{i_0} \cup \{\gamma_{i_0}\} \vdash_{\mathbf{ICK}} \beta \quad \text{and} \quad N_{i_1} \cup \{\gamma_{i_1}\} \vdash_{\mathbf{ICK}} \beta.$$

From that we conclude

$$\mathbf{ICK} \vdash N^*, \gamma_{i_0} \vee \gamma_{i_1} \supset \beta, \quad \text{i.e.} \quad \mathbf{ICK} \vdash N^*, \delta_0 \vee \delta_1 \supset \beta.$$

However, because of $N^* \vdash_{\mathbf{ICK}} \delta_0 \vee \delta_1$ a cut yields $N^* \vdash_{\mathbf{ICK}} \beta$. This is a contradiction, and also (P.3) is established. Hence our lemma is proved. □

In the following the capital Greek letters Γ, Δ, Π (possibly with subscripts) range over α-prime sets of $\mathcal{L}_{CK}$ formulas. In addition, we set

$$\Gamma^c := M(\alpha) \setminus \Gamma.$$

Definition 13 (Canonical model)**.** *Depending on the given $\mathcal{L}_{CK}$ formula α we now define:*

(C.1) $W^\alpha := \{N \subseteq M(\alpha) : N \text{ is } \alpha\text{-prime}\}$.

(C.2) For any $i = 1, \ldots, \ell$, f_i^α is defined to be the function from W^α to the power set of W^α given by

$$f_i^\alpha(\Gamma) := \{\Delta : \mathsf{K}_i^{-1}(\Gamma) \subseteq \Delta\},$$

where $\mathsf{K}_i^{-1}(N) := \{\beta : \mathsf{K}_i(\beta) \in N\}$ for any set of $\mathcal{L}_{CK}$ formulas N.

(C.3) V^α is the function from W^α to the power set of PROP given by

$$V^\alpha(\Gamma) := \{p : p \in \Gamma\}.$$

(C.4) $\mathfrak{M}(\alpha) \ := \ (W^\alpha, \subseteq, f_1^\alpha, \ldots, f_\ell^\alpha, V^\alpha)$.

We immediately observe that $\mathfrak{M}(\alpha)$ is an EK-structure. To simplify the notation we write, from now on, $\Gamma \models \beta$ instead of $\Gamma \in \|\beta\|_{\mathfrak{M}(\alpha)}$. The following lemma is the core of the completeness proof.

Lemma 14 (Truth lemma)**.** *We have for all formulas* $\beta \in M(\alpha)$ *and all* Γ *that*

$$\beta \in \Gamma \quad \Longleftrightarrow \quad \Gamma \models \beta.$$

Proof. It is clear that we can assign a rank to each $\mathcal{L}_{CK}$ formula such that the rank of the logical constant $\bot$ and of every atomic proposition is 0, the rank of a subformula δ_0 of a formula δ is smaller than that of δ, and the rank of a formula $\mathsf{K}_i(\delta)$ is smaller than that of $\mathsf{C}(\delta)$. We establish the equivalence of the truth lemma by induction on the rank of β and distinguish the following cases.

(i) It trivially holds in case that β is an atomic proposition or the logical constant $\bot$.

(ii) If β is a disjunction or a conjunction it follows from the induction hypothesis and the properties of α-prime sets.

(iii) β is the implication $\gamma_1 \to \gamma_2$. We first assume that

$$\gamma_1 \to \gamma_2 \in \Gamma, \qquad \Gamma \subseteq \Delta, \qquad \text{and} \qquad \Delta \models \gamma_1.$$

Then we have $\gamma_1 \to \gamma_2 \in \Delta$ and (by the induction hypothesis) $\gamma_1 \in \Delta$. Since Δ is deductively closed with respect to $M(\alpha)$ this yields $\gamma_2 \in \Delta$ and thus again by the induction hypothesis that $\Delta \models \gamma_2$. Since Δ has been an arbitrary superset of Γ, we conclude $\Gamma \models \gamma_1 \to \gamma_2$.

Now assume $\Gamma \models \gamma_1 \to \gamma_2$ and $\gamma_1 \to \gamma_2 \notin \Gamma$. Since Γ is deductively closed with respect to $M(\alpha)$, we have $\Gamma \cup \{\gamma_1\} \nvdash_{\mathbf{ICK}} \gamma_2$. By the prime lemma there exists a Δ such that

$$\Gamma \cup \{\gamma_1\} \subseteq \Delta \qquad \text{and} \qquad \Delta \nvdash_{\mathbf{ICK}} \gamma_2.$$

Together with the induction hypothesis we thus obtain

$$\Delta \models \gamma_1 \qquad \text{and} \qquad \Delta \not\models \gamma_2.$$

Since $\Gamma \subseteq \Delta$, this contradicts $\Gamma \models \gamma_1 \to \gamma_2$.

(iv) β is a formula $\mathsf{K}_i(\gamma)$. For the direction from left to right assume

$$\mathsf{K}_i(\gamma) \in \Gamma \qquad \text{and} \qquad \mathsf{K}_i^{-1}(\Gamma) \subseteq \Delta$$

for an arbitrary Δ. This implies $\gamma \in \Delta$, and in view of the induction hypothesis we thus have $\Delta \models \gamma$. Therefore, $\Gamma \models \mathsf{K}_i(\gamma)$.

For the converse direction we assume $\Gamma \models \mathsf{K}_i(\gamma)$. We first claim that

$$\mathsf{K}_i^{-1}(\Gamma) \vdash_{\mathbf{ICK}} \gamma. \tag{1}$$

To establish this claim, assume for contradiction that $\mathsf{K}_i^{-1}(\Gamma) \nvdash_{\mathbf{ICK}} \gamma$. According to the prime lemma we thus have a Δ such that $\mathsf{K}_i^{-1}(\Gamma) \subseteq \Delta$ and $\Delta \nvdash_{\mathbf{ICK}} \gamma$. In particular, $\gamma \notin \Delta$. By the induction hypothesis, this yields $\Delta \not\models \gamma$; a contradiction to $\Gamma \models \mathsf{K}_i(\gamma)$ and $\mathsf{K}_i^{-1}(\Gamma) \subseteq \Delta$.

From (1) we conclude that there are $\delta_1, \ldots, \delta_n \in \mathsf{K}_i^{-1}(\Gamma)$ with

$$\mathbf{ICK} \vdash \delta_1, \ldots, \delta_n \supset \gamma.$$

By applying the appropriate K rule we obtain

$$\mathbf{ICK} \vdash \mathsf{K}_i(\delta_1), \ldots, \mathsf{K}_i(\delta_n) \supset \mathsf{K}_i(\gamma)$$

with $\mathsf{K}_i(\delta_1), \ldots, \mathsf{K}_i(\delta_n) \in \Gamma$, implying that $\Gamma \vdash_{\mathbf{ICK}} \mathsf{K}_i(\gamma)$. Hence $\mathsf{K}_i(\gamma) \in \Gamma$ since Γ is deductively closed with respect to $M(\alpha)$.

(v) β is a formula $\mathsf{C}(\gamma)$. We first assume $\mathsf{C}(\gamma) \in \Gamma$ and check by simple induction on n that for all natural numbers $n \geq 1$ and all $\Delta \in \mathit{Reach}_{\mathfrak{M}(\alpha)}(\Gamma, n)$,

$$\gamma \in \Delta \qquad \text{and} \qquad \mathsf{C}(\gamma) \in \Delta. \tag{2}$$

Hence we have $\gamma \in \Delta$ for all $\Delta \in \mathit{Reach}_{\mathfrak{M}(\alpha)}(\Gamma)$ and by the induction hypothesis $\Delta \models \gamma$ for these sets Δ. Therefore, $\Gamma \models \mathsf{C}(\gamma)$.

Now we assume $\Gamma \models \mathsf{C}(\gamma)$. To show that then $\mathsf{C}(\gamma) \in \Gamma$ is the most interesting part of this proof. We set

$$\mathcal{W} \ := \ \{\Pi : \Pi \models \mathsf{C}(\gamma)\} \quad \text{and} \quad \varphi \ := \ d(\{c(\Pi) : \Pi \in \mathcal{W}\})$$

and prove a series of auxiliary assertions.

(I) For all $\Delta \in \mathcal{W}$: $\mathbf{ICK} \vdash c(\Delta) \rightarrow \mathsf{K}_i(\gamma)$.

Proof of (I). For $\Delta \in \mathcal{W}$ we have $\Delta \models \mathsf{K}_i(\gamma)$ and thus $\mathsf{K}_i(\gamma) \in \Delta$ by the induction hypothesis. The assertion follows immediately.

(II) For all $\Delta \in \mathcal{W}$ and $\Pi \in f_i^\alpha(\Delta)$: $\Pi \in \mathcal{W}$.

Proof of (II). For $\Delta \in \mathcal{W}$ we have $\Delta \models \mathsf{K}_i(\mathsf{C}(\gamma))$ and thus $\Pi \models \mathsf{C}(\gamma)$ for all $\Pi \in f_i^\alpha(\Delta)$. This is the assertion.

(III) For all $\Delta \in \mathcal{W}$: $\mathsf{K}_i^{-1}(\Delta) \vdash \varphi$.

Proof of (III). Let Δ be an element of $\mathcal{W}$ and assume that $\mathsf{K}_i^{-1}(\Delta) \nvdash \varphi$. By the prime lemma then there exists a Σ such that $\mathsf{K}_i^{-1}(\Delta) \subseteq \Sigma$ and $\Sigma \nvdash \varphi$. Hence $\Sigma \in f_i^{\alpha}(\Delta)$ and, in view of (II), $\Sigma \in \mathcal{W}$. This is a contradiction to $\Sigma \nvdash \varphi$.

(IV) For all $\Delta \in \mathcal{W}$: $\mathbf{ICK} \vdash c(\Delta) \to \mathsf{K}_i(\varphi)$.

Proof of (IV). Because of (III) we know that there are $\delta_1, \ldots, \delta_n \in \mathsf{K}_i^{-1}(\Delta)$ such that

$$\mathbf{ICK} \vdash \delta_1, \ldots, \delta_n \supset \varphi.$$

Thus we also have

$$\mathbf{ICK} \vdash \mathsf{K}_i(\delta_1), \ldots, \mathsf{K}_i(\delta_n) \supset \mathsf{K}_i(\varphi)$$

with $\mathsf{K}_i(\delta_1), \ldots, \mathsf{K}_i(\delta_n) \in \Delta$. Hence $\Delta \vdash \mathsf{K}_i(\varphi)$, and the assertion is an immediate consequence.

From (I) and (IV) we obtain

$$\mathbf{ICK} \vdash c(\Delta) \to \mathsf{E}(\gamma) \land \mathsf{E}(\varphi)$$

for all $\Delta \in \mathcal{W}$, hence

$$\mathbf{ICK} \vdash \varphi \to \mathsf{E}(\gamma) \land \mathsf{E}(\varphi). \tag{3}$$

By means of the induction rule or Theorem 8 we obtain from (3) that

$$\mathbf{ICK} \vdash \varphi \to \mathsf{C}(\gamma).$$

By assumption we have $\Gamma \in \mathcal{W}$ and thus $\Gamma \vdash_{\mathbf{ICK}} \varphi$. Hence $\Gamma \vdash_{\mathbf{ICK}} \mathsf{C}(\gamma)$ and so $\mathsf{C}(\gamma) \in \Gamma$ since Γ is deductively closed with respect to $M(\alpha)$. This finishes the proof of the truth lemma. □

With the truth lemma at our disposal, the proof of the completeness of **ICK** is now a trivial matter.

Theorem 15 (Completeness of **ICK**). *Suppose that α is an EK-valid $\mathcal{L}_{CK}$ formula. Then* $\mathbf{ICK} \vdash \alpha$.

Proof. Assume that $\mathbf{ICK} \nvdash \alpha$. Then there exists an α-prime Γ such that $\Gamma \nvdash_{\mathbf{ICK}} \alpha$. Hence $\alpha \notin \Gamma$, and thus the truth lemma implies $\Gamma \not\models \alpha$. However, then α is not valid in the canonical model $\mathfrak{M}(\alpha)$, contradicting our assumption. □

Now we turn to the completeness of **ICKT**. In principle, we proceed as before: We start off from an $\mathcal{L}_{CK}$ formula α, introduce the set $M(\alpha)$ and build a canonical model. The only difference is that we work with α-**T**-prime sets instead of α-prime sets. Here a set N of $\mathcal{L}_{CK}$ formulas is called α-**T**-prime iff it has the properties (P.1), (P.3), (P.4) of α-prime sets plus for all β the property

(P.2') $\beta \in M(\alpha)$ and $N \vdash_{\mathbf{ICKT}} \beta \quad \Longrightarrow \quad \beta \in N.$

Then we construct the canonical model as before, but with α-prime sets replaced by α-**T**-prime sets; we call it

$$\mathfrak{N}(\alpha) \; := \; (S^\alpha, \subseteq, g_1^\alpha, \ldots, g_\ell^\alpha, T^\alpha).$$

All we have to show in addition to what we did before is that $\mathfrak{N}(\alpha)$ is reflexive. Hence take an i with $1 \leq i \leq \ell$, an α-**T**-prime set Γ, and an arbitrary element β of $\mathsf{K}_i^{-1}(\Gamma)$. Then $\mathsf{K}_i(\beta) \in \Gamma$. Since

$$\mathbf{ICKT} \vdash \mathsf{K}_i(\beta) \supset \beta$$

this implies $\beta \in \Gamma$. Hence $\mathsf{K}_i^{-1}(\Gamma) \subseteq \Gamma$ and thus $\Gamma \in g_i^\alpha(\Gamma)$. The truth lemma for $\mathfrak{N}(\alpha)$ goes through as above, and the completeness of **ICKT** with respect to reflexive EK-structures is an immediate consequence.

Theorem 16 (Completeness of **ICKT**)**.** *Suppose that the $\mathcal{L}_{CK}$ formula α is valid in all reflexive EK-structures. Then* **ICKT** $\vdash \alpha$*.*

5 Disjunction property

Typically, intuitionistic formalisms possess the disjunction property. It is an immediate consequence of the previous soundness and completeness results that this is also the case for **ICK** and **ICKT**.

Theorem 17 (Disjunction property)**.** *For all $\mathcal{L}_{CK}$ formulas α and β we have:*

1. *If $\alpha \vee \beta$ is EK-valid, then α is EK-valid or β is EK-valid.*
2. *If $\alpha \vee \beta$ is valid in all reflexive EK-structures, then α is valid in all reflexive EK-structures or β is valid in all reflexive EK-structures.*
3. *If* $\mathbf{ICK}^\bullet$ *is the system* **ICK** *or the system* **ICKT***, then*

$$\mathbf{ICK}^\bullet \vdash \alpha \vee \beta \quad \Longrightarrow \quad \mathbf{ICK}^\bullet \vdash \alpha \;\; \textit{or} \;\; \mathbf{ICK}^\bullet \vdash \beta.$$

Proof. In view of the soundness and completeness of $\mathbf{ICK}^\bullet$, the third assertion is an immediate consequence of the first and the second. The proof of the second is exactly as the proof of the first, and to prove the first, we assume that neither α nor β are EK-valid. Then there exist EK-structures

$$\mathfrak{M}_1 = (W_1, \preceq_1, f_1^{(1)}, \dots, f_\ell^{(1)}, V_1) \quad \text{and} \quad \mathfrak{M}_2 = (W_2, \preceq_2, f_1^{(2)}, \dots, f_\ell^{(2)}, V_2)$$

together with $v \in W_1$ and $w \in W_2$ such that

$$(*) \qquad v \notin \|\alpha\|_{\mathfrak{M}_1} \quad \text{and} \quad w \notin \|\beta\|_{\mathfrak{M}_2}.$$

Now consider the structure $\mathfrak{M} := (W, \preceq, f_1, \dots, f_\ell, V)$ whose universe is the set

$$W := \{(0,0)\} \cup \{(1,x) : x \in W_1\} \cup \{(2,x) : x \in W_2\}$$

and the preorder $\preceq$ on W is defined by

$$(x_1, y_1) \preceq (x_2, y_2) \;:\Leftrightarrow\; \begin{cases} x_1 = 0 \text{ or} \\ (x_1 = x_2 = 1 \text{ and } y_1 \preceq_1 y_2) \text{ or} \\ (x_1 = x_2 = 2 \text{ and } y_1 \preceq_2 y_2). \end{cases}$$

Furthermore, for every i with $1 \le i \le \ell$, f_i is the function from W to the power set of W given by

$$f_i(x,y) \;:=\; \begin{cases} W & \text{if } x = 0, \\ \{(1,z) : z \in f_i^{(1)}(y)\} & \text{if } x = 1, \\ \{(2,z) : z \in f_i^{(2)}(y)\} & \text{if } x = 2. \end{cases}$$

Finally, V is defined to be the function from W to the power set of *PROP* defined by

$$V(x,y) \;:=\; \begin{cases} \varnothing & \text{if } x = 0, \\ V_1(y) & \text{if } x = 1, \\ V_2(y) & \text{if } x = 2. \end{cases}$$

Obviously, $\mathfrak{M}$ is an EK-structure. It is also easy to check that for all $\mathcal{L}_{CK}$ formulas γ, all $x \in W_1$, and all $y \in W_2$,

$$(1,x) \in \|\gamma\|_{\mathfrak{M}} \iff x \in \|\gamma\|_{\mathfrak{M}_1},$$
$$(2,y) \in \|\gamma\|_{\mathfrak{M}} \iff y \in \|\gamma\|_{\mathfrak{M}_2}.$$

Because of (*) this implies that

$$(1,v) \notin \|\alpha\|_{\mathfrak{M}} \quad \text{and} \quad (2,w) \notin \|\beta\|_{\mathfrak{M}}.$$

The monotonicity of $\mathfrak{M}$ with respect to $\preceq$, cf. Lemma 3, and the fact that $(0,0) \preceq (1,v)$ and $(0,0) \preceq (2,w)$ thus yield

$$(0,0) \notin \|\alpha\|_{\mathfrak{M}} \quad \text{and} \quad (0,0) \notin \|\beta\|_{\mathfrak{M}},$$

implying that $(0,0) \notin \|\alpha \vee \beta\|_{\mathfrak{M}}$. From this we conclude that $\alpha \vee \beta$ is not EK-valid. □

It is also fairly easy to extend the results of this article to semantics and deductive systems that reflect positive introspection; details are left to the reader. Negative introspection, on the other hand, is a different matter. Typically, intuitionistic **S5** is formulated by making use of the box and the diamond operator; see, e.g., Fischer Servi [5] and Simpson [15] and in intuitionistic modal logic $\Diamond(\alpha)$ is not equivalent to $\neg\Box(\neg\alpha)$. In our present framework the operators $\mathsf{K}_1, \ldots, \mathsf{K}_\ell$ are boxes, but the corresponding diamonds are not available. It is planned for the future to look at negative introspection from an intuitionistic perspective and to analyze the emerging issues from a technical and conceptual perspective.

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