A proof-theoretic analysis of theories for stratified inductive definitions

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Abstract

In this article we study subsystems SID_ν of the theory ID_1 in which fixed point induction is restricted to properly stratified formulas.

Keywords: Proof theory, inductive definitions, stratification.

1 Introduction

Several years ago Leivant presented a conceptually interesting form of stratified complete induction that can be roughly described as follows: Let T be some weak base theory about the natural numbers with the constant 0 and the unary successor function S. In the extension T^* of T we have, for every natural number i, a fresh unary relation symbols N_i and the axiom

$$\forall x(N_i(x) \leftrightarrow (x = 0 \vee \exists y(x = S(y) \wedge N_i(y)))),$$

stating that N_i has the usual closure properties of the natural numbers. The new aspect is that in T^* complete induction is restricted to a form of stratified induction. For all natural numbers i and all formulas $A[u]$,

$$A[0] \wedge \forall x(A[x] \to A[S(x)]) \to \forall x(N_i(x) \to A[x]),$$

provided that $A[u]$ does not contain relation symbols N_j with $j \geq i$. As a simple consequence, T^* proves $\forall x(N_{i+1}(x) \to N_i(x))$, and so the sequence $(N_i : i \in \mathbb{N})$ provides finer and finer approximations of the "real" natural numbers.

Leivant is interested in this system in connection with Nelson's predicative arithmetic and his own research on systems tailored for representing feasible complexity classes. See Leivant [14, 15] for all details. Wainer and Williams [22] begins from a similar standpoint and analyzes inductive definitions over a predicative arithmetic.

In this article, we apply Leivant's idea in the context of inductive definitions: we study an analogous stratifiction of the fixed point induction axioms of the theory ID_1 of one inductive definition, incontrovertibly one of the best studied theories at the borderline between predicative and impredicative proof theory. There exist numerous articles on ID_1, good introductory texts are, for example, Buchholz, Feferman, Pohlers, and Sieg [7] and Pohlers [16].

The language $\mathcal{L}(\mathsf{ID}_1)$ of ID_1 is the extension of the language $\mathcal{L}$ of first order arithmetic by a new (unary) relation symbol $\mathcal{P}^{\mathfrak{A}}$ for every X positive arithmetic formula $\mathfrak{A}[X, x]$. The axioms of ID_1 are the axioms of Peano arithmetic PA with the schema of complete induction for all $\mathcal{L}(\mathsf{ID}_1)$ formulas plus the fixed point axioms

$$(Fix) \qquad \forall x(\mathcal{P}^{\mathfrak{A}}(x) \leftrightarrow \mathfrak{A}[\mathcal{P}^{\mathfrak{A}}, x])$$

and the axioms for fixed point induction

$$(FI) \qquad \forall x(\mathfrak{A}[B, x] \to B[x]) \to \forall x(\mathcal{P}^{\mathfrak{A}}(x) \to B[x]),$$

where $B[u]$ ranges over all $\mathcal{L}(\mathsf{ID}_1)$ formulas. Together, they formalize that $\mathcal{P}^{\mathfrak{A}}$ represents the least definable fixed point of the monotone operator $\Gamma_{\mathfrak{A}}$ defined by

$$\Gamma_{\mathfrak{A}}(M) := \{n \in \mathbb{N} : \mathbb{N} \models \mathfrak{A}[M, n]\}$$

for any $M \subseteq \mathbb{N}$. The proof-theoretic ordinal of ID_1 is the Bachmann-Howard ordinal $\theta\varepsilon_{\Omega+1}0$.

Currently studied subsystems of ID_1 with classical logic are obtained by weakening fixed point induction (FI). For instance, in ID^* (FI) is restricted to formulas positive in the fixed point constants, and $\widehat{\mathsf{ID}}_1$ is obtained by dropping (FI) completely. Both systems are significantly weaker than ID_1 and have the proof-theoretic ordinal $\varphi\varepsilon_0 0$; see, e.g., Aczel [2], Afshari and Rathjen [3], Feferman [9], Friedman [10], Probst [17].

In contrast to these approaches, the subsystems of ID_1 studied in this article are obtained by a stratification of the fixed point axioms (FI). The idea is simple: Let $\prec$ be a primitive recursive wellordering of order type ν, given from outside. For any X positive arithmetic $\mathfrak{A}[X, x]$ and any $\alpha \prec \nu$ we add a fresh unary relation symbol $\mathcal{P}^{\mathfrak{A}}_{\alpha}$ to the language $\mathcal{L}$ and consider the fixed point axiom

$$(Fix[\nu]) \qquad (\forall a \prec \nu)\forall x(\mathcal{P}^{\mathfrak{A}}_a(x) \leftrightarrow \mathfrak{A}[\mathcal{P}^{\mathfrak{A}}_a, x]),$$

stating that any $\mathcal{P}^{\mathfrak{A}}_a$ with $a \prec \nu$ is a fixed point of $\mathfrak{A}[X, x]$. However, instead of (FI) we only permit *stratified fixed point induction*

$$(SI[\nu]) \qquad (\forall a \prec \nu)(\forall x(\mathfrak{A}[B, x] \to B[x]) \to \forall x(\mathcal{P}^{\mathfrak{A}}_a(x) \to B[x])),$$

where now the formula $B[u]$ must not speak about any $\mathcal{P}_b^{\mathfrak{A}}$ with $a \preceq b$. Call the resulting theory SID_ν. Analogous to the Leivant case, we now have $\forall x(\mathcal{P}_b^{\mathfrak{A}}(x) \to \mathcal{P}_a^{\mathfrak{A}}(x))$ for all $a \prec b \prec \nu$, provable in SID_ν, and the sequence of relations $(\mathcal{P}_a^{\mathfrak{A}} : a \prec \nu)$ thus provides a decreasing approximation of the least fixed point of the operator $\Gamma_{\mathfrak{A}}$.

Obviously, all $\mathsf{SID}_{1+\nu}$ contain $\widehat{\mathsf{ID}}_1$. However, for finite ν, the theories SID_ν are surprisingly weak. As shown in Ranzi and Strahm [18], $\mathsf{SID}_1, \mathsf{SID}_2, \dots$ have the same strength and are all proof-theoretically equivalent to the fixed point theory $\widehat{\mathsf{ID}}_1$.

The first interesting increase of strength happens when we move to SID_ω and at the limit ordinals later. In the following we present a complete proof-theoretic analysis of the theories SID_ν and obtain the following result: Given an ordinal $\nu > 0$, we write it as

$$\nu \;=\; \omega^{\alpha_m} + \ldots + \omega^{\alpha_1} \quad \text{with} \quad \alpha_1 \leq \ldots \leq \alpha_m,$$

let $\varepsilon(\nu)$ be the least ε-number greater that ν and define

$$\Lambda_\nu \;:=\; \varphi\alpha_m(\ldots(\varphi\alpha_1\varepsilon(\nu))\ldots).$$

Then $\varphi\Lambda_\nu 0$ is the proof-theoretic ordinal of SID_ν. This implies, for example, that we need all stratifications less than ε_0 in order to reach the strength of $\widehat{\mathsf{ID}}_2$, i.e. $\mathsf{SID}_{<\varepsilon_0} \equiv \widehat{\mathsf{ID}}_2$.

Very much in the spirit of Gentzen-style proof theory, the lower bounds are established by carrying out wellordering proofs within the systems SID_ν. For determining the upper bounds, we use a combination of various forms of cut elimination and asymmetric interpretations. For obtaining the full picture and since it is needed for the general reduction, we also sketch the finite case, using an approach slightly different from that in [18].

2 Ordinal-theoretic preliminaries

Every theory SID_ν is based on stratifications of inductive definitions along a primitive recursive wellordering of order type ν. In order to concentrate on the essential proof-theoretics aspects of the theories SID_ν and to make our approach as perspicuous as possible we fix a specific primitive recursive wellordering $\prec$ of order type Γ_0 right in advance and iterate stratifications along its initial segments.

It will become evident that $\prec$ and Γ_0 could be replaced by ordinal notations system generated, for example, from the ternary Veblen functions (cf. Jäger and Strahm [13]), the Veblen functions of all finite arities, Schütte's

Klammersymbole (cf. Schütte [20]), Feferman's θ functions (cf. Aczel [1] and Buchholz [4]), or Buchholz's ψ functions (cf. Buchholz [5]). The choice of $\prec$ and Γ_0 is only motivated by notational simplicity.

The standard notation system up to the Feferman-Schütte ordinal Γ_0 is provided by the usual Veblen hierarchy $(\varphi\alpha : \alpha \in On)$ of ordinal functions from On to On, inductively defined as follows:

(i) $\varphi 0 \beta \;:=\; \omega^\beta$,

(ii) if $\alpha > 0$, then $\varphi\alpha$ enumerates $\{\xi \in On : (\forall \gamma < \alpha)(\varphi\gamma\xi = \xi)\}$.

Γ_0 is the least ordinal α with $\varphi\alpha 0 = \alpha$. We write

$$LI \;:=\; \{\omega\xi : 0 < \xi < \Gamma_0\} \qquad \text{and} \qquad AP \;:=\; \{\omega^\xi : \xi < \Gamma_0\}$$

for the sets of *limit numbers* and *additive principal numbers* less than Γ_0; the ordinals ξ with $\omega^\xi = \xi$ are called *ε-numbers.* Also, we set for all ordinals α,

$$\varepsilon_\alpha \;:=\; \varphi 1 \alpha \quad \text{and} \quad \varepsilon(\alpha) \;=\; \text{least element of } \{\xi > \alpha : \omega^\xi = \xi\}.$$

Hence $\varepsilon(\alpha)$ is the least ε-number greater than α. From now on we expect that the reader is familiar with ordinal computations and the basic properties of the Veblen hierarchy; all relevant details can be found, for example, in Pohlers [16] and Schütte [21]. In particular, there are two important decomposition properties:

(D.1) For any ordinal $\alpha > 0$ there are uniquely determined $\alpha_1, \ldots, \alpha_m$ such that $\alpha = \omega^{\alpha_m} + \ldots + \omega^{\alpha_1}$ and $\alpha_1 \leq \ldots \leq \alpha_m$.

(D.2) For any ε-number $\alpha < \Gamma_0$ there are uniquely determined β and γ such that $\alpha = \varphi\beta\gamma$, $0 < \beta < \alpha$, and $\gamma < \alpha$.

Making use of these decompositions, we inductively assign a *fundamental sequence* $(\alpha[n] : n < \omega)$ to any limit number less than Γ_0:

(FS.1) If $\alpha = \omega^{\alpha_m} + \ldots + \omega^{\alpha_1}$ with $0 < \alpha_1 \leq \ldots \leq \alpha_m$ and $\omega^{\alpha_1} < \alpha$, then

$$\alpha[n] \;:=\; \omega^{\alpha_m} + \ldots + \omega^{\alpha_1}[n].$$

(FS.2) If $\alpha = \omega^{\beta+1}$, then $\alpha[n] := \omega^\beta n$.

(FS.3) If $\alpha = \omega^\beta$ with $\beta < \alpha$ and $\beta \in LI$, then $\alpha[n] := \omega^{\beta[n]}$.

(FS.4) If $\alpha = \varphi(\beta + 1)0$, then $\alpha[0] := 0$ and $\alpha[n+1] := \varphi\beta\alpha[n]$.

(FS.5) If $\alpha = \varphi(\beta+1)(\gamma+1)$, then

$$\alpha[0] := \varphi(\beta+1)\gamma+1 \quad \text{and} \quad \alpha[n+1] := \varphi\beta\alpha[n].$$

(FS.6) If $\alpha = \varphi\beta 0$ with $\beta \in LI$, then $\alpha[n] := \varphi\beta[n]0$.

(FS.7) If $\alpha = \varphi\beta(\gamma+1)$ with $\beta \in LI$, then

$$\alpha[0] := \varphi\beta\gamma+1 \quad \text{and} \quad \alpha[n+1] := \varphi\beta[n]\alpha[n].$$

(FS.8) If $\alpha = \varphi\beta\gamma$ with $\gamma \in LI$, then $\alpha[n] := \varphi\beta\gamma[n]$.

As it is easy to check, for all $\alpha \in LI$ and $n < \omega$ we have $\alpha[n] < \alpha[n+1] < \alpha$ and $\alpha = \sup(\{\alpha[n] : n < \omega\})$.

Decomposition (D1) is used once more. Given $\nu > 0$, we write it as $\nu = \omega^{\alpha_m} + \ldots + \omega^{\alpha_1}$ with $\alpha_1 \leq \ldots \leq \alpha_m$. Depending on this presentation of ν, we inductively define a sequence of ordinals $\nu_0, \ldots, \nu_m$ by

$$\nu_0 \ := \ \varepsilon(\nu) \qquad \text{and} \qquad \nu_{i+1} \ := \ \varphi\alpha_{i+1}\nu_i$$

and then set

$$\Lambda_\nu := \nu_m.$$

To finish the definition of Λ we set $\Lambda_0 := 1$. As we will see, the ordinals Λ_ν play a crucial role in the proof-theoretic analysis of the theories SID_ν.

Simple computations show that, for example, $\Lambda_n = \varepsilon_0$ for all $0 < n < \omega$, $\Lambda_\omega = \varepsilon_{\varepsilon_0}$, $\Lambda_{\omega+\omega} = \varepsilon_{\varepsilon_{\varepsilon_0}}$, $\Lambda_{\varepsilon_\alpha} = \varphi\varepsilon_\alpha\varepsilon_{\alpha+1}$ for all α, and $\Lambda_{\varphi\alpha\beta} = \varphi(\varphi\alpha\beta)\varepsilon_{\varphi\alpha\beta+1}$ for all $\alpha > 1$ and all β. The following lemma is immediate from the definition of Λ_ν.

Lemma 1 *If $\nu = \mu + k$ for some $k < \omega$, then $\varepsilon(\nu) = \varepsilon(\mu)$; if, in addition, $\mu > 0$, then $\Lambda_\nu = \Lambda_\mu$.*

In the textbooks by Pohlers and Schütte it is also explained in detail that there exist a primitive recursive wellordering $\prec$ on the natural numbers corresponding to the ordinals less than Γ_0; $m \preceq n$ is written iff $m \prec n$ or $m = n$.

Each natural number codes exactly one ordinal less than Γ_0, and given an $n \in \mathbb{N}$, we write $ot(n)$ for this ordinal; $ot(n)$ is called the *order type* of n with respect to $\prec$. The inverse of ot, let us call it nr, assigns a natural number $nr(\alpha)$ to any $\alpha < \Gamma_0$. The sets $\{nr(\xi) : \xi \in LI\}$ and $\{nr(\xi) : \xi \in AP\}$ are primitive recursive subsets of the natural numbers.

Furthermore, for all ordinal functions f on $(\Gamma_0, <)$ such as addition, multiplication, exponentiation, ε, φ, fundamental sequences, Λ, ... there exist primitive recursive functions f_{code} acting on $(\mathbb{N}, \prec)$ that correspond to these ordinal operations. Without loss of generality we can assume that $ot(0) = 0$.

3 The theories SID_ν

In the following we let $\mathcal{L}$ denote our language of first order arithmetic. It includes *number variables* $a, b, c, d, u, v, w, x, y, z$ (possibly with subscripts), symbols for all primitive recursive functions and relations as well as the unary relation symbol W. This relation symbol W plays the role of an anonymous relation variable with no specific meaning. Its role will become clear in Definition 2 below. Furthermore, there is a symbol $\sim$ for forming negative literals. When dealing with primitive recursive functions and relations we often write the same expression for the primitive recursive function (relation) and for the associated function (relation) symbol.

The *number terms* p, q, r, s, t (possibly with subscripts) of $\mathcal{L}$ are defined as usual; in particular, the numeral associated to the natural number n is denoted by $\bar{n}$. The *positive literals* of $\mathcal{L}$ are all expressions of the form $R(s_1, \ldots, s_n)$ where R is a symbol for an n-ary primitive recursive relation and all expressions $W(s)$. The *negative literals* of $\mathcal{L}$ are all expressions $\sim E$ such that E is a positive literal of $\mathcal{L}$. Infix notation is used whenever convenient, and $(s = t)$ stands for $R_=(s, t)$ if $R_=$ is the symbol for primitive recursive equality.

The *formulas* A, B, C, D (possibly with subscripts) of $\mathcal{L}$ are generated from the positive and negative literals of $\mathcal{L}$ by closing against disjunctions, conjunctions as well as existential and universal number quantifications. The negation $\neg A$ of an $\mathcal{L}$ formula A is defined by making use of De Morgan's laws and the law of double negation; the remaining logical connectives are abbreviated as usual. We will often omit parentheses and brackets whenever there is no danger of confusion. Also, we frequently make use of the vector notation $\vec{\mathfrak{e}}$ as shorthand for a finite string $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$ of expressions whose length is not important or evident from the context.

Suppose now that $\vec{a} = a_1, \ldots, a_n$ and $\vec{s} = s_1, \ldots, s_n$. Then $A[\vec{s}/\vec{a}]$ is the $\mathcal{L}$ formula that is obtained from the $\mathcal{L}$ formula A by simultaneously replacing all free occurrences of the variables $\vec{a}$ by the $\mathcal{L}$ terms $\vec{s}$ (in order to avoid collision of variables, a renaming of bound variables may be necessary). If the $\mathcal{L}$ formula A is written as $B[\vec{a}]$, then we often simply write $B[\vec{s}]$ instead of $A[\vec{s}/\vec{a}]$; variants of this notation will be self-explaining.

If X is a fresh unary relation symbol, we let $\mathcal{L}(X)$ denote the extension of $\mathcal{L}$ by X; i.e. expressions of the forms $X(s)$ and $\sim X(s)$ are additional literals. Given a formula $A[X]$ of $\mathcal{L}(X)$ and a formula $B[u]$ of $\mathcal{L}$, we write $A[\{x{:}B[x]\}]$ to indicate the result of substituting $B[s]$ for each occurrence of $X(s)$ and $\neg B[s]$ for each occurrence of $\sim X(s)$ in $A[X]$ (again, bound variables are renamed if necessary). If $\mathcal{L}'$ is a language extending $\mathcal{L}$, then $\mathcal{L}'(X)$ is defined accordingly.

We will be interested in determining the proof-theoretic ordinals of the theories SID_ν. For this purpose we fix the auxiliary notions of progressiveness and transfinite induction. Given a primitive recursive relation $\lhd$, an $\mathcal{L}$ term s, and a formula $A[a]$ of $\mathcal{L}$ (or of some extension of $\mathcal{L}$ to be introduced later), we set:

$$\begin{aligned} \mathit{Prog}[\lhd, \{x{:}A[x]\}] &:= \forall x((\forall y \lhd x)A[y] \to A[x]), \\ \mathit{TI}[\lhd, \{x{:}A[x]\}] &:= \mathit{Prog}[\lhd, \{x{:}A[x]\}] \to \forall x A[x], \\ \mathit{TI}[\lhd, s, \{x{:}A[x]\}] &:= \mathit{Prog}[\lhd, \{x{:}A[x]\}] \to (\forall x \lhd s)A[x]. \end{aligned}$$

In the following we often work with the primitive recursive wellordering $\prec$ introduced in the previous section and thus simply write $\mathit{Prog}[\{x{:}A[x]\}]$ for $\mathit{Prog}[\prec, \{x{:}A[x]\}]$ and $\mathit{TI}[s, \{x{:}A[x]\}]$ for $\mathit{TI}[\prec, s, \{x{:}A[x]\}]$.

Definition 2 *Let T be a theory formulated in $\mathcal{L}$ or an extension of $\mathcal{L}$.*

1. *An ordinal α is called* provable in T *iff there exists a primitive recursive wellordering $\lhd$ of order type α such that $T \vdash \mathit{TI}[\lhd, \{x{:}W(x)\}]$.*

2. *The* proof-theoretic ordinal $|T|$ of T *is the least ordinal that is not provable in T.*

We call an $\mathcal{L}(X)$ formula X *positive* if it has no subformulas of the form $\sim X(s)$. An X positive $\mathcal{L}(X)$ formula that contains at most the variable x free is called an *inductive operator form*, and we let $\mathfrak{A}[X, x]$ range over such forms.

From now on ν always stands for an ordinal less than Γ_0, and $\bar{\nu}$ denotes the numeral $\overline{nr(\nu)}$ corresponding to the element $nr(\nu) \in OT$. For the formulation of the theories SID_ν we add to the first order language $\mathcal{L}$ a new unary relation symbol $\mathcal{P}^{\mathfrak{A}}$ for every inductive operator form $\mathfrak{A}[X, x]$ and call this new language $\mathcal{L}_S$. We write $\mathcal{P}^{\mathfrak{A}}_s(t)$ for $\mathcal{P}^{\mathfrak{A}}(\langle s, t\rangle)$ and $\mathcal{P}^{\mathfrak{A}}_{\prec s}(t)$ for $(t = \langle (t)_0, (t)_1\rangle \wedge (t)_0 \prec s \wedge \mathcal{P}^{\mathfrak{A}}(t))$, where $\langle .,. \rangle$ denotes a primitive recursive pairing function with the associated primitive recursive projection functions $(.)_0$ and $(.)_1$. Also, $\mathfrak{A}[\mathcal{P}^{\mathfrak{A}}_a, b]$ and $\mathfrak{A}[\mathcal{P}^{\mathfrak{A}}_{\prec a}, b]$ are short for $\mathfrak{A}[\{x{:}\mathcal{P}^{\mathfrak{A}}_a(x)\}, b]$ and $\mathfrak{A}[\{x{:}\mathcal{P}^{\mathfrak{A}}_{\prec a}(x)\}, b]$, respectively.

We express the closure of an $\mathcal{L}_S$ formula $B[a]$ under the inductive operator form $\mathfrak{A}[X, x]$ by the formula

$$\mathit{Cl}_{\mathfrak{A}}[\{a{:}B[a]\}] := \forall x(\mathfrak{A}[\{a{:}B[a]\}, x] \to B[x]).$$

For formulating stratified fixed point induction, a further shorthand notation is useful. Given a number variable u, we call an $\mathcal{L}_S$ formula A *bounded by u*

iff all relation symbols $\mathcal{P}^{\mathfrak{A}}$ occur in A only in the form $\mathcal{P}^{\mathfrak{A}}_{\prec u}(t)$ or $\sim\mathcal{P}^{\mathfrak{A}}_{\prec u}(t)$. More formally, $\mathcal{BL}_S(u)$ is the collection of $\mathcal{L}_S$ formulas inductively generated as follows:

($\mathcal{B}$.1) All atomic formulas of $\mathcal{L}$ as well as all formulas $\mathcal{P}^{\mathfrak{A}}_{\prec u}(t)$ and $\sim\mathcal{P}^{\mathfrak{A}}_{\prec u}(t)$ belong to $\mathcal{BL}_S(u)$ (for all inductive operator forms $\mathfrak{A}[X, x]$).

($\mathcal{B}$.2) If A and B belong to $\mathcal{BL}_S(u)$, then $(A \vee B)$ and $(A \wedge B)$ belong to $\mathcal{BL}_S(u)$.

($\mathcal{B}$.3) If A belongs to $\mathcal{BL}_S(u)$ and x is a number variable different from u, then $\exists xA$ and $\forall xA$ belong to $\mathcal{BL}_S(u)$.

Therefore, if a is an element of OT, then the formulas in $\mathcal{BL}_S(a)$ are $\mathcal{L}_S$ formulas in which only stratifications less than a play a role.

Every theory SID_ν is formulated in the language $\mathcal{L}_S$ for stratified inductive definitions. Its axioms and rules of inference are the usual axioms and rules of inference of first order order logic, the usual equality axioms formulated for all $\mathcal{L}_S$ formulas plus the following four classes of non-logical axioms.

I. Peano axioms. All axioms of Peano arithmetic PA with the schema of complete induction for all formulas of $\mathcal{L}_S$.

II. Transfinite induction up to ν. For all formulas $A[u]$ of $\mathcal{L}_S$:

$$(TI[\nu]) \qquad\qquad TI[\bar{\nu}, \{x{:}A[x]\}].$$

III. Fixed point axioms. For all inductive operator forms $\mathfrak{A}[X, x]$:

$$(Fix[\nu]) \qquad\qquad (\forall a \prec \bar{\nu})\forall x(\mathcal{P}^{\mathfrak{A}}_a(x) \leftrightarrow \mathfrak{A}[\mathcal{P}^{\mathfrak{A}}_a, x]).$$

IV. Stratified fixed point induction. For all inductive operator forms $\mathfrak{A}[X, x]$ and all formulas $B[u, v]$ from $\mathcal{BL}_S(u)$:

$$(SI[\nu]) \qquad\qquad (\forall a \prec \bar{\nu})(Cl_{\mathfrak{A}}[\{x{:}B[a, x]\}] \rightarrow \forall x(\mathcal{P}^{\mathfrak{A}}_a(x) \rightarrow B[a, x])).$$

The theories SID_ν have the important property that the stratifications of the fixed points form a weakly decreasing sequence of relations.

Lemma 3 *For all inductive operator forms $\mathfrak{A}[X, x]$ we can prove in SID_ν that*

$$(\forall a, b \prec \bar{\nu})\forall x(a \prec b \wedge \mathcal{P}^{\mathfrak{A}}_b(x) \rightarrow \mathcal{P}^{\mathfrak{A}}_a(x)).$$

PROOF. We define $B[u, v, w] := (v \prec u \wedge P^{\mathfrak{A}}_{\prec u}(\langle v, w\rangle))$. Obviously, this formula belongs to $\mathcal{BL}_S(u)$ and SID_ν proves

$$a \prec b \prec \bar{\nu} \;\rightarrow\; \forall x(\mathcal{P}^{\mathfrak{A}}_a(x) \;\leftrightarrow\; B[b, a, x]).$$

In view of $(Fix[\nu])$ and $(SI[\nu])$ our assertion follows immediately. □

Besides the theories SID_ν, also their unions are of some interest. If $\nu > 0$, we write $\mathsf{SID}_{<\nu}$ for the union of the theories SID_μ with $\mu < \nu$,

$$\mathsf{SID}_{<\nu} \;:=\; \bigcup_{\mu<\nu} \mathsf{SID}_\mu.$$

In the following sections we will show that $|\mathsf{SID}_\nu| = \varphi\Lambda_\nu 0$. The theory SID_0 contains neither fixed point axioms nor axioms for stratified fixed point induction and simply is a variant of Peano arithmetic PA. The theories SID_ν for $\nu > 0$ are more interesting. Here some specific examples of theories and their proof-theoretic ordinals:

- $|\mathsf{SID}_1| \;=\; |\mathsf{SID}_{<\omega}| \;=\; \varphi\varepsilon_0 0$,
- $|\mathsf{SID}_\omega| \;=\; |\mathsf{SID}_{<\omega+\omega}| \;=\; \varphi\varepsilon_{\varepsilon_0} 0$ and $|\mathsf{SID}_{\omega+\omega}| \;=\; \varphi\varepsilon_{\varepsilon_{\varepsilon_0}} 0$,
- $|\mathsf{SID}_{<\omega^\omega}| \;=\; \varphi(\varphi\omega 0)0$ and $|\mathsf{SID}_{\omega^\omega}| \;=\; \varphi(\varphi\omega\varepsilon_0)0$,
- $|\mathsf{SID}_{<\varepsilon_0}| \;=\; \varphi(\varphi\varepsilon_0 0)0$ and $|\mathsf{SID}_{\varepsilon_0}| \;=\; \varphi(\varphi\varepsilon_0\varepsilon_1)0$,
- $|\mathsf{SID}_{<\Gamma_0}| \;=\; \Gamma_0$.

4 Lower proof-theoretic bound for SID_ν

The lower bounds of the theories SID_ν will be established by carrying out wellordering proofs within the theories SID_ν. To increase readability we shall use in our formal language $\mathcal{L}_S$ the ordinal-theoretic functions f on $(\Gamma_0, <)$ introduced in Section 2 instead of their primitive recursive analogues f_{code} on $(OT, \prec)$. We also write α instead of $\overline{nr(\alpha)}$ in terms and formulas of $\mathcal{L}_S$. Thus, for instance $s + t$, ω^s, $\varphi\omega 0$ are to be considered as terms of $\mathcal{L}_S$ and $(\forall x \prec \omega^\omega)(\exists y \prec \omega)(x \prec \omega^y)$ is to be considered as a formula of $\mathcal{L}_S$. LI and AP are used as relation symbols for the sets of (the codes of) the limit numbers and additive principal numbers below Γ_0.

For the following considerations, the provably accessible parts of the relation $\prec$ play the decisive role. We only need the inductive operator form

$$\mathfrak{Ap}[X, x] \;:=\; (\forall y \prec x)X(y).$$

Then, given any $\mathcal{L}_S$ formula $B[u]$, the closure assertion $Cl_{\mathfrak{Ap}}[\{x{:}B[x]\}]$ simply means $Prog[\{x{:}B[x]\}]$. For all number terms s and t we introduce as abbreviations:

$$\begin{aligned} AC_s[t] &:= \mathcal{P}_s^{\mathfrak{Ap}}(t), \\ AC_{\prec s}[t] &:= \mathcal{P}_{\prec s}^{\mathfrak{Ap}}(t), \\ AC^s[t] &:= (\forall x \prec s) AC_x[t]. \end{aligned}$$

Thus AC^s describes the intersection of all stratifications of the inductive operator form $\mathfrak{Ap}[X,x]$ less than s. As we can easily conclude from Lemma 3 these intersections have the following property.

Lemma 4 *We can prove in* SID_ν *that*

$$\forall a(a \prec \bar{\nu} \;\rightarrow\; \forall x(AC^{a+1}[x] \;\leftrightarrow\; AC_a[x])).$$

Recall some notation: If the number terms s and t code the ordinal α and the number $n < \omega$, respectively, then $s[t]$ codes the nth component of the fundamental sequence of α. The following useful observation is directly implied by the fixed point axioms.

Lemma 5 *We can prove in* SID_ν*:*

1. $a \prec \bar{\nu} \wedge (\forall x \prec \omega) AC_a[s[x]]) \;\rightarrow\; AC_a[s]$.
2. $a \preceq \bar{\nu} \wedge (\forall x \prec \omega) AC^a[s[x]]) \;\rightarrow\; AC^a[s]$.

After these preparatory remarks we now turn to the wellordering proofs. We begin with considering a property of the stratifications AC_a that will turn out to be central for what follows.

Lemma 6 *We can prove in* SID_ν *that*

$$a+1 \prec \bar{\nu} \wedge AC_{a+1}[b] \;\rightarrow\; AC_a[\omega^b].$$

Proof. Working in SID_ν, we fix an a such that $a+1 \prec \bar{\nu}$. Then define

$$A[x] \;:=\; \forall z(AC_a[z] \;\rightarrow\; AC_a[z+\omega^x]).$$

We show

$$(\forall y \prec x) A[y] \;\rightarrow\; A[x]. \tag{1}$$

for an arbitrary x by distinguishing the following cases.

(i) For $x = 0$ or $LI(x)$, assertion (1) is an immediate consequence of the closure properties of AC_a or of the previous lemma.

(ii) Now assume that $x = y + 1$ for some y. Then complete induction yields

$$A[y] \wedge AC_a[z] \;\rightarrow\; (\forall e \prec \omega) AC_a[z + \omega^y e],$$

and we conclude

$$A[y] \wedge AC_a(z) \;\rightarrow\; AC_a[z + \omega^{y+1}].$$

This establishes (1) also in this case and finishes the proof of this auxiliary consideration. We further observe that the formula

$$B[u, v, w] \;:=\; v \prec u \;\wedge\; \forall z(AC_{\prec u}[\langle v, z\rangle] \;\rightarrow\; AC_{\prec u}[\langle v, z + \omega^w\rangle])$$

belongs to $\mathcal{BL}_S(u)$ and that

(2) $$\forall x(A[x] \;\leftrightarrow\; B[a + 1, a, x]).$$

From (1) we have $Prog[\{x{:}A[x]\}]$, i.e. $Cl_{\mathfrak{Ap}}[\{x : B[a + 1, a, x]\}]$, hence stratified fixed point induction implies

$$AC_{a+1}[b] \;\rightarrow\; B[a + 1, a, b]$$

and thus

$$AC_{a+1}[b] \;\rightarrow\; \forall z(AC_a[z] \;\rightarrow\; AC_a[z + \omega^b])$$

according to (2) and the definition of A. For $z = 0$ this is the assertion of our lemma. $\square$

We continue with introducing a formula, depending on ν, that describes a specific property of stratifications of the accessible parts with respect to the Veblen functions,

$$A_\nu[u] \;:=\; \forall x \forall y(x + \omega^u \prec \bar{\nu} \;\wedge\; AC_{x+\omega^u}[y] \;\rightarrow\; AC^{x+\omega^u}[\varphi u y]).$$

The following two lemmas isolate some technical properties that will be needed in the proofs of Theorem 9 and Corollary 10 below.

Lemma 7 *We can prove in* SID_ν *that*

$$A_\nu[s] \wedge s \prec r \wedge t + \omega^r \preceq \bar{\nu} \;\rightarrow\; \forall x(AC^{t+\omega^r}[x] \;\rightarrow\; AC^{t+\omega^r}[\varphi s x]).$$

Proof. Assume $A_\nu[s]$, $s \prec r$, and $t + \omega^r \preceq \bar{\nu}$ and pick an x such that $AC^{t+\omega^r}[x]$. Now we distinguish the following cases:

(i) $r = p+1$. Choose an arbitrary $u \prec \omega$. Then $t+\omega^p u+\omega^s \prec t+\omega^r$, and so $AC^{t+\omega^r}[x]$ implies $AC_{t+\omega^p u+\omega^s}[x]$. Using the assumption $A_\nu[s]$, we conclude that $AC^{t+\omega^p u+\omega^s}[\varphi sx]$, hence also $AC^{t+\omega^p u}[\varphi sx]$. So we have shown that

$$(\forall u \prec \omega)AC^{t+\omega^p u}[\varphi sx].$$

In view of Lemma 5 this implies $AC^{t+\omega^r}[\varphi sx]$.

(ii) $r \in LI$. Again we choose an arbitrary $u \prec \omega$ and observe that now $t+\omega^{r[u]}+\omega^s \prec t+\omega^r$. Because of $AC^{t+\omega^r}[x]$ we thus have $AC_{t+\omega^{r[u]}+\omega^s}[x]$, and the assumption $A_\nu[s]$ implies $AC^{t+\omega^{r[u]}+\omega^s}[\varphi sx]$, hence also $AC^{t+\omega^{r[u]}}[\varphi sx]$. This means that we have

$$(\forall u \prec \omega)AC^{t+\omega^{r[u]}}[\varphi sx],$$

and again a simple application of Lemma 5 yields $AC^{t+\omega^r}[\varphi sx]$.

Since for $r = 0$ nothing is to show, the proof of our assertion is complete. □

Lemma 8 *We can prove in* SID_ν *that*

$$(\forall z \prec r)A_\nu[z] \land 0 \prec r \land t+\omega^r \preceq \bar{\nu} \rightarrow Prog[\{x{:}AC^{t+\omega^r}[\varphi rx]\}].$$

Proof. For any a we show that under the assumptions $(\forall z \prec r)A_\nu[z]$, $0 \prec r$, and $t + \omega^r \preceq \bar{\nu}$,

$$(1) \qquad (\forall x \prec a)AC^{t+\omega^r}[\varphi rx] \rightarrow AC^{t+\omega^r}[\varphi ra].$$

For this purpose, fix an arbitrary a and consider the fundamental sequence $(\varphi ra)[u]$, for $u \prec \omega$, of φra. By complete induction on u, making essential use of the previous lemma, we then show

$$(2) \qquad (\forall x \prec a)AC^{t+\omega^r}[\varphi rx] \rightarrow (\forall u \prec \omega)AC^{t+\omega^r}[(\varphi ra)[u]].$$

In this proof a case distinction with respect to φra is carried through. We discuss one case and leave the others to the reader. So assume $r = p+1$ and $a = b+1$. Then $(\varphi ra)[0] = \varphi rb+1$ and $(\varphi ra)[u+1] = \varphi p(\varphi ra)[u]$. Clearly,

$$(3) \qquad (\forall x \prec a)AC^{t+\omega^r}[\varphi rx] \rightarrow AC^{t+\omega^r}[(\varphi ra)[0]].$$

Applying Lemma 7, we also obtain

$$(4) \qquad AC^{t+\omega^r}[(\varphi ra)[u]] \rightarrow AC^{t+\omega^r}[(\varphi ra)[u+1]].$$

Assertion (2) is immediate from (3) and (4) by complete induction. All other cases are similar.

So we have (2). But this completes the proof of our lemma since (1) is an immediate consequence of (2) and Lemma 5. $\square$

Theorem 9 *The theory* SID_ν *proves* $Prog[\{z{:}A_\nu[z]\}]$.

PROOF. Given any a, we have to show in SID_ν that

$$(\forall z \prec a)A_\nu[z] \;\to\; A_\nu[a].$$

If $a = 0$, then $A_\nu[a]$ follows immediately from Lemma 4 and Lemma 6. So let $0 \prec a$ and assume $(\forall z \prec a)A_\nu[z]$. In view of the previous lemma we have

$$\forall x(x + \omega^a \prec \bar{\nu} \;\to\; Prog[\{z{:}AC^{x+\omega^a}[\varphi az]\}]).$$

Since $AC^{x+\omega^a}[\varphi az]$ is (equivalent to) a formula bounded by $x+\omega^a$, stratified fixed point induction yields

$$\forall x(x + \omega^a \prec \bar{\nu} \;\to\; \forall y(AC_{x+\omega^a}[y] \to AC^{x+\omega^a}[\varphi ay])),$$

and this formula is equivalent to $A_\nu[a]$. This completes the proof of this lemma. $\square$

Corollary 10 *The theory* SID_ν *proves*

$$t + \omega^r \preceq \bar{\nu} \wedge 0 \prec r \;\to\; Prog[\{x{:}AC^{t+\omega^r}[\varphi rx]\}].$$

PROOF. Since transfinite induction up to ν is available in SID_ν, the previous theorem gives us $(\forall z \prec \bar{\nu})A_\nu[z]$. By Lemma 8 this implies what we want. $\square$

Theorem 11 *Let* ν *be a limit number. If* $\alpha < \Lambda_\nu$, *then* SID_ν *proves* $AC^\omega[\bar{\alpha}]$ *for the code* $\bar{\alpha}$ *of* α.

PROOF. Since ν is a limit ordinal, there are uniquely determined ordinals $\sigma_1, \ldots, \sigma_m$ such that $\nu = \omega^{\sigma_m} + \ldots + \omega^{\sigma_1}$ and $1 \le \sigma_1 \le \ldots \le \sigma_m$. Pick an arbitrary $\alpha < \Lambda_\nu$. Then simple ordinal computation shows that there exists an ordinal $\beta < \varepsilon(\nu)$ for which

(1) $$\alpha \;<\; \varphi\sigma_m(\varphi\sigma_{m-1}(\ldots(\varphi\sigma_1\beta)\ldots).$$

Since SID_ν comprises transfinite induction up to ν, standard proof-theoretic techniques yield

(2) $$TI[\bar{\beta}, \{x{:}A[x]\}]$$

for all $\mathcal{L}_S$ formulas $A[u]$ where $\bar{\beta}$ is the code of β.

For $i = 1, \dots, m$ we let s_i be the code of σ_i, and for $j = 0, \dots, m-1$ we let t_j be the code of $\omega^{\sigma_m} + \dots + \omega^{\sigma_{j+1}}$. Then we define, for $i = 1, \dots, m$ and $j = 1, \dots, m-1$:

$$B_i[u] \;:=\; AC^{t_{i-1}}[\varphi s_i u] \qquad \text{and} \qquad C_j[u] \;:=\; AC_{t_j}[u].$$

With these notations we have immediately that, for $i = 1, \dots, m-1$,

(3) $$\forall x(B_i[x] \;\rightarrow\; C_i[\varphi s_i x]).$$

By Corollary 10 we also know that

(4) $$Prog[\{x{:}B_i[x]\}]$$

for all $i = 1, \dots, m$. As in previous proofs we observe that every formula $B_{j+1}[u]$, for $j = 1, \dots, m-1$, is (equivalent to) a formula bounded by t_j such that stratified fixed point induction implies

(5) $$\forall x(C_i[x] \;\rightarrow\; B_{i+1}[x])$$

for $i = 1, \dots, m-1$. Now we proceed as follows. From (2) and (4) we obtain $B_1[\bar{\beta}]$, and then iterative applications of (3) and (5) lead to

$$B_m[\varphi s_{m-1}(\dots(\varphi s_1 \bar{\beta})\dots)], \text{ i.e. to } AC^{t_{m-1}}[\varphi s_m(\dots(\varphi s_1 \bar{\beta})\dots)].$$

Since $\omega \preceq \omega^{s_m} = t_{m-1}$ and in view of (1) we finally conclude that $AC^{\omega}[\bar{\alpha}]$, as desired. □

Now the stage is set for determining the lower proof-theoretic bounds of SID_ν. For finite ν, the situation is trivial. In the transfinite cases, the previous theorem and some methods of predicative proof theory do the job.

Theorem 12 $\varphi\Lambda_\nu 0 \;\leq\; |\mathsf{SID}_\nu|$.

Proof. SID_0 is equivalent to Peano arithmetic PA, and $\Lambda_0 = \varepsilon_0$. If ν is finite and greater than 0, then SID_ν contains the theory $\widehat{\mathsf{ID}}_1$, whose proof-theoretic ordinal is $\varphi\varepsilon_0 0$ (cf.,e.g., Aczel [2] or Feferman [9]), and $\Lambda_\nu = \varphi\varepsilon_0 0$. So the theorem is clear for $\nu < \omega$.

Let us turn to the more interesting situation and assume that $\nu = \mu + n$ for some limit number μ and some $n < \omega$. Then $\Lambda_\nu = \Lambda_\mu$, and given any ordinal $\alpha < \Lambda_\nu$, the previous theorem yields

$$\mathsf{SID}_\nu \vdash AC^\omega[\bar{\alpha}] \tag{1}$$

for the code $\bar{\alpha}$ of α. Consider an arbitrary formula $A[u,v]$ belonging to $\mathcal{BL}_S(u)$. Because of the axioms about stratified fixed point induction we have

$$\mathsf{SID}_\nu \vdash (\forall a \prec \bar{\nu})(Prog[\{z{:}A[a,z]\}] \;\rightarrow\; \forall x(AC_a[x] \;\rightarrow\; A[a,x])). \tag{2}$$

In particular, if we write 1 for the code of the ordinal $1 \prec \omega$, then (2) implies

$$\mathsf{SID}_\nu \vdash \forall x(AC_1[x] \;\rightarrow\; TI[x, \{z{:}A[1,z]\}]).$$

Together with (1) we thus have

$$\mathsf{SID}_\nu \vdash TI[\bar{\alpha}, \{x{:}A[1,x]\}] \tag{3}$$

for all formulas $A[u,v]$ from $\mathcal{BL}_S(u)$.

The following is a standard result of predicative proof-theory, rephrased in the terminology of this article:

> Let r be a closed number term. If $\mathsf{SID}_\nu \vdash TI[r, \{z{:}A[1,z]\}]$ for all formulas $A[u,v]$ from $\mathcal{BL}_S(u)$, then $\mathsf{SID}_\nu \vdash TI[\varphi r0, \{z{:}B[z]\}]$ for all formulas $B[v]$ of $\mathcal{L}$.

This assertion is proved in detail in Buchholz [6] for the theory $\widehat{\mathsf{ID}}_1$, but since SID_ν contains $\widehat{\mathsf{ID}}_1$ it transfers without problems. Applying this result to (3) yields

$$\mathsf{SID}_\nu \vdash TI[\varphi\bar{\alpha}0, \{z{:}B[z]\}]$$

for all $\mathcal{L}$ formulas $B[v]$, hence, in particular, $\mathsf{SID}_\nu \vdash TI[\varphi\bar{\alpha}0, \{x{:}W(x)\}]$. Therefore, $\varphi\alpha0$ is provable in SID_ν.

We have shown that for any $\alpha < \Lambda_\nu$ the ordinal $\varphi\alpha0$ is provable in SID_ν. This implies the assertion $\varphi\Lambda_\nu 0 \leq |\mathsf{SID}_\nu|$. □

5 Upper proof-theoretic bound for SID_ν

In this section we establish the (sharp) upper proof-theoretic bounds of the theories SID_ν. Our strategy is similar to that used in Jäger, Kahle, Setzer, and Strahm [12] for the proof-theoretic analysis of transfinitely iterated fixed point theories.

We first introduce auxiliary semiformal systems H_ν, in which the theories SID_ν can be embedded. The ordinal analysis of these H_ν via the methods of partial cut elimination and asymmetric interpretation finally yields the desired result. To simplify the notation, we assume from now on that in SID_ν we work with one inductive operator form $\mathfrak{A}[X, x]$ only; the generalization of everything to a finite number of such forms is obvious.

The language $\mathcal{L}_\infty$ extends the basic first order language $\mathcal{L}$ by unary relation symbols $P^{\mathfrak{A}}_{<\alpha}$, $P^{\mathfrak{A}}_{\alpha}$, and $Q^{<\alpha}_{\mathfrak{A}}$ for all ordinals $\alpha < \Gamma_0$. In the systems H_ν the relation symbols $P^{\mathfrak{A}}_{<\alpha}$ and $P^{\mathfrak{A}}_{\alpha}$ will be used to deal with the stratifications $\mathcal{P}^{\mathfrak{A}}_{\prec a}$ and $\mathcal{P}^{\mathfrak{A}}_{a}$ of the theories SID_ν, whereas the sets defined by the $Q^{<\alpha}_{\mathfrak{A}}$ represent the initial stages

$$I^{<\alpha}_{\mathfrak{A}} = \bigcup_{\xi<\alpha} \{n \in \mathbb{N} : \mathbb{N} \models \mathfrak{A}[I^{<\xi}_{\mathfrak{A}}, n]\}$$

of the inductive definition that is associated to the inductive operator form $\mathfrak{A}[X, x]$ up to the ordinals $\alpha < \Gamma_0$.

Definition 13 *The* formulas $(A, B, C, A_0, B_0, C_0, \dots)$ *of* $\mathcal{L}_\infty$ *together with their lengths are inductively generated as follows:*

1. *Every closed literal of* $\mathcal{L}$ *is an* $\mathcal{L}_\infty$ *formula of length* 0.

2. *If* t *is a closed number term and* $\alpha < \Gamma_0$*, then* $P^{\mathfrak{A}}_{<\alpha}(t)$ *and* $\sim P^{\mathfrak{A}}_{<\alpha}(t)$ *are* $\mathcal{L}_\infty$ *formulas of length* 1.

3. *If* t *is a closed number term and* $\alpha < \Gamma_0$*, then* $P^{\mathfrak{A}}_{\alpha}(t)$ *and* $\sim P^{\mathfrak{A}}_{\alpha}(t)$ *are* $\mathcal{L}_\infty$ *formulas of length* 0.

4. *If* t *is a closed number term and* $\alpha < \Gamma_0$*, then* $Q^{<\alpha}_{\mathfrak{A}}(t)$ *and* $\sim Q^{<\alpha}_{\mathfrak{A}}(t)$ *are* $\mathcal{L}_\infty$ *formulas of length* 0.

5. *If* A *is an* $\mathcal{L}_\infty$ *formula of length* m *and if* B *is an* $\mathcal{L}_\infty$ *formula of length* n*, then* $(A \vee B)$ *and* $(A \wedge B)$ *are* $\mathcal{L}_\infty$ *formulas of length* $\max(m, n) + 1$.

6. *If* $A[0]$ *is an* $\mathcal{L}_\infty$ *formula of length* m*, then* $\exists x A[x]$ *and* $\forall x A[x]$ *are* $\mathcal{L}_\infty$ *formulas of length* $m + 1$.

$\mathcal{L}_\infty$ formulas of length 0 are called $\mathcal{L}_\infty$ *literals*. Clearly, as in the language $\mathcal{L}$, a literal of the form $\sim E$ acts as negation of the literal E, the negations $\neg A$ of arbitrary $\mathcal{L}_\infty$ formulas A are defined by making use of De Morgan's laws plus the law of double negation, and the remaining logical connectives are abbreviated as usual.

For the later proof-theoretic considerations we assign two ordinals to any $\mathcal{L}_\infty$ formula A: its level $lev(A)$ provides a bound of the stratifications of the fixed point occurring in A, and its stage $stg(A)$ informs us about the maximal α for which $Q_{\mathfrak{A}}^{<\alpha}(t)$ or $\neg Q_{\mathfrak{A}}^{<\alpha}(t)$ is a subformula of A.

Definition 14 *The* level $lev(A)$ *and the* stage $stg(A)$ *of an* $\mathcal{L}_\infty$ *formula* A *are inductively defined as follows:*

1. *If A is a closed literal of $\mathcal{L}$, then $lev(A) := stg(A) := 0$.*
2. *If A is of the form $P^{\mathfrak{A}}_{<\alpha}(t)$ or $\neg P^{\mathfrak{A}}_{<\alpha}(t)$, then*
$$lev(A) := \alpha \ \text{ and } \ stg(A) := 0.$$
3. *If A is of the form $P^{\mathfrak{A}}_{\alpha}(t)$ or $\neg P^{\mathfrak{A}}_{\alpha}(t)$, then*
$$lev(A) := \alpha + 1 \ \text{ and } \ stg(A) := 0.$$
4. *If A is of the form $Q_{\mathfrak{A}}^{<\alpha}(t)$ or $\neg Q_{\mathfrak{A}}^{<\alpha}(t)$, then*
$$lev(A) := 0 \ \text{ and } \ stg(A) := \alpha.$$
5. *If A is of the form $(B \vee C)$ or $(B \wedge C)$, then*
$$lev(A) := \max(lev(B), lev(C)) \ \text{ and } \ stg(A) := \max(stg(B), stg(C)).$$
6. *If A is of the form $\exists x B[x]$ or $\forall x B[x]$, then*
$$lev(A) := lev(B[0]) \ \text{ and } \ stg(A) := stg(B[0]).$$

If $\nu < \Gamma_0$, we write $\mathcal{L}_\nu$ for the collection of all $\mathcal{L}_\infty$ formulas of levels less than or equal to ν.

Observe that $\mathcal{L}_\infty$ formulas do not contain free number variables. As a consequence, any number term t occurring in an $\mathcal{L}_\infty$ formula has a specific numerical value $t^{\mathbb{N}}$, and we denote two $\mathcal{L}_\infty$ literals as *numerically equivalent* iff they are syntactically identical modulo number terms of the same value.

Furthermore, we write $pair[t]$ iff the closed number term t codes a pair, i.e. iff $t^{\mathbb{N}}$ is equal to $\langle (t^{\mathbb{N}})_0, (t^{\mathbb{N}})_1 \rangle$. Finally, we extend the function ot mentioned in Section 2 to all closed number terms by setting $ot(t) := ot(t^{\mathbb{N}})$. Hence

$ot(t)$ is the unique ordinal less than Γ_0 that is associated to the closed number term t with respect to the wellordering $\prec$.

Every semiformal system H_ν is formulated as a Tait-style calculus for finite subsets $(\Delta, \Pi, \Sigma, \Delta_0, \Pi_0, \Sigma_0, \dots)$ of $\mathcal{L}_\nu$. If $\Delta \subseteq \mathcal{L}_\nu$ and $A \in \mathcal{L}_\nu$, then Δ, A is shorthand for $\Delta \cup \{A\}$; similarly for expressions such as Δ, A, B and Δ, Π, A. Every system H_ν comprises the following axioms and rules of inference.

I. Axioms, group 1. For all finite $\Delta \subseteq \mathcal{L}_\nu$, all numerically equivalent literals $A, B \in \mathcal{L}_\nu$, and all true literals C of $\mathcal{L}$:

$$\Delta,\ \neg A,\ B \qquad \text{and} \qquad \Delta,\ C.$$

II. Axioms, group 2. For all finite $\Delta \subseteq \mathcal{L}_\nu$, all $\alpha \leq \nu$, all closed number terms s such that $pair[s]$ is false, and all closed number terms t such that $pair[t]$ is true and $\alpha \leq ot((t)_0)$:

$$\Delta,\ \neg P^{\mathfrak{A}}_{<\alpha}(s) \qquad \text{and} \qquad \Delta,\ \neg P^{\mathfrak{A}}_{<\alpha}(t).$$

III. Induction axioms. For all finite $\Delta \subseteq \mathcal{L}_\nu$, all $\alpha < \nu$, all closed number terms t, and all formulas $B[0] \in \mathcal{L}_\alpha$:

$$\Delta,\ \neg Cl_{\mathfrak{A}}[B],\ \neg P^{\mathfrak{A}}_{\alpha}(t),\ B[t].$$

IV. Stage axioms. For all finite $\Delta \subseteq \mathcal{L}_\nu$, all $\alpha \leq \beta < \Gamma_0$, and all closed number terms s, t that have the same value:

$$\Delta,\ \neg Q^{<\alpha}_{\mathfrak{A}}(s),\ Q^{<\beta}_{\mathfrak{A}}(t).$$

V. Fixed point rules, group 1. For all finite $\Delta \subseteq \mathcal{L}_\nu$, all $\alpha < \beta \leq \nu$, and all closed number terms t such that $pair[t]$ is true and $ot((t)_0) = \alpha$:

$$\frac{\Delta,\ P^{\mathfrak{A}}_{\alpha}((t)_1)}{\Delta,\ P^{\mathfrak{A}}_{<\beta}(t)} \qquad \text{and} \qquad \frac{\Delta,\ \neg P^{\mathfrak{A}}_{\alpha}((t)_1)}{\Delta,\ \neg P^{\mathfrak{A}}_{<\beta}(t)}.$$

VI. Fixed point rules, group 2. For all finite $\Delta \subseteq \mathcal{L}_\nu$, all $\alpha < \nu$, and all closed number terms t:

$$\frac{\Delta,\ \mathfrak{A}[P^{\mathfrak{A}}_{\alpha}, t]}{\Delta,\ P^{\mathfrak{A}}_{\alpha}(t)} \qquad \text{and} \qquad \frac{\Delta,\ \neg\mathfrak{A}[P^{\mathfrak{A}}_{\alpha}, t]}{\Delta,\ \neg P^{\mathfrak{A}}_{\alpha}(t)}.$$

VII. Stage rules. For all finite $\Delta \subseteq \mathcal{L}_\nu$, all $\alpha < \beta < \Gamma_0$, and all closed number terms t:

$$\frac{\Delta,\ \mathfrak{A}[Q^{<\alpha}_{\mathfrak{A}}, t]}{\Delta,\ Q^{<\beta}_{\mathfrak{A}}(t)} \qquad \text{and} \qquad \frac{\Delta,\ \neg\mathfrak{A}[Q^{<\xi}_{\mathfrak{A}}, t] \quad \text{for all } \xi < \beta}{\Delta,\ \neg Q^{<\beta}_{\mathfrak{A}}(t)}.$$

VIII. Propositional rules. For all finite $\Delta \subseteq \mathcal{L}_\nu$ and all $A, B \in \mathcal{L}_\nu$:

$$\frac{\Delta,\ A,\ B}{\Delta,\ A \vee B} \qquad \text{and} \qquad \frac{\Delta,\ A \qquad \Delta,\ B}{\Delta,\ A \wedge B}.$$

IX. Quantifier rules. For all finite $\Delta \subseteq \mathcal{L}_\nu$ and all $A[s] \in \mathcal{L}_\nu$:

$$\frac{\Delta,\ A[s]}{\Delta,\ \exists x A[x]} \qquad \text{and} \qquad \frac{\Delta,\ A[t] \quad \text{for all closed number terms } t}{\Delta,\ \forall x A[x]}.$$

X. Cut rules. For all finite $\Delta \subseteq \mathcal{L}_\nu$ and all $A \in \mathcal{L}_\nu$:

$$\frac{\Delta,\ A \qquad \Delta,\ \neg A}{\Delta},$$

where the formulas A and $\neg A$ are called the *cut formulas* of this cut.

Before turning to the proof-theoretic analysis of the systems H_ν, we need some further auxiliary notions. We first fix the collection of those formulas that still may be used as cuts after partial cut elimination has been carried through.

Definition 15 *$Simp_\nu$ is defined to be the subset of $\mathcal{L}_\nu$ that comprises $\mathcal{L}_0$, all $\mathcal{L}_\infty$ formulas of levels less than ν, and all elements of $\mathcal{L}_\nu$ of the form $P^{\mathfrak{A}}_\alpha(t)$ and $\neg P^{\mathfrak{A}}_\alpha(t)$ for $\alpha < \nu$.*

According to this definition, $Simp_0 = \mathcal{L}_0$. Hence $Simp_\nu$ is an interesting set of formulas only for $\nu > 0$. Now, depending on $Simp_\nu$, we introduce a complexity measure for all formulas in $\mathcal{L}_\nu$ that measures their complexities "above" $Simp_\nu$. This is the measure to be used for partial cut elimination.

Definition 16 *The ν-*rank *$rk_\nu(A)$ of an $A \in \mathcal{L}_\nu$ is inductively defined as follows:*

1. *If $A \in Simp_\nu$, then $rk_\nu(A) := 0$.*

2. *If A is a formula $P^{\mathfrak{A}}_{<\nu}(t)$ or $\neg P^{\mathfrak{A}}_{<\nu}(t)$, then $rk_\nu(A) := 1$.*

3. *If A does not belong to $Simp_\nu$ and is of the form $(B \vee C)$ or $(B \wedge C)$, then $rk_\nu(A) := \max(rk_\nu(B), rk_\nu(C)) + 1$.*

4. *If A does not belong to $Simp_\nu$ and is of the form $\exists x B[x]$ or $\forall x B[x]$, then $rk_\nu(A) := rk_\nu(B[0]) + 1$.*

Obviously, the ν-rank of any $A \in \mathcal{L}_\nu$ is finite and less than or equal to the length of A, and $rk_\nu(A) = 0$ iff $A \in Simp_\nu$; also $rk_\nu(A) = rk_\nu(\neg A)$ for any $A \in \mathcal{L}_\nu$. Since $Simp_0 = \mathcal{L}_0$ we have $rk_0(A) = 0$ for all $A \in \mathcal{L}_0$.

Definition 17 *We define* $\mathsf{H}_\nu \vdash^{\alpha}_{(\sigma,m,n)} \Delta$ *for all finite* $\Delta \subseteq \mathcal{L}_\nu$*, all* $m, n < \omega$*, and all ordinals* $\alpha, \sigma < \Gamma_0$ *by induction on* α*.*

1. *If* Δ *is an axiom of* H_ν*, then we have* $\mathsf{H}_\nu \vdash^{\alpha}_{(\sigma,m,n)} \Delta$ *for all* $m, n < \omega$ *and all* $\alpha, \sigma < \Gamma_0$*.*

2. *If* $\mathsf{H}_\nu \vdash^{\alpha_i}_{(\sigma,m,n)} \Delta_i$ *and* $\alpha_i < \alpha$ *for every premise* Δ_i *of a fixed point rule, a stage rule, a propositional rule, or a quantifier rule of* H_ν*, then we have* $\mathsf{H}_\nu \vdash^{\alpha}_{(\sigma,m,n)} \Delta$ *for the conclusion* Δ *of this rule.*

3. *Under the assumptions*

   - $\mathsf{H}_\nu \vdash^{\alpha_0}_{(\sigma,m,n)} \Delta, A$ *and* $\alpha_0 < \alpha$*,*
   - $\mathsf{H}_\nu \vdash^{\alpha_1}_{(\sigma,m,n)} \Delta, \neg A$ *and* $\alpha_1 < \alpha$*,*
   - $stg(A) < \sigma$*,* $rk_\nu(A) < m$*, and the length of* A *is less than* n

   we have $\mathsf{H}_\nu \vdash^{\alpha}_{(\sigma,m,n)} \Delta$*.*

In addition, $\mathsf{H}_\nu \vdash^{<\alpha}_{(\sigma,m,n)} \Delta$ *means* $\mathsf{H}_\nu \vdash^{\beta}_{(\sigma,m,n)} \Delta$ *for some* $\beta < \alpha$*.*

Thus $\mathsf{H}_\nu \vdash^{\alpha}_{(\sigma,m,n)} \Delta$ states that there exists a proof of Δ in the system H_ν whose depth is bounded by α such that any cut formula in this proof is of stage less than σ, of ν-rank smaller than m, and of length smaller than n. Consequently, $\mathsf{H}_\nu \vdash^{\alpha}_{(\sigma,1,n)} \Delta$ implies that there is a proof of Δ in H_ν with all its cut formulas belonging to the set $Simp_\nu$.

It is easy to verify that the axioms and rules of H_ν and the notion of ν-rank are tailored in such a way that all cuts but the ones from $Simp_\nu$ can be eliminated. The following lemma, whose proof is left to the reader, is shown by standard proof-theoretic methods. You may also consult Ranzi and Strahm [18] for a similar result.

Lemma 18 *For all ordinals* $\alpha, \sigma < \Gamma_0$*, all* $m, n < \omega$*, and all finite* $\Delta \subseteq \mathcal{L}_\nu$ *we have*

$$\mathsf{H}_\nu \vdash^{\alpha}_{(\sigma,m+2,n)} \Delta \quad \Longrightarrow \quad \mathsf{H}_\nu \vdash^{\omega^\alpha}_{(\sigma,m+1,n)} \Delta.$$

Since $\omega_n(\alpha) < \varepsilon(\alpha)$ for all $n < \omega$, the previous lemma immediately yields the following partial cut elimination result for the systems H_ν.

Theorem 19 (Partial cut elimination) *For all* $\alpha, \sigma < \Gamma_0$*, all* $m, n < \omega$*, and all finite* $\Delta \subseteq \mathcal{L}_\nu$ *we have*

$$\mathsf{H}_\nu \vdash^{\alpha}_{(\sigma,m+1,n)} \Delta \quad \Longrightarrow \quad \mathsf{H}_\nu \vdash^{<\varepsilon(\alpha)}_{(\sigma,1,n)} \Delta.$$

Our next aim is a reduction result for $\mathsf{H}_{\nu+1}$: if a finite $\Delta \subseteq \mathcal{L}_\nu$ is provable in $\mathsf{H}_{\nu+1}$, then it can already be proved in H_ν. To show this reduction theorem and several other properties of the systems H_ν, several auxiliary lemmas are needed.

Lemma 20 *If A and B are numerically equivalent elements of $\mathcal{L}_\nu$ of length k, then we have $\mathsf{H}_\nu \vdash^{2k}_{(0,0,0)} \neg A,\, B$.*

Lemma 21 *Let X be a fresh unary relation symbol and $C[X]$ an X positive $\mathcal{L}(X)$ formula of length k (in the usual sense); in addition, assume that $\Delta, A[u], B[u] \subseteq \mathcal{L}_\nu$, $\alpha, \sigma < \Gamma_0$, and $m, n < \omega$. If*

$$\mathsf{H}_\nu \vdash^{\alpha}_{(\sigma,m,n)} \Delta,\, \neg A[t],\, B[t]$$

for all closed number terms t, then we have

$$\mathsf{H}_\nu \vdash^{\alpha+2k}_{(\sigma,m,n)} \Delta,\, \neg C[\{x{:}A[x]\}],\, C[\{x{:}B[x]\}].$$

Lemma 22 *Let $B[u]$ be any element of $\mathcal{L}_\nu$. Then we have for all $\alpha < \Gamma_0$ and all closed number terms t that*

$$\mathsf{H}_\nu \vdash^{\omega\alpha}_{(0,0,0)} \neg Cl_{\mathfrak{A}}[\{x{:}B[x]\}],\, \neg Q^{<\alpha}_{\mathfrak{A}}(t),\, B[t].$$

PROOF. We show this assertion by induction on α. In the case $\alpha = 0$ we simply use the appropriate right stage rule. Given any $0 < \alpha < \Gamma_0$, the induction hypothesis yields

$$\mathsf{H}_\nu \vdash^{\omega\xi}_{(0,0,0)} \neg Cl_{\mathfrak{A}}[\{x{:}B[x]\}],\, \neg Q^{<\xi}_{\mathfrak{A}}(t),\, B[t].$$

for all $\xi < \alpha$ and all closed number terms t. Now we apply the previous lemma and conclude

$$\mathsf{H}_\nu \vdash^{<\omega\xi+\omega}_{(0,0,0)} \neg Cl_{\mathfrak{A}}[\{x{:}B[x]\}],\, \neg\mathfrak{A}[Q^{<\xi}_{\mathfrak{A}}, t],\, \mathfrak{A}[B, t].$$

Making use of Lemma 20 and some propositional rules we thus have

$$\mathsf{H}_\nu \vdash^{<\omega\xi+\omega}_{(0,0,0)} \neg Cl_{\mathfrak{A}}[\{x{:}B[x]\}],\, \neg\mathfrak{A}[Q^{<\xi}_{\mathfrak{A}}, t],\, \mathfrak{A}[B, t] \wedge \neg B[t],\, B[t].$$

By an existential quantification over t we therefore obtain

$$\mathsf{H}_\nu \vdash^{<\omega\xi+\omega}_{(0,0,0)} \neg Cl_{\mathfrak{A}}[\{x{:}B[x]\}],\, \neg\mathfrak{A}[Q^{<\xi}_{\mathfrak{A}}, t],\, B[t].$$

Since $\omega\xi + \omega \leq \omega\alpha$ for all $\xi < \alpha$, a final application of a right stage rule implies what we want. □

We write $\mathcal{L}^-_{\nu+1}$ for the collection of all $A \in \mathcal{L}_{\nu+1}$ that do not contain subformulas of the form $P^{\mathfrak{A}}_{<\nu+1}(t)$ or $\neg P^{\mathfrak{A}}_{<\nu+1}(t)$. For the following considerations it is convenient (and sufficient) to restrict our attention to such formulas.

Definition 23 *Assume $\alpha, \beta < \Gamma_0$.*

1. *For any $A \in \mathcal{L}_{\nu+1}^{-}$ we define the set $Var_\nu(A, \alpha, \beta)$ of the (α, β)*-variants of A with respect to ν *by induction on the build-up of A as follows:*

   (a) *If $A \in \mathcal{L}_\nu$, then A belongs to $Var_\nu(A, \alpha, \beta)$.*

   (b) *If A is the formula $\neg P_\nu^{\mathfrak{A}}(t)$ and $\eta \leq \alpha$, then $\neg Q_{\mathfrak{A}}^{<\eta}(t)$ belongs to $Var_\nu(A, \alpha, \beta)$.*

   (c) *If A is the formula $P_\nu^{\mathfrak{A}}(t)$ and $\beta \leq \xi$, then $Q_{\mathfrak{A}}^{<\xi}(t)$ belongs to $Var_\nu(A, \alpha, \beta)$.*

   (d) *If A is the formula $(B_0 \vee B_1)$ and $C_i \in Var_\nu(B_i, \alpha, \beta)$ for $i = 0, 1$, then also $(C_0 \vee C_1)$ belongs to $Var_\nu(A, \alpha, \beta)$.*

   (e) *If A is the formula $(B_0 \wedge B_1)$ and $C_i \in Var_\nu(B_i, \alpha, \beta)$ for $i = 0, 1$, then also $(C_0 \wedge C_1)$ belongs to $Var_\nu(A, \alpha, \beta)$.*

   (f) *If A is the formula $\exists x B[x]$ and $C[0] \in Var_\nu(B[0], \alpha, \beta)$, then also $\exists x C[x]$ belongs to $Var_\nu(A, \alpha, \beta)$.*

   (g) *If A is the formula $\forall x B[x]$ and $C[0] \in Var_\nu(B[0], \alpha, \beta)$, then also $\forall x C[x]$ belongs to $Var_\nu(A, \alpha, \beta)$.*

2. *Now let Δ be a finite subset of $\mathcal{L}_{\nu+1}^{-}$. We call a finite $\Pi \subseteq \mathcal{L}_\nu$ an (α, β)-variant of Δ with respect to ν iff for every $A \in \Delta$ there exists a $B \in \Pi$ such that $B \in Var_\nu(A, \alpha, \beta)$. The set of all (α, β)-variants of Δ with respect to ν is denoted by $Var_\nu(\Delta, \alpha, \beta)$.*

For $A \in \mathcal{L}_{\nu+1}^{-}$ any (α, β)-variant of A with respect to ν is an asymmetric interpretation of A: (i) every occurrence of a negative literal $\sim P_\nu^{\mathfrak{A}}(t)$ in A is replaced by $\sim Q_{\mathfrak{A}}^{<\eta}(t)$ for some $\eta \leq \alpha$ and (ii) every occurrence of a positive literal $P_\nu^{\mathfrak{A}}(t)$ in A is replaced by $Q_{\mathfrak{A}}^{<\xi}(t)$ for some $\xi \geq \beta$. This form of asymmetric interpretation is instrumental for the reduction of $\mathsf{H}_{\nu+1}$ to H_ν, as carried through now.

The method of asymmetric interpretation belongs to the standard repertoire of predicative proof theory. One of its first applications is in Schütte [21], but it has been used in numerous other contexts since then; cf., for example, Cantini [8], Jäger [11], or Rathjen [19]. Ranzi and Strahm [18] also proves a similar result.

Lemma 24 *For all $\alpha, \beta, \sigma < \Gamma_0$, all $n < \omega$, all $\tau \geq \max(\sigma, \beta + \omega^\alpha)$, all finite $\Delta \subseteq \mathcal{L}_{\nu+1}^{-}$, and all $\Pi \in Var_\nu(\Delta, \beta, \beta + \omega^\alpha)$ we have that*

$$\mathsf{H}_{\nu+1} \mathrel{\vert\!\frac{\alpha}{(\sigma,1,n)}} \Delta \quad \Longrightarrow \quad \mathsf{H}_\nu \mathrel{\vert\!\frac{\omega\beta+\omega^{2\alpha}}{(\tau,n,n)}} \Pi.$$

PROOF. We proceed by induction on α and distinguish the following cases.

(i) If Δ is an axiom of group 1 or group 2, or a stage axiom, then our assertion is obvious. If Δ is an induction axiom, then Π is either an induction axiom, or we simply apply Lemma 22.

(ii) Δ is the conclusion of a fixed point rule whose main formula belongs to $\mathcal{L}_\nu$, the conclusion of a propositional rule, the conclusion of a quantifier rule, or the conclusion of a cut whose cut formulas belong to $\mathcal{L}_\nu$. Then our assertion immediately follows from the induction hypothesis.

(iii) Δ is the conclusion of a left group 2 fixed point rule of the form

$$\frac{\Sigma,\, \mathfrak{A}[P_\nu^{\mathfrak{A}}, t]}{\Sigma,\, P_\nu^{\mathfrak{A}}(t)}.$$

In this case Π is of the form $\Pi', Q_{\mathfrak{A}}^{<\gamma}(t)$ with $\Pi' \in \mathit{Var}_\nu(\Delta, \beta, \beta + \omega^\alpha)$ and $\beta + \omega^\alpha \leq \gamma$, and there exists a $\delta < \alpha$ such that

$$\mathsf{H}_{\nu+1} \vdash^{\delta}_{(\sigma,1,n)} \Sigma,\, \mathfrak{A}[P_\nu^{\mathfrak{A}}, t].$$

Hence the induction hypothesis implies

$$\mathsf{H}_{\nu} \vdash^{\omega\beta+\omega^{2\delta}}_{(\tau,n,n)} \Pi',\, \mathfrak{A}[Q_{\mathfrak{A}}^{<\beta+\omega^\delta}, t].$$

Since $\omega\beta + \omega^{2\delta} + 2 \leq \omega\beta + \omega^{2\alpha}$ and $\beta + \omega^\delta < \beta + \omega^\alpha \leq \gamma$, it follows

$$\mathsf{H}_{\nu} \vdash^{\omega\beta+\omega^{2\alpha}}_{(\tau,n,n)} \Pi',\, Q_{\mathfrak{A}}^{<\gamma}(t)$$

by a left stage rule.

(iv) Δ is the conclusion of a right group 2 fixed point rule of the form

$$\frac{\Sigma,\, \neg\mathfrak{A}[P_\nu^{\mathfrak{A}}, t]}{\Sigma,\, \neg P_\nu^{\mathfrak{A}}(t)}.$$

Now Π is of the form $\Pi', \neg Q_{\mathfrak{A}}^{<\gamma}(t)$ with $\Pi' \in \mathit{Var}_\nu(\Delta, \beta, \beta + \omega^\alpha)$ and $\gamma \leq \beta$, and there exists a $\delta < \alpha$ such that

$$\mathsf{H}_{\nu+1} \vdash^{\delta}_{(\sigma,1,n)} \Sigma,\, \neg\mathfrak{A}[P_\nu^{\mathfrak{A}}, t].$$

If $\gamma = 0$, then our assertion is immediate by an application of a right stage rule. Otherwise, we obtain in view of the induction hypothesis that

$$\mathsf{H}_{\nu} \vdash^{\omega\beta+\omega^{2\delta}}_{(\tau,n,n)} \Pi',\, \neg\mathfrak{A}[Q_{\mathfrak{A}}^{<\xi}, t]$$

for all $\xi < \gamma$. Consequently, since $\omega\beta + \omega^{2\delta} + 2 \leq \omega\beta + \omega^{2\alpha}$, we have

$$\mathsf{H}_\nu \vdash^{\omega\beta+\omega^{2\alpha}}_{(\tau,n,n)} \Pi',\ \neg Q^{<\gamma}_{\mathfrak{A}}(t)$$

by a left stage rule.

(iv) Δ is the conclusion of a cut of the form

$$\frac{\Delta,\ P^{\mathfrak{A}}_\nu(t) \qquad \Delta,\ \neg P^{\mathfrak{A}}_\nu(t)}{\Delta}.$$

Then there exist $\gamma, \delta < \alpha$ such that

(1) $$\mathsf{H}_{\nu+1} \vdash^{\gamma}_{(\sigma,1,n)} \Delta,\ P^{\mathfrak{A}}_\nu(t),$$

(2) $$\mathsf{H}_{\nu+1} \vdash^{\delta}_{(\sigma,1,n)} \Delta,\ \neg P^{\mathfrak{A}}_\nu(t).$$

From (1) we conclude with the induction hypothesis that

(3) $$\mathsf{H}_\nu \vdash^{\omega\beta+\omega^{2\gamma}}_{(\tau,n,n)} \Pi,\ Q^{<\beta+\omega^\gamma}_{\mathfrak{A}}(t).$$

Since $\beta + \omega^\gamma + \omega^\delta < \beta + \omega^\alpha$, we have $\Pi \in \mathit{Var}_\nu(\Delta, \beta + \omega^\gamma, \beta + \omega^\gamma + \omega^\delta)$. Hence we obtain from (2) by means of the induction hypothesis that

(4) $$\mathsf{H}_\nu \vdash^{\omega(\beta+\omega^\gamma)+\omega^{2\delta}}_{(\tau,n,n)} \Pi,\ \neg Q^{<\beta+\omega^\gamma}_{\mathfrak{A}}(t).$$

Since $\omega\beta+\omega^{2\gamma} < \omega(\beta+\omega^\gamma)+\omega^{2\delta} < \omega\beta+\omega^{2\alpha}$, $\mathit{stg}(Q^{<\beta+\omega^\gamma}_{\mathfrak{A}}(t)) < \tau$, and both, the ν-rank and the length of $Q^{<\beta+\omega^\gamma}_{\mathfrak{A}}(t)$, are smaller than n, a cut applied to (3) and (4) yields

$$\mathsf{H}_\nu \vdash^{\omega\beta+\omega^{2\alpha}}_{(\tau,n,n)} \Pi.$$

Since now all possible cases have been covered, the proof of our lemma is completed. $\square$

The following reduction theorem for the systems $\mathsf{H}_{\nu+1}$ is an immediate consequence of Theorem 19 and the previous lemma.

Theorem 25 (Reduction) *For all finite $\Delta \subseteq \mathcal{L}_\nu$, all $\alpha, \sigma < \Gamma_0$, and all $n < \omega$ we have that*

$$\mathsf{H}_{\nu+1} \vdash^{\alpha}_{(\sigma,n,n)} \Delta \quad \Longrightarrow \quad \mathsf{H}_\nu \vdash^{<\varepsilon(\alpha)}_{(\sigma+\omega^\alpha,1,n)} \Delta.$$

In a next step we turn to the reduction of systems H_ν for limit ordinals ν. We are interested in finding out what it means that a finite $\Delta \subseteq \mathcal{L}_\mu$ for $\mu < \nu$ is provable in H_ν. The following lemma gives the correct answer.

Lemma 26 *For all $\alpha, \beta, \sigma < \Gamma_0$, all γ with $0 < \gamma < \Gamma_0$, all $\delta < \omega^\gamma$, all $n < \omega$, and all finite $\Delta \subseteq \mathcal{L}_{\beta+\delta}$ we have that*

$$\mathsf{H}_{\beta+\omega^\gamma} \vdash^{\alpha}_{(\sigma,1,n)} \Delta \quad \Longrightarrow \quad \mathsf{H}_{\beta+\delta} \vdash^{\varphi\gamma\alpha}_{(\varphi\gamma(\sigma+\alpha),1,n)} \Delta.$$

Proof. We prove this assertion by main induction on γ and side induction on α. If Δ is an axiom of $\mathsf{H}_{\beta+\omega^\gamma}$, then it is also an axiom of $\mathsf{H}_{\beta+\delta}$ and our claim is trivially satisfied. If Δ is the conclusion of a rule different from a cut rule, our claim is immediate from the induction hypothesis. Finally, if Δ is the conclusion of a cut rule, then there exist $\alpha_0, \alpha_1 < \alpha$ and a formula $A \in \mathit{Simp}_{\beta+\omega^\gamma}$ such that

$$(*) \qquad \mathsf{H}_{\beta+\omega^\gamma} \vdash^{\alpha_0}_{(\sigma,1,n)} \Delta, A \qquad \text{and} \qquad \mathsf{H}_{\beta+\omega^\gamma} \vdash^{\alpha_1}_{(\sigma,1,n)} \Delta, \neg A.$$

Now we distinguish whether $\gamma = 1$, γ is a successor ordinal greater than 1, or γ is a limit ordinal.

(i) $\gamma = 1$. In this case δ is smaller than ω. Since $A \in \mathit{Simp}_{\beta+\omega}$, we know that there exists a natural number $m \geq \delta$ for which $\Delta \subseteq \mathcal{L}_{\beta+m}$ and $A, \neg A \in \mathcal{L}_{\beta+m}$. By the side induction hypothesis we obtain from (*) that

$$\mathsf{H}_{\beta+m} \vdash^{\varphi\gamma\alpha_0}_{(\varphi\gamma(\sigma+\alpha_0),1,n)} \Delta, A \qquad \text{and} \qquad \mathsf{H}_{\beta+m} \vdash^{\varphi\gamma\alpha_1}_{(\varphi\gamma(\sigma+\alpha_1),1,n)} \Delta, \neg A,$$

and a cut yields

$$\mathsf{H}_{\beta+m} \vdash^{\eta}_{(\xi,n,n)} \Delta$$

for $\eta := \max(\varphi\gamma\alpha_0, \varphi\gamma\alpha_1)+1$ and $\xi := \max(\varphi\gamma(\sigma+\alpha_0), \varphi\gamma(\sigma+\alpha_1))$. Finitely many applications of Theorem 25 therefore yield

$$\mathsf{H}_{\beta+\delta} \vdash^{\tau}_{(\varphi(\gamma(\sigma+\alpha),1,n)} \Delta$$

for some $\tau < \varphi\gamma\alpha$. This proves our claim for the case $\gamma = 1$.

(ii) $\gamma = \rho + 1$ for some $\rho \geq 1$. Now $\delta = \omega^\rho m + \zeta$ for some $m < \omega$ and $\zeta < \omega^\rho$. From $A \in \mathit{Simp}_{\beta+\omega^{\rho+1}}$ we now conclude that there exist a natural number $k > m$ such that $\Delta \subseteq \mathcal{L}_{\beta+\omega^\rho k}$ and $A, \neg A \in \mathcal{L}_{\beta+\omega^\rho k}$. Therefore the side induction hypothesis applied to (*) yields

$$\mathsf{H}_{\beta+\omega^\rho k} \vdash^{\varphi\gamma\alpha_0}_{(\varphi\gamma(\sigma+\alpha_0)1,n)} \Delta, A \qquad \text{and} \qquad \mathsf{H}_{\beta+\omega^\rho k} \vdash^{\varphi\gamma\alpha_1}_{(\varphi\gamma(\sigma+\alpha_1)1,n)} \Delta, \neg A,$$

and a cut gives us

$$\mathsf{H}_{\beta+\omega^\rho k} \vdash^{\eta}_{(\xi,n,n)} \Delta$$

for $\eta := \max(\varphi\gamma\alpha_0, \varphi\gamma\alpha_1) + 1$ and $\xi := \max(\varphi\gamma(\sigma+\alpha_0), \varphi\gamma(\sigma+\alpha_1)$. In view of Theorem 19 we have

$$\mathsf{H}_{\beta+\omega^\rho k} \vdash^{\tau}_{(\xi,1,n)} \Delta$$

for some $\tau < \varepsilon(\eta)$. For $i < \omega$ we now set

$$\tau_0 := \tau, \quad \tau_{i+1} := \varphi\rho\tau_i \qquad \text{and} \qquad \xi_0 := \xi, \quad \xi_{i+1} := \varphi\rho(\xi_i + \tau_i).$$

Then $k - m$ applications of the main induction hypothesis yield

$$\mathsf{H}_{\beta+\omega^\rho(k-1)} \vdash^{\tau_1}_{(\xi_1,1,n)} \Delta,$$

$$\mathsf{H}_{\beta+\omega^\rho(k-2)} \vdash^{\tau_2}_{(\xi_2,1,n)} \Delta,$$

$$\vdots$$

$$\mathsf{H}_{\beta+\omega^\rho(m+1)} \vdash^{\tau_{k-m-1}}_{(\xi_{k-m-1},1,n)} \Delta,$$

$$\mathsf{H}_{\beta+\omega^\rho m+\zeta} \vdash^{\tau_{k-m}}_{(\xi_{k-m},1,n)} \Delta.$$

In these reductions we have successively replaced the β of the main induction hypothesis by

$$\beta + \omega^\rho(k-1),\ \beta + \omega^\rho(k-2),\ \ldots,\ \beta + \omega^\rho(m+1),\ \beta + \omega^\rho m.$$

In addition, observe that $\tau_i < \varphi\gamma\alpha$ and $\xi_i < \varphi\gamma(\sigma + \alpha)$ for all $i < \omega$. Hence we have shown that

$$\mathsf{H}_{\beta+\delta} \vdash^{\varphi\gamma\alpha}_{(\varphi\gamma(\sigma+\alpha),1,n)} \Delta,$$

as desired, finishing the treatment of this case.

(iii) γ is a limit number. Because of $A \in Simp_{\beta+\omega^\gamma}$ we know that there exists a $\rho < \gamma$ satisfying $\Delta \subseteq \mathcal{L}_{\beta+\omega^\rho}$ and $A, \neg A \in \mathcal{L}_{\beta+\omega^\rho}$. Thus the side induction hypothesis applied to (*) asserts that

$$\mathsf{H}_{\beta+\omega^\rho} \vdash^{\varphi\gamma\alpha_0}_{(\varphi\gamma(\sigma+\alpha_0),1,n)} \Delta, A \qquad \text{and} \qquad \mathsf{H}_{\beta+\omega^\rho} \vdash^{\varphi\gamma\alpha_1}_{(\varphi\gamma(\sigma+\alpha_1),1,n)} \Delta, \neg A.$$

For $\eta := \max(\varphi\gamma\alpha_0, \varphi\gamma\alpha_1) + 1$ and $\xi := \max(\varphi\gamma(\sigma + \alpha_0), \varphi\gamma(\sigma + \alpha_1))$ we deduce by a cut that

$$\mathsf{H}_{\beta+\omega^\rho} \vdash^{\eta}_{(\xi,n,n)} \Delta.$$

As before, by making use of Theorem 19, we find a $\tau < \varepsilon(\eta)$ such that

$$\mathsf{H}_{\beta+\omega^\rho} \vdash^{\tau}_{(\xi,1,n)} \Delta$$

Now we are in the position to apply the main induction hypothesis and conclude

$$\mathsf{H}_{\beta+\delta} \vdash^{\varphi\rho\tau}_{(\varphi\rho(\xi+\tau),1,n)} \Delta.$$

Since $\varphi\rho\tau < \varphi\gamma\alpha$ and $\varphi\rho(\xi + \tau) < \varphi(\sigma + \alpha)$, this establishes our claim, finishing the proof of case (iii) and also the verification of our lemma. $\square$

Theorem 27 *Assume that Δ is a finite subset of $\mathcal{L}_0$ and $\mathsf{H}_\nu \vdash^{<\varepsilon(\nu)}_{(0,n,n)} \Delta$ for some $n < \omega$. Then there exist $\alpha, \beta < \Lambda_\nu$ such that $\mathsf{H}_0 \vdash^{\alpha}_{(\beta,1,n)} \Delta$.*

PROOF. We first observe that ν can be uniquely written as

$$\nu = \omega^{\nu_m} + \ldots + \omega^{\nu_1} + k$$

with $k < \omega$ and ordinals $\nu_1 \leq \ldots \leq \nu_m \leq \nu$. Set $\mu := \omega^{\nu_m} + \ldots \omega^{\nu_1}$. By our assumptions we know that there exists an ordinal $\gamma < \varepsilon(\nu)$ for which

$$\mathsf{H}_{\mu+k} \vdash^{\gamma}_{(0,n,n)} \Delta.$$

In view of (k applications of) Theorem 25 this implies

(1) $$\mathsf{H}_\mu \vdash^{\delta}_{(\rho,1,n)} \Delta$$

for suitable $\delta, \rho < \varepsilon(\nu)$. By induction on i we now define for all natural numbers i such that $0 \leq i \leq m-1$:

$$\begin{array}{lllllll} \mu_0 & := & \varepsilon(\mu) & \text{and} & \mu_{i+1} & := & \varphi\nu_{i+1}\mu_i, \\ \sigma_0 & := & \delta & \text{and} & \sigma_{i+1} & := & \varphi\nu_{i+1}\sigma_i, \\ \tau_0 & := & \rho & \text{and} & \tau_{i+1} & := & \varphi\nu_{i+1}(\tau_i + \sigma_i). \end{array}$$

A simple induction on i then shows for all $i \leq m$ that

(2) $$\sigma_i < \mu_i \quad \text{and} \quad \tau_i < \mu_i.$$

In view of Lemma 1 and the choice of μ we also know that

(3) $$\Lambda_\nu = \Lambda_\mu = \mu_m.$$

It remains to apply Lemma 26 several times. More precisely, starting off from (1) and making use of Lemma 26 repeatedly we obtain

$$\begin{array}{c} \mathsf{H}_{0+\omega^{\nu_m}+\ldots+\omega^{\nu_1}} \vdash^{\sigma_0}_{(\tau_0,1,n)} \Delta, \\ \mathsf{H}_{0+\omega^{\nu_m}+\ldots+\omega^{\nu_2}} \vdash^{\sigma_1}_{(\tau_1,1,n)} \Delta, \\ \vdots \\ \mathsf{H}_{0+\omega^{\nu_m}} \vdash^{\sigma_{m-1}}_{(\tau_{m-1},1,n)} \Delta, \\ \mathsf{H}_0 \vdash^{\sigma_m}_{(\tau_m,1,n)} \Delta. \end{array}$$

Because of (2) and (3) the last line immediately gives us our assertion for $\alpha := \sigma_m$ and $\beta := \tau_m$. □

This theorem provides a reduction of the systems H_ν to H_0 with respect to all finite $\Delta \subseteq \mathcal{L}_0$. To complete the proof-theoretic analysis of the H_ν we now turn to complete cut elimination for H_0. To this end we first assign a rank $rk(A)$ to any $A \in \mathcal{L}_0$.

Definition 28 *The* rank $rk(A)$ *of an* $A \in \mathcal{L}_0$ *is inductively defined as follows:*

1. *If A is a closed literal of $\mathcal{L}$, then $rk(A) := 0$.*
2. *If A is of the form $P^{\mathfrak{A}}_{<0}(t)$ or $\neg P^{\mathfrak{A}}_{<0}(t)$, then $rk(A) := 1$.*
3. *If A is of the form $Q^{<\alpha}_{\mathfrak{A}}(t)$ or $\neg Q^{<\alpha}_{\mathfrak{A}}(t)$, then $rk(A) := \omega\alpha$.*
4. *If A is of the form $(B \vee C)$ or $(B \wedge C)$, then*
$$rk(A) := \max(rk(B), rk(C)) + 1.$$
5. *If A is of the form $\exists x B[x]$ or $\forall x B[x]$, then $rk(A) := rk(B[0]) + 1$.*

By straightforward induction on the build-up of the formulas in $\mathcal{L}_0$ we can easily verify a close relationship between their stages and ranks.

Lemma 29 *If A is a formula from $\mathcal{L}_0$ of length n and $stg(A) = \alpha$, then $rk(A) < \omega\alpha + n$.*

If we restrict ourselves to the system H_0, then (obviously) the ranks of the cut formulas are the appropriate parameters for measuring the complexities of cuts; levels and lengths of cut formula are no longer interesting. To make this precise, we introduce a slightly modified notion of derivability within H_0.

Definition 30 *We define $\mathsf{H}_0 \overset{\alpha}{\underset{\sigma}{\vdash}} \Delta$ for all finite $\Delta \subseteq \mathcal{L}_0$ and all ordinals $\alpha, \sigma < \Gamma_0$ by induction on α.*

1. *If Δ is an axiom of H_0, then we have $\mathsf{H}_\nu \overset{\alpha}{\underset{\sigma}{\vdash}} \Delta$ for all $\alpha, \sigma < \Gamma_0$.*
2. *If $\mathsf{H}_0 \overset{\alpha_i}{\underset{\sigma}{\vdash}} \Delta_i$ and $\alpha_i < \alpha$ for every premise Δ_i of a stage rule, a propositional rule, or a quantifier rule or a cut of H_0 whose cut formulas have rank less than σ, then we have $\mathsf{H}_0 \overset{\alpha}{\underset{\sigma}{\vdash}} \Delta$ for the conclusion Δ of this rule.*

Thus $\mathsf{H}_0 \vdash^{\alpha}_{0} \Delta$ means that Δ is cut-free provable in the system H_0. Our two methods of measuring derivations in H_0 are, of course, closely linked. In particular, we can easily transform the former into the latter. The proof of the following lemma is trivial by induction on α.

Lemma 31 *For all $\alpha, \beta < \Gamma_0$, all $n < \omega$, and all finite $\Delta \subseteq \mathcal{L}_0$ we have that*

$$\mathsf{H}_0 \vdash^{\alpha}_{(\beta,n,n)} \Delta \quad \Longrightarrow \quad \mathsf{H}_0 \vdash^{\alpha}_{\omega\beta+n} \Delta.$$

The last step in our proof-theoretic analysis of the systems H_ν is complete cut elimination for H_0 with respect to our new derivability relation. However, it is easy to check that the assignment of ranks and the rules of inference are tailored such that the methods of predicative proof theory yield full cut elimination for H_0. Therefore we omit the proof of the following theorem and refer to the standard literature, for example, Pohlers [16] or Schütte [21].

Theorem 32 *For all $\alpha, \beta, \gamma < \Gamma_0$ and all finite $\Delta \subseteq \mathcal{L}_0$ we have that*

$$\mathsf{H}_0 \vdash^{\alpha}_{\beta+\omega^\gamma} \Delta \quad \Longrightarrow \quad \mathsf{H}_0 \vdash^{\varphi\gamma\alpha}_{\beta} \Delta.$$

From Theorem 27, Lemma 31, Theorem 32 and the fact that $\varphi\eta\xi < \varphi\Lambda_\nu 0$ for all $\eta < \Lambda_\nu$ and $\xi < \varphi\Lambda_\nu 0$ we deduce the following key result.

Corollary 33 *Assume that Δ is a finite subset of $\mathcal{L}_0$ and $\mathsf{H}_\nu \vdash^{<\varepsilon(\nu)}_{(0,n,n)} \Delta$ for some $n < \omega$. Then there exists an $\alpha < \varphi\Lambda_\nu 0$ such that $\mathsf{H}_0 \vdash^{\alpha}_{0} \Delta$.*

This corollary brings us very close to the desired computation of the upper proof-theoretic bounds of the theories SID_ν. It now remains only to embed the theories SID_ν into the systems H_ν. The following lemma deals with transfinite induction within H_ν.

Lemma 34 *For all $\alpha < \Gamma_0$, all closed number terms t such that $ot(t) = \alpha$, and all $A[0] \in \mathcal{L}_\nu$ we have:*

1. $\mathsf{H}_\nu \vdash^{<\omega\alpha+\omega}_{(0,0,0)} \neg Prog[\prec, \{x{:}A[x]\}],\ A[t]$.
2. $\mathsf{H}_\nu \vdash^{\omega\alpha+\omega}_{(0,0,0)} Prog[\prec, \{x{:}A[x]\}] \rightarrow (\forall x \prec t)A[x]$.

Proof. As to be expected, we prove the first assertion by induction on α. Let t be a closed term such that $ot(t) = \alpha$. We first consider all closed number terms r for which $\alpha \leq ot(r)$. In this case $r \not\prec t$ is a true closed literal, hence

$$(1) \qquad \mathsf{H}_\nu \vdash^{0}_{(0,0,0)} \neg Prog[\prec, \{x{:}A[x]\}],\ r \not\prec t,\ A[r].$$

Secondly, if s is a closed number term for which $ot(s) = \xi < \alpha$, then the induction hypothesis implies

(2) $$\mathsf{H}_\nu \overset{\omega\xi+\omega}{\underset{(0,0,0)}{\vdash}} \neg Prog[\prec, \{x{:}A[x]\}],\ s \not\prec t,\ A[s].$$

By some simple applications of propositional and quantifier rules, (1) and (2) yield
$$\mathsf{H}_\nu \overset{\omega\alpha+3}{\underset{(0,0,0)}{\vdash}} \neg Prog[\prec, \{x{:}A[x]\}],\ (\forall x \prec t)A[x].$$
Making use of Lemma 20, we can continue with

$$\mathsf{H}_\nu \overset{<\omega\alpha+\omega}{\underset{(0,0,0)}{\vdash}} \neg Prog[\prec, \{x{:}A[x]\}],\ (\forall x \prec t)A[x] \wedge \neg A[t],\ A[t]$$

and obtain
$$\mathsf{H}_\nu \overset{\omega\alpha+\omega}{\underset{(0,0,0)}{\vdash}} \neg Prog[\prec, \{x{:}A[x]\}],\ A[t]$$
by a further application of a quantifier rule, as desired. This completes the proof of the first assertion. The second assertion is a simple consequence of the first. □

The embedding of SID_ν into H_ν can now be defined straightforwardly: To each closed formula A in the language $\mathcal{L}_S$ of SID_ν we associate its interpretation $A^{(\nu)}$, given by replacing all subexpressions $\mathcal{P}^{\mathfrak{A}}(t)$ and $\sim\mathcal{P}^{\mathfrak{A}}(t)$ of A by $P^{\mathfrak{A}}_{<\nu}(t)$ and $\neg P^{\mathfrak{A}}_{<\nu}(t)$, respectively. Based on this translation of the closed $\mathcal{L}_S$ formulas into formulas belonging to $\mathcal{L}_\nu$, we can formulate our embedding theorem.

Theorem 35 (Embedding) *Let A be a closed formula of $\mathcal{L}_S$ and assume $\mathsf{SID}_\nu \vdash A$. Then we have $\mathsf{H}_\nu \overset{<\varepsilon(\nu)}{\underset{(0,n,n)}{\vdash}} A^{(\nu)}$ for some $n < \omega$.*

Proof. Since $\omega\nu + \omega < \varepsilon(\nu)$, the previous lemma takes care of the axioms about transfinite induction up to ν. The translations of (the universal closures of) all other axioms of SID_ν are obviously provable in H_ν and closure under the translations of the inference rules of SID_ν is guaranteed; always respecting the required bounds. □

The previous embedding theorem and Corollary 33 finally provide the reduction of SID_ν with respect to all closed $\mathcal{L}$ formulas to the cut-free part of the system H_0.

Theorem 36 (Final reduction) *If A is a closed $\mathcal{L}$ formula and $\mathsf{SID}_\nu \vdash A$, then there exists an $\alpha < \varphi\Lambda_\nu 0$ such that $\mathsf{H}_0 \overset{\alpha}{\underset{0}{\vdash}} A$.*

Corollary 37 $|\mathsf{SID}_\nu| = \varphi\Lambda_\nu 0$.

PROOF. $\varphi\Lambda_\nu 0 \leq |\mathsf{SID}_\nu|$ is Theorem 12. To show $|\mathsf{SID}_\nu| \leq \varphi\Lambda_\nu 0$ assume that SID_ν proves $TI[\lhd, \{x{:}W(x)\}]$ for some primitive recursive wellordering $\lhd$. By the previous theorem we thus have an ordinal $\alpha < \varphi\Lambda_\nu 0$ such that $\mathsf{H}_0 \,\overset{\alpha}{\underset{0}{\vdash}}\, TI[\lhd, \{x{:}W(x)\}]$. Standard boundedness techniques as presented, for example, in Pohlers [16] and Schütte [21] then imply that the order-type of $\lhd$ is less than or equal to $\omega\alpha < \varphi\Lambda_\nu 0$. Hence every ordinal provable in SID_ν is smaller than $\varphi\Lambda_\nu 0$. □

This finishes our proof-theoretic analysis of the theories SID_ν. Of course, this result also provides the proof-theoretic ordinals of the systems $\mathsf{SID}_{<\nu}$, namely

$$|\mathsf{SID}_{<\nu}| \;=\; \sup(\{\varphi\Lambda_\mu 0 : \mu < \nu\}).$$

The proof-theoretic ordinals of some important such theories have been mentioned in Section 3.

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