Two interpretations of WKL_0 in subsystems of PA

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Introduction

In this thesis we will present two completely different approaches to obtain conservation results of WKL_0 over subsystems of Peano arithmetic.

The first one uses mainly proof-theoretic methods; we will first embed WKL_0 in $\mathsf{s\text{-}RCA}_0$, that is RCA_0 together with the strict Π^1_1 reflection principle (which implies (WKL)). Then we will asymmetrically interpret $\mathsf{s\text{-}RCA}_0$ in Δ_0-CA. From this we obtain Π^0_2-conservation. By model-theoretic arguments (although it could be done purely proof-theoretically) we will show full conservation of Δ_0-CA over PRA, and thus Π^0_2-conservation of WKL_0 over PRA.

The second approach is recursion-theoretic; at first we will define several satisfaction predicates (using a Gödel numbering of the language $\mathcal{L}_0$) and give the definition of the meaningful class of low $\Sigma^\star_0(\Sigma_1)$ sets. The low basis theorem will be the basis from which we will be able to define an operation $B^\star$ which will give rise to defining two predicates, number and class, which will eventually yield the ω-interpretation of WKL_0 in Σ_1-PA. Hence we get full conservation of WKL_0 over Σ_1-PA.

WKL_0 and RCA_0 have their meaningfullness in the foundations of mathematics and reverse mathematics. The main question asks which set existence axioms are needed to support ordinary mathematical reasoning. RCA_0 is related to Bishop's program of constructivism, while on the other hand WKL_0 has relations to Hilbert's finitistic reductionism. In RCA_0 one can develop already a large part of ordinary mathematics (e.g., real or complex analysis) which does not rely on set-theoretic mathematics. RCA_0 is strong enough to prove basic results of analysis such as Baire's category theorem, Urysohn's and Tietze's lemma. On the other hand RCA_0 does not prove weak König's lemma (WKL). Within RCA_0 we can show that (WKL) is equivalent to Heine–Borel covering lemma or Gödel's completeness theorem. From the viewpoint of mathematical practice WKL_0 is much stronger than RCA_0. In fact WKL_0 is strong enough to prove many non-constructive theorems which are important for mathematical practice.

I am grateful to Prof. Gerhard Jäger for introducing me to proof- and recursion-theory and Dr. Thomas A. Strahm for guiding me during my work. I have always appreciated his competent advise. I also wish to acknowledge support and assistance I received from my friends and all others who contributed in one or the other way to the completion of this work.

Thomas Schweizer

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Our difficulty is not in the proofs,
but in learning what to prove.

— EMIL ARTIN

Proof-theoretic Approach

In this chapter we will present a syntactical way to achieve the conservation result of WKL_0 over PRA. We first define the logical systems which we will work in and then the relevant theories we need. As (WKL) is a rather complicated rule, we will introduce $\mathsf{s\text{-}RCA}_0$, that is RCA_0 together with strict Π^1_1-reflection, which implies weak König's lemma (WKL), so we can embed WKL_0 in $\mathsf{s\text{-}RCA}_0$. Then we will asymmetrically interpret $\mathsf{RCA}_0 + (\mathsf{s}\text{-}\Pi^1_1)$, and hence also WKL_0, in the weaker theory $\Delta_0\text{-}\mathsf{CA}$. From this interpretation we will obtain Π^0_2-conservativity of WKL_0 over $\Delta_0\text{-}\mathsf{CA}$

In the last section we will use model-theoretic arguments to show full conservation of $\Delta_0\text{-}\mathsf{CA}$ over PRA, even though it could also be obtained using proof-theoretical methods.

The way we proceed is inspired by Cantini's *Asymmetric Interpretations for Bounded Theories*[2].

1.1 Logical framework

The subsystems $\Delta_0\text{-}\mathsf{CA}$, RCA_0, $\mathsf{s\text{-}RCA}_0$ and WKL_0 of analysis are formulated in the second order language $\mathcal{L}_2$, which consists of number and set variables, symbols for all primitive recursive functions and three relation symbols.

1.1.1 Language $\mathcal{L}_2$ of second order arithmetic

Definition 1.1.1. Let $\mathcal{L}_2$ denote the language of second order arithmetic which contains the following symbols:

(1) countably many free number variables $u1, u2, \ldots$

(2) countably many bound number variables $x1, x2, \ldots$

(3) countably many free set variables $U1, U2, \ldots$

(4) countably many bound set variables $X1, X2, \ldots$

(5) the function symbols are defined inductively by:

  (i) 0 is a 0-ary function symbol and S is a unary function symbol,

  (ii) for all natural numbers n, m and i with $0 \leq i \leq n$ Cs^n_m and Pr^n_i are n-ary function symbols,

  (iii) if f is an m-ary function symbol and $g_1, \ldots, g_m$ are n-ary function symbols, then $\mathsf{Comp}^n(f, g_0, \ldots, g_m)$ is an n-ary function symbol,

  (iv) if f is an n-ary function symbol and g an $(n+2)$-ary function symbol, then $\mathsf{Rec}^{n+1}(f, g)$ is an $(n+1)$-ary function symbol,

(6) the binary relation symbols $=$, $\leq$, and $\in$,

(7) the symbol $\sim$ to express complementary propositions,

(8) the logical connectives $\vee, \wedge, \forall, \exists$,

(9) auxiliary symbols.

In this definition we require that the symbols are syntactically different. The 0-ary function symbols are also called constants of $\mathcal{L}_2$. Because of (5), $\mathcal{L}_2$ contains symbols for all primitive recursive functions. Furthermore we will use $+, \cdot$ as symbols for the primitive recursive function symbols representing addition and multiplication.

Let $\mathfrak{Z}$, $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be a finite sequence of symbols and $u_1, \ldots, u_n$ be a sequence of pairwise distinct free number or set variables. So we write

$$\mathfrak{Z}[\mathfrak{a}_1, \ldots, \mathfrak{a}_n / u_1, \ldots, u_n]$$

for the sequence of symbols, which we obtain by simultaneously replacing all free variables u_i by $\mathfrak{a}_\mathfrak{i}$ for all $1 \leq i \leq n$. We will often use the notion $\mathfrak{Z}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$ instead of $\mathfrak{Z}[\mathfrak{a}_1, \ldots, \mathfrak{a}_n / u_1, \ldots, u_n]$

Definition 1.1.2. $\mathcal{L}_2$-terms are inductively defined by:

(1) all free number variables are terms,

(2) if $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol ($n \geq 1$), then $f(t_1, \ldots, t_n)$ is a term.

Numerals $\bar{n}$ for all natural numbers n are variable-free terms, defined by $\bar{n} :\equiv \mathsf{S}(\ldots\mathsf{S}(0)\ldots)$ where S occurs n-times. They are used to represent natural numbers in $\mathcal{L}_2$.

The positive atomic formulas of $\mathcal{L}_2$ are expressions of the form $t_1 = t_2$, $t_1 \leq t_2$ and $t_1 \in U$ where t_1, t_2 are terms, and U is a set variable. The negative atomic formulas of $\mathcal{L}_2$ are expressions of the form $\sim R$ where R is a positive atomic formula.

Literals are positive or negative atomic formulas.

Definition 1.1.3. $\mathcal{L}_2$-formulas are defined inductively by:

(1) every literal is a formula,

(2) if A, B are formulas, so are $(A \wedge B)$ and $(A \vee B)$,

(3) if A is a formula, u a free number variable and x a bound number variable, which does not occur in A, then $\exists x A[x/u]$ and $\forall x A[x/u]$ are formulas,

(4) if A is a formula, U a free set variable and X a bound set variable, which does not occur in A, then $\exists X A[X/U]$ and $\forall X A[X/U]$ are formulas.

By $\mathrm{FV}(t)$, $\mathrm{FV}(A)$ or $\mathrm{FV}(\Gamma)$ we denote the set of free variables which occur in the term t, in the formula A or the set of formulas Γ respectively. A term or formula is called closed or variable-free if $\mathrm{FV}(t) = \varnothing$ resp. $\mathrm{FV}(A) = \varnothing$. Closed formulas are often called sentences.

Definition 1.1.4 (Negation). The negation $\neg A$ of a formula A is defined inductively by:

(1) if A is a positive atomic formula, then $\neg A :\equiv {\sim} A$,

(2) if $A \equiv {\sim} B$ and B positive atomic, then $\neg A :\equiv B$,

(3) if $A \equiv (B \vee C)$, then $\neg A :\equiv (\neg B \wedge \neg C)$,

(4) if $A \equiv (B \wedge C)$, then $\neg A :\equiv (\neg B \vee \neg C)$,

(5) if $A \equiv \exists x B[x/u]$, then $\neg A :\equiv \forall x \neg B[x/u]$,

(6) if $A \equiv \forall x B[x/u]$, then $\neg A :\equiv \exists x \neg B[x/u]$,

(5) if $A \equiv \exists X B[X/U]$, then $\neg A :\equiv \forall X \neg B[X/U]$,

(6) if $A \equiv \forall X B[X/U]$, then $\neg A :\equiv \exists X \neg B[X/U]$.

The logical implication $(A \rightarrow B)$, logical equivalence $(A \leftrightarrow B)$ and the binary relations $<$, $\neq$ are introduced as abbreviations:

$$(A \rightarrow B) :\equiv (\neg A \vee B) \qquad (A \leftrightarrow B) :\equiv (A \rightarrow B) \wedge (B \rightarrow A)$$

$$(x \neq y) :\equiv \neg(x = y) \qquad (x < y) :\equiv (x \leq y \wedge x \neq y)$$

As we will deal a lot with bounded formulas we introduce the following abbreviations which we will use very often.

(1) $(\forall x \leq t)A(x) :\equiv \forall x(x \leq t \rightarrow A(x))$

(2) $(\exists x \leq s)A(x) :\equiv \exists x(x \leq s \wedge A(x))$

Further we will use the vector notion $\vec{z}$ for finite sequences $z_1, \ldots, z_n$. The arity will always be clear from the context.

1.1.2 Arithmetical hierarchy and asymmetric translation

The quantifiers $(\forall x \leq t)$ and $(\exists x \leq s)$ are called *bounded quantifiers*. By $\Delta^0_0 = \Sigma^0_0 = \Pi^0_0$ we denote the smallest collection of formulas generated from literals by means of conjunction, disjunction and bounded number quantification. Δ^0_0-formulas may contain free set and free number variables, the so-called parameters.

The arithmetical hierarchy is inductively defined by:

(1) A is Σ^0_1 if $A \equiv \exists x B$ for a Δ^0_0-formula B or A is Δ^0_0,
A is Π^0_1 if $A \equiv \forall x B$ for a Δ^0_0-formula B or A is Δ^0_0.

(2) A is Σ^0_{n+1} if $A \equiv \exists x B$ for a Π^0_n-formula B or A is Σ^0_n,
A is Π^0_{n+1} if $A \equiv \forall x B$ for a Σ^0_n-formula B or A is Π^0_{n+1}.

Furthermore the collection of strict Π^1_1- and strict Σ^1_1-formulas will be of a certain interest in the sequel.

Definition 1.1.5 (s-Π^1_1/s-Σ^1_1-formulas).
By s-Π^1_1 we denote the smallest collection of formulas which are generated from literals by means of $\wedge, \vee, \exists x \leq t, \forall x \leq s, \forall X$ and $\exists x$.
By s-Σ^1_1 we denote the smallest collection of formulas which are generated from literals by means of $\wedge, \vee, \exists x \leq t, \forall x \leq s, \exists X$ and $\forall x$.

The asymmetric translation transforms every $\mathcal{L}_2$-formula A into a bounded (i.e., Δ^0_0-) formula $A[t,s]$; existential and universal quantifiers are usually treated differently. It will be the key instrument to prove Π^0_2-conservation of WKL_0 over PRA.

Definition 1.1.6 (Asymmetric translation). Let A be an $\mathcal{L}_2$-formula and t, s be $\mathcal{L}_2$-terms, then $A[t,s]$ is the formula obtained from A according to the following transformation:

(1) each unbounded universal quantifier $(\forall x)$ is replaced by $(\forall x \leq t)$,

(2) each unbounded existential quantifier $(\exists x)$ is replaced by $(\exists x \leq s)$.

$A[t,s]$ is called *asymmetric translation* of A.

By $A^{\leq s}$ we denote the formula obtained from A by simply replacing every unbounded number quantifier $(\mathsf{Q}x)$ by $(\mathsf{Q}x \leq s)$ (for $\mathsf{Q} = \forall, \exists$). $A^{\leq s}$ is called *relativization* of A.

From the definitions of s-Π^1_1- and s-Σ^1_1-formulas and the asymmetric translation we immediately get

Lemma 1.1.7.

(1) If A is s-Π^1_1 then $A[t,s] \equiv A^{\leq s}$

(2) If A is s-Σ^1_1 then $A[t,s] \equiv A^{\leq t}$

1.1.3 Axioms and rules of inference

The relevant theories formulated in $\mathcal{L}_2$ we will consider in this part of the thesis are all presented as Tait-style calculi. By capital Greek letters $\Gamma, \Delta, \ldots$ we denote finite sets of $\mathcal{L}_2$-formulas. The intended meaning of $\Gamma = \{A_1, \ldots, A_n\}$ is the finite disjunction $\bigvee_{i=1}^n A_i$.

The expression Γ, Δ stands for the set theoretic union $\Gamma \cup \Delta$. For sake of simplicity we omit set-braces around single formulas (i.e., we write Γ, A instead of $\Gamma, \{A\}$).

By $\Gamma_{\vec{u}}$ we denote a set Γ of formulas in which at most the variables $\vec{u}$ occur freely. If we let $\Gamma = \{A_1, \ldots, A_n\}$ be a set of formulas, t, s be terms, then we write $\Gamma[t,s]$ as an abbreviation for the set $\{A_1[t,s], \ldots, A_n[t,s]\}$. We may also combine these two notions (i.e., $\Gamma_{\vec{u}}[t,s]$).

The rank of a formula is a measure for its complexity and will be needed in the definition of the derivability relation.

Definition 1.1.8 (Rank). The *rank* $\mathrm{rk}(A)$ of a formula A is defined by

(1) $\mathrm{rk}(A) = 0$, if A is $\mathsf{s}\text{-}\Pi^1_1$ or $\mathsf{s}\text{-}\Sigma^1_1$,

(2) otherwise the rank is:

(i) $\mathrm{rk}(A \circ B) = \max\{\mathrm{rk}(A), \mathrm{rk}(B)\} + 1$, if $\circ = \wedge, \vee$,

(ii) $\mathrm{rk}(\mathsf{Q}x A(x)) = \mathrm{rk}(A(u)) + 1$, if $\mathsf{Q} = \forall, \exists$,

(iii) $\mathrm{rk}(\mathsf{Q}X A(X)) = \mathrm{rk}(A(U)) + 1$, if $\mathsf{Q} = \forall, \exists$.

The following definition describes the axioms and rules of inference that are present in all theories T we will consider. The mathematical rules of inference and axioms are theory-dependent; this is: a theory consists of some additional *mathematical* rules of inference (or axioms) which make up the theory. Examples of the latter are: induction and comprehension rules for specific collections of formulas. In the following two sections we define the theories we will make use of.

Definition 1.1.9. The *axioms* of theories T formulated in $\mathcal{L}_2$ consist of the substitution closure of the following sets:

(A.1) Logical Axioms.

$$\Gamma, u = u$$
$$\Gamma, \neg u = v, \neg A(u), A(v) \quad (A \text{ atomic})$$
$$\Gamma, \neg A, A \quad (A \text{ atomic}).$$

(A.2) Axioms for primitive recursion.

$$\Gamma, \neg \mathsf{S}(u) = 0 \qquad \Gamma, \neg \mathsf{S}(u) = \mathsf{S}(v), u = v$$
$$\Gamma, \neg u < 0 \qquad \Gamma, \neg u < \mathsf{S}(v), u < v, u = v$$
$$\Gamma, u < v, u < \mathsf{S}(v), u = v \qquad \Gamma, \neg u < v, u < \mathsf{S}(v)$$
$$\Gamma \neg u = v, u < v \qquad \Gamma, u < v, u = v, v < u$$
$$\Gamma, \mathsf{Cs}^n_m(u_1, \ldots, u_n) = m \qquad \Gamma, \mathsf{Pr}^n_i(u_1, \ldots, u_n) = u_i$$
$$\Gamma, \mathsf{Comp}^n(f, g_1, \ldots, g_n)(\vec{u}) = f(g_1(\vec{u}), \ldots, g_n(\vec{u}))$$
$$\Gamma, \mathsf{Rec}^{n+1}(f, g)(\vec{u}, 0) = f(\vec{u})$$
$$\Gamma, \mathsf{Rec}^{n+1}(f, g)(\vec{u}, \mathsf{S}(v)) = g(u, v, \mathsf{Rec}^{n+1}(f, g)(\vec{u}, v))$$

The logical *rules of inference* are given by.

(R.1) Logical Rules

$$\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B}\ (\wedge) \qquad \frac{\Gamma, A}{\Gamma, A \vee B}\ (\vee_1) \qquad \frac{\Gamma, B}{\Gamma, A \vee B}\ (\vee_2)$$

$$\frac{\Gamma, A(u)}{\Gamma, \forall x A(x)}\ (\forall^0), \text{ provided } u \text{ is not a free variable in } \Gamma, \forall x A(x)$$

$$\frac{\Gamma, A(t)}{\Gamma, \exists x A(x)}\ (\exists^0), \text{ where } t \text{ is an arbitrary term.}$$

$$\frac{\Gamma, A \quad \Gamma, \neg A}{\Gamma}\ (\mathsf{cut})$$

$$\frac{\Gamma, A(U)}{\Gamma, \forall X A(X)}\ (\forall^1), \text{ provided } U \text{ is not a free variable in } \Gamma, \forall X A(X)$$

$$\frac{\Gamma, A(U)}{\Gamma, \exists X A(X)}\ (\exists^1)$$

1.1.4 Theories RCA_0 and Δ_0-CA

In this section we will define two second-order theories which will be important: RCA_0 with recursive comprehension (RCA stands for Recursive Comprehension Axiom, the zero indicates restricted induction), and the weaker theory Δ_0-CA in which we will interpret $(\mathsf{s}\text{-}\Pi^1_1) + \mathsf{RCA}_0$ asymmetrically. Both theories are weak subsystems of ACA_0.

Definition 1.1.10 (RCA_0). The theory RCA_0 is formulated in $\mathcal{L}_2$ and contains the axioms and logical rules of inference given in definition 1.1.9 and the following mathematical rules:

(R.2) Mathematical Rules.

For any Σ^0_1-formula $A(u)$:

$$\frac{\Gamma, A(0) \quad \Gamma, \forall x(A(x) \rightarrow A(\mathsf{S}(x))}{\Gamma, A(t)}\ (\Sigma^0_1\text{-}\mathsf{IND}), \quad t \text{ any term}$$

For any Σ^0_1-formula $A(u)$ and Π^0_1-formula $B(u)$:

$$\frac{\Gamma, \forall x(A(x) \rightarrow B(x)) \quad \Gamma, \forall x(B(x) \rightarrow A(x))}{\Gamma, \exists X[\forall x(x \in X \rightarrow B(x)) \wedge \forall x(A(x) \rightarrow x \in X)]}\ (\Delta^0_1\text{-}\mathsf{CR})$$

Δ^0_1-comprehension is often also called recursive comprehension, since a set is recursive iff itself and its complement are recursively enumerable and recursively enumerable sets correspond to Σ_1-definable sets in $\mathbb{N}$.

In the literature about RCA_0 or reverse mathematics, such as Simpson [10], we will find Δ^0_1-comprehension and Σ^0_1-induction formulated as axiom-schemes; these axioms are logical consequences of the rules presented here (because we have formulated them with side-formulas).

Definition 1.1.11 (Δ_0-CA). The theory Δ_0-CA is formulated in $\mathcal{L}_2$ and contains the axioms and logical rules of inference given in definition 1.1.9 and the following mathematical axioms and rules:

(A.2) Axiom for Δ^0_0-comprehension.

For any Δ^0_0-formula $A(u)$:

$$\Gamma, \exists X \forall x(x \in X \leftrightarrow A(x)) \quad (\Delta^0_0\text{-}\mathsf{CA})$$

(R.2) Mathematical Rules.

For any Δ^0_0-formula $A(u)$:

$$\frac{\Gamma, A(0) \quad \Gamma, \forall x(A(x) \rightarrow A(\mathsf{S}(x))}{\Gamma, A(t)} \quad (\Delta^0_0\text{-}\mathsf{IND}), \quad t \text{ any term}$$

1.2 WKL: Weak König's lemma

In ordinary mathematics weak König's Lemma is stated as follows:

Weak König's Lemma.[1] *Given an infinite binary tree T, there exists an infinite path P through the tree T.*

We need a formulation of "binary tree" and "path" in our second order arithmetic. Since our language $\mathcal{L}_2$ contains symbols for all primitive recursive functions, there is a symbol of a primitive recursive function that maps finite sequences $x_1, \ldots, x_n$ of natural numbers to the so-called sequence number $\langle x_1, \ldots, x_n \rangle$ which is a standard result from basic recursion theory (cf. for example Jäger [7]). On the other hand we can also find a primitive recursive "decoding" function $(\cdot)_i$ with the property $(\langle x_1, \ldots, x_n \rangle)_i = x_i$ and a length function $\mathsf{lh}(\cdot)$ defined on sequence numbers. Furthermore we can define a primitive recursive predicate $\mathsf{Seq}(x)$ which holds iff x is a sequence number.

Now we are able to define a binary relation $\subseteq$ on the set of all sequence numbers stating that u is a subsequence of v, $u \subseteq v$, formally:

$$u \subseteq v :\equiv \mathsf{Seq}(u) \wedge \mathsf{Seq}(v) \wedge \forall x \leq \mathsf{lh}(u)[(u)_x = (v)_x]$$

[1] which is named after the Hungarian mathematician Dénes König (1884–1944)

We require all sequences, that build up the tree to be binary (i.e., to consist only of 0 and 1). So we define an additional unary predicate $\mathsf{Seq}_2(\cdot)$, ensuring that a given sequence s consists only of 0's and 1's:

$$\mathsf{Seq}_2(s) \equiv \mathsf{Seq}(s) \wedge \forall x \leq \mathsf{lh}(s)[(s)_x = 0 \vee (s)_x = 1]$$

An infinite binary tree is therefore a set consisting of 0-1-sequence numbers of arbitrary length that are closed under initial subsequences.

Definition 1.2.1. Let U be a set of sequence-numbers. U defines an infinite binary tree, if U consists only of 0-1 sequence numbers, is closed under subsequences and contains sequences of arbitrary length; formally:

$$\begin{aligned} \mathsf{Tree}_\infty(U) :\equiv \quad & \forall x(x \in U \rightarrow \mathsf{Seq}_2(x)) \wedge \\ & \forall x \forall y(x \in U \wedge y \subseteq x \rightarrow y \in U) \wedge \\ & \forall x \exists y \leq \langle 1 \rangle(x)(y \in U \wedge \mathsf{lh}(y) = x) \end{aligned}$$

where $\langle 1 \rangle$ denotes the unary primitive-recursive function symbol with the property $\langle 1 \rangle(x) = \langle \underbrace{1, \ldots, 1}_{x\text{-times}} \rangle$.

A path is an infinite tree linearly ordered with respect to the subsequence relation:

$$\mathsf{Path}_\infty(U) :\equiv \mathsf{Tree}_\infty(U) \wedge \forall x \forall y(x \in U \wedge y \in U \rightarrow x \subseteq y \vee y \subseteq x)$$

We remark that $\mathsf{RCA}_0 \nvdash (\mathsf{WKL})$, as the standard model $\mathcal{M} = (\omega, REC, \leq^{\mathcal{M}}, \mathsf{S}^{\mathcal{M}}, \ldots)$ of RCA_0 is a not model of (WKL). There exist infinite recursive trees with no recursive paths (e.g., Kleene-Tree).

1.2.1 Theory WKL_0 and strict Π^1_1-reflection

In this section we formally define the theory WKL_0 which consists of the same axioms and rules as RCA_0 plus weak König's lemma principle (WKL). We will not directly work within WKL_0, but we define an additional theory based on RCA_0 with the strict Π^1_1-reflection rule $(\mathsf{s}\text{-}\Pi^1_1)$ which will prove weak König's lemma, so we can embed WKL_0 in this theory. Strict Π^1_1-reflection is an important reflection principle which is equivalent to weak König's lemma. A predicate P on $\mathbb{N}$ is strict Π^1_1 iff it is recursively enumerable. For more information and details on $\mathsf{s}\text{-}\Pi^1_1$-reflection and $\mathsf{s}\text{-}\Pi^1_1$-sets we refer to Barwise [1].

Definition 1.2.2. The theory WKL_0 is formulated in $\mathcal{L}_2$ and contains the axioms and logical rules of inference given in definition 1.1.9 and the following mathematical rules:

(R.2) Mathematical Rules.

For any Σ^0_1-formula $A(u)$:

$$\frac{\Gamma, A(0) \quad \Gamma, \forall x(A(x) \rightarrow A(\mathsf{S}(x))}{\Gamma, A(t)} \quad (\Sigma^0_1\text{-}\mathsf{IND}), \quad t \text{ any term}$$

For any Σ^0_1-formula $A(u)$ and Π^0_1-formula $B(u)$:

$$\frac{\Gamma, \forall x(A(x) \rightarrow B(x)) \quad \Gamma, \forall x(B(x) \rightarrow A(x))}{\Gamma, \exists X[\forall x(x \in X \rightarrow B(x)) \wedge \forall x(A(x) \rightarrow x \in X)]} \quad (\Delta^0_1\text{-}\mathsf{CR})$$

$$\frac{\Gamma, \mathsf{Tree}_\infty(U)}{\Gamma, \exists X[\mathsf{Path}_\infty(X) \wedge \forall x(x \in X \rightarrow x \in U)]} \quad (\mathsf{WKL})$$

Definition 1.2.3. The theory $\mathsf{s\text{-}RCA}_0$ is formulated in $\mathcal{L}_2$ and contains the axioms and logical rules of inference given in definition 1.1.9 and the following mathematical rules:

(R.2) Mathematical Rules.

For any Σ^0_1-formula $A(u)$:

$$\frac{\Gamma, A(0) \quad \Gamma, \forall x(A(x) \rightarrow A(\mathsf{S}(x))}{\Gamma, A(t)} \quad (\Sigma^0_1\text{-}\mathsf{IND}), \quad t \text{ any term}$$

For any Σ^0_1-formula $A(u)$ and Π^0_1-formula $B(u)$:

$$\frac{\Gamma, \forall x(A(x) \rightarrow B(x)) \quad \Gamma, \forall x(B(x) \rightarrow A(x))}{\Gamma, \exists X[\forall x(x \in X \rightarrow B(x)) \wedge \forall x(A(x) \rightarrow x \in X)]} \quad (\Delta^0_1\text{-}\mathsf{CR})$$

For any $\mathsf{s}\text{-}\Pi^1_1$-formula A

$$\frac{\Gamma, A}{\Gamma, \exists x A^{\leq x}} \quad (\mathsf{s}\text{-}\Pi^1_1)$$

According to this definition, $\mathsf{s\text{-}RCA}_0$ equals $\mathsf{RCA}_0 + (\mathsf{s}\text{-}\Pi^1_1)$ in which we will embed WKL_0.

1.3 Derivability relation and useful results from proof-theory

In this section we will state the definition of the derivability relation in our Tait-calculus and some basic results which come from ordinary proof-theory that we will be often using. For further information on proof theory we refer to Schütte [8], Girard [4] or Takeuti [11].

Definition 1.3.1 (Derivability). The derivability relation $T \mathrel{|\!\frac{m}{n}} \Gamma$ $(m, n \in \omega)$ for theories T formulated in $\mathcal{L}_2$ is inductively defined by the clauses:

(1) If Γ is an axiom, $T \mid\!\frac{m}{n}\, \Gamma$ for every m, n.

(2) Assume that Γ is the conclusion from the premises Γ_i of a logical or mathematical rule, or of a cut of rank $< n$ with $T \mid\!\frac{m_i}{n}\, \Gamma_i$ $(i < 3)$ and $m_i < m$. Then $T \mid\!\frac{m}{n}\, \Gamma$.

$T \mid\!\frac{m}{n}\, \Gamma$ means that there exists a proof of Γ whose depth is bound by m and which contains only cuts of rank smaller than n.

Lemma 1.3.2 (Weakening). *If* $\mathsf{s\text{-}RCA_0} \mid\!\frac{m}{n}\, \Gamma$ *and* $\Gamma \subset \Delta$*, then* $\mathsf{s\text{-}RCA_0} \mid\!\frac{m}{n}\, \Delta$

Lemma 1.3.3. $\mathsf{s\text{-}RCA_0} \vdash \Gamma, A$ *and* $\mathsf{s\text{-}RCA_0} \vdash \Delta, \neg A$ *imply* $\mathsf{s\text{-}RCA_0} \vdash \Gamma, \Delta$.

Proof. Obvious. □

Lemma 1.3.4 (Inversion).

(1) If $\mathsf{s\text{-}RCA_0} \mid\!\frac{m}{n}\, \Gamma, A_1 \wedge A_2$ *and* $\mathrm{rk}(A_1 \wedge A_2) > 0$ *then* $\mathsf{s\text{-}RCA_0} \mid\!\frac{m}{n}\, \Gamma, A_i$ *($i = 1, 2$).*

(2) If $\mathsf{s\text{-}RCA_0} \mid\!\frac{m}{n}\, \Gamma, A_1 \vee A_2$ *and* $\mathrm{rk}(A_1 \vee A_2) > 0$ *then* $\mathsf{s\text{-}RCA_0} \mid\!\frac{m}{n}\, \Gamma, A_1, A_2$.

(3) If $\mathsf{s\text{-}RCA_0} \mid\!\frac{m}{n}\, \Gamma, \forall x A(x)$ *and* $\mathrm{rk}(\forall x A(x)) > 0$ *then* $\mathsf{s\text{-}RCA_0} \mid\!\frac{m}{n}\, \Gamma, A(t)$ *(t an individual term).*

(4) If $\mathsf{s\text{-}RCA_0} \mid\!\frac{m}{n}\, \Gamma, \forall X A(X)$ *and* $\mathrm{rk}(\forall X A(X)) > 0$ *then* $\mathsf{s\text{-}RCA_0} \mid\!\frac{m}{n}\, \Gamma, A(U)$.

Weak cut elimination gives us the information that any proof can be translated into one using only formulas of rank < 1 in the cut rule, even though the depth of the proof will increase. "Weak" in this context means that we only eliminate cuts of rank ≥ 1. Since the principal formulas of conclusions of mathematical rules are always $\mathsf{s}\text{-}\Pi^1_1$ or $\mathsf{s}\text{-}\Sigma^1_1$ and thus have a cut-rank of zero, we do not eliminate these cuts. In the cut-elimination procedure we replace cuts occuring in the proof by cuts with a smaller rank (which lets the proof-depth increase).

Further information on cut elimination can be found in Schwichtenberg [9], for instance.

Theorem 1.3.5 (Weak cut elimination). *If* $\mathsf{s\text{-}RCA_0} \vdash \Gamma$*, then* $\mathsf{s\text{-}RCA_0} \mid\!\frac{k}{1}\, \Gamma$ *for some* $k \in \omega$.

The following lemma is very helpful in the proceeding; it is so-to-speak the technical tool to handle the asymmetric interpretation of the rules of inference quite easily. As sets of formulas are interpreted as the disjunction of its members, it is not required to apply it to all members of the set. Thus we may leave some formulas of a set untouched if we wish so.

Lemma 1.3.6 (Persistence). *Let $\Gamma \cup \{A\}$ be a set of $\mathcal{L}_2$-formulas.*

(1) $\Delta_0\text{-}\mathsf{CA} \vdash \neg t' \leq t, s \leq s', \neg A[t,s], A[t',s']$

(2) If A is $\mathsf{s}\text{-}\Pi^1_1$, then $A[t,s] \equiv A^{\leq s}$ and $\Delta_0\text{-}\mathsf{CA} \vdash \neg s \leq t, \neg A^{\leq s}, A^{\leq t}$

(3) If A is $\mathsf{s}\text{-}\Sigma^1_1$, then $A[t,s] \equiv A^{\leq t}$ and $\Delta_0\text{-}\mathsf{CA} \vdash \neg t \leq s, \neg A^{\leq s}, A^{\leq t}$

where $\neg A[t,s]$ is an abbreviation for $\neg(A[t,s])$.

Proof. (2) and (3) are easy consequences of (1) and lemma 1.1.7. We prove (1) on the build-up of formulas:

Let A be atomic, then $A \equiv A[t,s] \equiv A[t',s']$ and $\neg t' \leq t, \neg s \leq s', \neg A[t,s], A[t',s']$ is a logical axiom by definition 1.1.9 (A.1).

<u>$A \equiv B \wedge C$</u>: By induction hypothesis we have

$$\neg t' \leq t, \neg s \leq s', \neg A[t,s], A[t',s'] \text{ and } \neg t' \leq t, \neg s \leq s', \neg B[t,s], B[t',s']$$

Applying $(\vee_i)$ to both sets of formulas yields

$$\neg t' \leq t, \neg s \leq s', \neg A[t,s] \vee \neg B[t,s], A[t',s']$$

$$\neg t' \leq t, \neg s \leq s', \neg A[t,s] \vee \neg B[t,s], B[t',s']$$

A final $(\wedge)$ between these two lines gives

$$\neg t' \leq t, \neg s \leq s', \neg(A[t,s] \wedge B[t,s]), A[t',s'] \wedge B[t',s']$$

<u>$A \equiv B \vee C$</u>: analogously to the previous case.

<u>$A \equiv \forall x B$</u>: Let $v \notin \mathrm{FV}(\neg t' \leq t, \neg s \leq s', \neg B[t,s], B[t',s'])$. Then $\neg t' \leq t$, $\neg s \leq s', v \leq t, \neg v \leq t'$ is provable, since $\leq$ is transitive. Hence by weakening:

$$\neg t' \leq t, \neg s \leq s', v \leq t, \neg v \leq t', B[t',s'] \tag{1.3.1}$$

By induction hypothesis we are allowed to assume:

$$\neg t' \leq t, \neg s \leq s', \neg B[t,s], B[t',s']$$

using weakening on the previous sequent leads to:

$$\neg t' \leq t, \neg s \leq s', \neg B[t,s], B[t',s'], \neg v \leq t' \tag{1.3.2}$$

Applying ($\wedge$) between (1.3.1) and (1.3.2) yields:

$$\neg t' \le t, \neg s \le s', v \le t \wedge \neg B[t,s], B[t',s'], \neg v \le t' \tag{1.3.3}$$

By ($\exists^0$)-introduction and using ($\vee_2$) we obtain from the preceding lines:

$$\neg t' \le t, \neg s \le s', \neg \forall x \le tB[t,s], \neg v \le t' \vee B[t',s']$$

A last application of ($\forall^0$) is needed to get to the desired result

$$\neg t' \le t, \neg s \le s', \neg \forall x \le tB[t,s], \forall x \le tB[t',s']$$

<u>$A \equiv \exists xB$</u>: This case is similar to ($\forall$); assume $v \notin \mathrm{FV}(\neg t' \le t, \neg s \le s', B[t,s], B[t',s'])$ and by transitivity of $\le$, $\neg s \le s', \neg v \le s, v \le s'$ is provable. Hence, by weakening:

$$\neg s \le s', \neg v \le s, v \le s', \neg B[t,s] \tag{1.3.4}$$

By induction hypothesis we are allowed to assume:

$$\neg t' \le t, \neg s \le s', \neg B[t,s], B[t',s']$$

using weakening on the previous sequent leads to:

$$\neg t' \le t, \neg s \le s', \neg B[t,s], B[t',s'], \neg v \le s \tag{1.3.5}$$

An application of ($\wedge$) between the sequents in (1.3.4) and (1.3.5) leads to

$$\neg t' \le t, \neg s \le s', \neg v \le s, \neg B[t,s], v \le s' \wedge B[t',s']$$

By ($\exists^0$)-introduction and using ($\vee_1$) we obtain from the preceding sequent:

$$\neg t' \le t, \neg s \le s', \neg v \le s \vee \neg B[t,s], \exists x \le s'B[t',s']$$

Eventually we apply ($\forall^0$) and get the required result:

$$\neg t' \le t, \neg s \le s', \exists x \le sB[t,s], \exists x \le s'B[t',s']$$

<u>$A \equiv \forall XB(X)$</u>, <u>$A \equiv \exists XB(X)$</u>: these cases are obvious. □

1.4 Asymmetric interpretation theorem

The following proposition shows the already mentioned fact that RCA_0 + (s-Π^1_1) proves weak König's lemma. So it is sufficient to show all subsequent results for s-RCA_0 and henceforth it applies to WKL_0 as well. The asymmetric interpretation of s-RCA_0 is more elegant than the one for WKL_0 even though a direct asymmetric interpretation of WKL_0 could be accomplished, as Cantini shows in [2] for his bounded arithmetic.

Proposition 1.4.1. RCA_0 *and* s-Π^1_1*-reflection imply weak König's lemma:* $\mathsf{RCA}_0 + (\text{s-}\Pi^1_1) \vdash (\mathsf{WKL})$.

Proof. We show the contra positive, i.e., $\mathsf{RCA}_0 + (\text{s-}\Pi^1_1)$ proves

$$\forall Y[\mathsf{Path}_\infty(Y) \to \exists w(w \in Y \wedge \neg w \in U)] \to \neg\, \mathsf{Tree}_\infty(U).$$

Therefore we assume

$$\forall Y[\mathsf{Path}_\infty(Y) \to \exists w(w \in Y \wedge \neg w \in U)] \tag{1.4.1}$$
$$\forall x \forall y(\mathsf{Seq}_2(x) \wedge x \in U \wedge y \subseteq x \to y \in U) \tag{1.4.2}$$
$$\forall x(x \in U \to \mathsf{Seq}_2(x)) \tag{1.4.3}$$

By the strict Π^1_1-reflection rule we obtain a bound b such that

$$\forall Y[\mathsf{Path}_\infty{}^{\leq b}(Y) \to \exists w \leq b(w \in Y \wedge \neg w \in U)].$$

We claim that the tree defined by U is finite and all its paths have length $< b$, i.e.,

$$\forall x(\mathsf{Seq}_2(x) \wedge \mathsf{lh}(x) = b \to \neg x \in U) \tag{1.4.4}$$

Let z be arbitrary with $\mathsf{lh}(z) = b \wedge \mathsf{Seq}_2(z)$ then by Δ^0_1-comprehension there exists the set $X(z) := \{u : u \subseteq z\}$. Then $X(z)$ satisfies

$$(\forall x \leq b)[x \in X \to \mathsf{Seq}_2(x)] \tag{1.4.5}$$
$$(\forall x \leq b)(\forall y \leq b)[x \in X(z) \wedge y \subseteq x \to y \in X(z)] \tag{1.4.6}$$
$$(\forall x \leq b)(\exists y \leq \langle 1\rangle(x))[y \in X(z) \wedge \mathsf{lh}(y) = x] \tag{1.4.7}$$
$$(\forall x \leq b)(\forall y \leq b)[x \in X(z) \wedge y \in X(z) \to x \subseteq y \vee y \subseteq x] \tag{1.4.8}$$

Hence, $\mathsf{Path}_\infty{}^{\leq b}(X(z))$ holds and thus, since

$$\mathsf{Path}_\infty(X(z)) \to \exists w \leq b(w \in X(z) \wedge \neg w \in U),$$

we conclude there exists w such that $w \subseteq z \wedge w \in X(z) \wedge \neg w \in U$ with length $\leq b$. By (1.4.2) we conclude $\neg z \in U$, the verification of (1.4.4). □

On our way to the Π^0_2-conservativity we prove the asymmetric interpretation theorem of $\mathsf{s\text{-}RCA}_0$ in Δ_0-CA. Based on this, we will show the conservativity. If we apply the transformation of asymmetric translation to any formula provable in $\mathsf{s\text{-}RCA}_0$ (and thus also in WKL_0) we will get a bounded formula which will be provable in the weaker system Δ_0-CA. The bound of an existential quantifier depends on the given bound of the universal quantifier.

Theorem 1.4.2 (Asymmetric interpretation). *Let* $\mathsf{s\text{-}RCA}_0 \mid\!\frac{k}{1}\, \Gamma_{\vec{z}}$. *Then one can find a unary primitive recursive function symbol g of $\mathcal{L}_2$ such that, provably in* Δ_0-CA*:*

(1) $\forall x \forall \vec{z} (\vec{z} \leq x \rightarrow \bigvee \Gamma_{\vec{z}}[x, g(x)])$*;*

(2) $\forall x (x \leq g(x))$

Proof. We prove the claim by induction on the depth of the derivation. As the logical axioms and axioms for primitive recursion are atomic, we may let $g(u) = u$ and thus the claim holds.

$(\wedge)$: The $(\wedge)$-rule applies if we can prove Γ, A and Γ, B in $\mathsf{s\text{-}RCA}_0$, then by induction hypothesis we may assume that Δ_0-CA proves the asymmetric translation of these two premises of $(\wedge)$ and thus we have $u \leq g_i(u)$ $(i = 1, 2)$ and

$$\neg \vec{z} \leq u, \Gamma_{\vec{z}}[u, g_1(u)], A[u, g_1(u)] \tag{1.4.9}$$
$$\neg \vec{z} \leq u, \Gamma_{\vec{z}}[u, g_2(u)], B[u, g_2(u)] \tag{1.4.10}$$

Define $g(u) := g_1(u) + g_2(u)$, then clearly $u \leq g(u)$ and by persistence $(g_i(u) \leq g(u)$ for $i = 1, 2)$ we get:

$$\neg \vec{z} \leq u, \Gamma_{\vec{z}}[u, g(u)], A[u, g(u)] \tag{1.4.11}$$
$$\neg \vec{z} \leq u, \Gamma_{\vec{z}}[u, g(u)], B[u, g(u)] \tag{1.4.12}$$

Applying $(\wedge)$ to (1.4.11) and (1.4.12) yields the result.

$(\vee_{1,2})$: analogously to $(\wedge)$

$(\forall^0)$: Then $\Gamma_{\vec{z}} = \Delta_{\vec{z}}, \forall x A(x)$ and thus we have for some $v \notin \mathrm{FV}(\Gamma_{\vec{z}})$ and some $k_0 < k$

$$\mathsf{s\text{-}RCA}_0 \mid\!\frac{k_0}{1}\, \Gamma_{\vec{z}}, A(v)$$

By the induction hypothesis we have provably in Δ_0-CA

$$\neg v, \vec{z} \leq u, \Gamma_{\vec{z}}[u, g_0(u)], A[u, g_0(u)](v)$$

for some term $g_0(u)$ such that $u \le g_0(u)$. We define $g(u) = g_0(u)$ and hence

$$\neg v, \vec{z} \le u, \Gamma_{\vec{z}}[u, g(u)], A[u, g(u)](v)$$

Since $v \notin \mathrm{FV}(\Gamma_{\vec{z}})$ we may apply $(\forall^0)$ and obtain:

$$\neg \vec{z} \le u, \Gamma_{\vec{z}}[u, g(u)], \forall x \le u A[u, g(u)](x)$$

<u>$(\exists^0)$</u>: Then $\Gamma_{\vec{z}} = \Delta_{\vec{z}}, \exists x A(x)$ and $\mathsf{s\text{-}RCA}_0 \mid\!\frac{k}{1}\; \Gamma_{\vec{z}}, A(s)$ for some $k \in \omega$ and some term $s = s(\vec{z})$. By induction hypothesis, we have, provably in Δ_0-CA, $u \le g(u)$ and under the assumption $\vec{z} \le u$

$$\Gamma_{\vec{z}}[u, g(u)], A[u, g(u)](s)$$

Since every primitive-recursive function f can be majorized by a monotone primitive-recursive one (e.g., a branch of the Ackermann function), we can find a term $s'(\vec{z})$ which is monotone with respect to $\le$, such that $s(\vec{z}) \le s'(\vec{z}) \le s'(u, \ldots, u)$. Then we choose $h(u) := g(u) + s'(u, \ldots, u)$. Obviously $s(\vec{z}), g(u) \le h(u)$. The persistence lemma and $(\exists^0)$ imply the required conclusion.

<u>$(\mathsf{s}\text{-}\Pi^1_1)$</u>: Let A be $\mathsf{s}\text{-}\Pi^1_1$ i.e., $A[t, s] \equiv A^{\le s}$, and assume $\vec{z} \le u$; then by induction hypothesis we may assume

$$\Gamma_{\vec{z}}[u, g(u)], A^{\le g(u)}.$$

Choose $h(u) := g(u)$; with an application of $(\wedge)$ we get

$$\Gamma_{\vec{z}}[u, g(u)], g(u) \le h(u) \wedge A^{\le g(u)} \tag{1.4.13}$$

Applying $(\exists^0)$ yields

$$\Gamma_{\vec{z}}[u, h(u)], \exists x (x \le h(u) \wedge A^{\le x}) \tag{1.4.14}$$
$$\equiv \Gamma_{\vec{z}}[u, h(u)], \exists x \le h(u) A^{\le x} \tag{1.4.15}$$

<u>$(\Delta^0_1\text{-}\mathsf{CR})$</u>: By induction hypothesis we may assume that Δ_0-CA proves the asymmetric translation of the two premises of $(\Delta^0_1\text{-}\mathsf{CR})$ and thus there exist terms $g_0(u), g_1(u)$ with $u \le g_0(u), g_1(u)$ such that under the assumption $\vec{z} \le u$

$$\Gamma_{\vec{z}}[u, g_0(u)], \forall y \le u[\exists x \le u A(x, y) \to \forall x \le u B(x, y)] \tag{1.4.16}$$
$$\Gamma_{\vec{z}}[u, g_1(u)], \forall y \le u[\forall x \le g_1(u) B(x, y) \to \exists x \le g_1(u) A(x, y)] \tag{1.4.17}$$

If we choose $g(u) := g_0(g_1(u))$ and let $u = g_1(u)$ then (1.4.16) implies

$$\Gamma_{\vec{z}}[g_1(u), g(u)], \forall y \leq g_1(u)[\exists x \leq g_1(u)A(x,y) \rightarrow \forall x \leq g_1(u)B(x,y)] \quad (1.4.18)$$

whence by persistence we get from (1.4.17) and (1.4.18)

$$\Gamma_{\vec{z}}[u, g(u)], \forall y \leq u[\exists x \leq g_1(u)A(x,y) \rightarrow \forall x \leq g_1(u)B(x,y)] \quad (1.4.19)$$

$$\Gamma_{\vec{z}}[u, g(u)], \forall y \leq u[\forall x \leq g_1(u)B(x,y) \rightarrow \exists x \leq g_1(u)A(x,y)] \quad (1.4.20)$$

As $w \leq u \wedge \exists x \leq g_1(u)A(x,w)$ is a Δ^0_0-formula, we obtain by the (Δ^0_0-CA) axiom the following sequent

$$\Gamma_{\vec{z}}[u, g(u)], \exists X \forall y[y \in X \leftrightarrow (y \leq u \wedge \exists x \leq g_1(u)A(x,y))] \quad (1.4.21)$$

Hence by logic

$$\Gamma_{\vec{z}}[u, g(u)], \exists X[\forall y(y \in X \rightarrow (y \leq u \wedge \exists x \leq g_1(u)A(x,y)) \wedge \forall y((y \leq u \wedge \exists x \leq g_1(u)A(x,y)) \rightarrow y \in X)] \quad (1.4.22)$$

Then from (1.4.22) together with (1.4.19) we obtain by logic

$$\Gamma_{\vec{z}}[u, g(u)], \exists X[\forall y(y \in X \rightarrow (y \leq u \wedge \forall x \leq g_1(u)B(x,y)) \wedge \forall y((y \leq u \wedge \exists x \leq g_1(u)A(x,y)) \rightarrow y \in X)] \quad (1.4.23)$$

A last application of persistence and some logic on the preceding sequent yields the asymmetric interpretation of the conclusion of (Δ^0_1-CR)

$$\Gamma_{\vec{z}}[u, g(u)], \exists X[\forall y \leq u(y \in X \rightarrow \forall x \leq uB(x,y)) \wedge \forall y \leq u(\exists x \leq uA(x,y) \rightarrow y \in X)] \quad (1.4.24)$$

Surprisingly we only need one asymmetrically interpreted premise of (Δ^0_1-CR). At a first look this may seem strange, since if we only take $\forall x(A(x) \rightarrow B(x))$ ($A(u)$ is Σ^0_1, $B(u)$ is Π^0_1) from the conjunction in the premise of Δ^0_1-CA, we can prove the existence of some separable sets, which would not be possible in RCA_0. But it is a consequence of (WKL) and hence we do not get too much comprehension.

<u>(Σ^0_1-IND)</u>: By induction hypothesis there exist terms $g_0(u), g_1(u)$ with $u \leq g_0(u), g_1(u)$ such that Δ_0-CA proves the asymmetrically translated premises of (Σ^0_1-IND):

$$\Gamma_{\vec{z}}[u, g_0(u)], \exists x \leq g_0(u)A(x,0) \quad (1.4.25)$$

$$\Gamma_{\vec{z}}[u, g_1(u)], \forall y \leq u[\exists x \leq uA(x,y) \rightarrow \exists x \leq g_1(u)A(x,\mathsf{S}(y))] \quad (1.4.26)$$

By primitive recursion, we define a term f as $f(u,0) = g_0(u)$ and $f(u,\mathsf{S}(v)) = g_1(f(u,v)) + 1$. By Δ_0^0-induction on v we will prove

$$\Gamma_{\vec{z}}[u, f(u,v)], \exists x \leq f(u,v) A(x,v). \tag{1.4.27}$$

By definition of f and persistence we have

$$\Gamma_{\vec{z}}[u, f(u,v)], \exists x \leq f(u,0) A(x,0) \tag{1.4.28}$$

Now we assume

$$\Gamma_{\vec{z}}[u, f(u,v)], \exists x \leq f(u,v) A(x,v) \tag{1.4.29}$$

If we let $u = f(u,v)$, then we obtain from (1.4.26) using persistence ($u \leq f(u,v)$ on (1.4.26))

$$\begin{aligned}\Gamma_{\vec{z}}[u, f(u,\mathsf{S}(v))], \forall y \leq f(u,v)[\exists x \leq f(u,v) A(x,y) \to \\ \exists x \leq f(u,\mathsf{S}(v)) A(x,\mathsf{S}(y))]\end{aligned} \tag{1.4.30}$$

Then by ($\forall^0$)-inversion we obtain

$$\begin{aligned}\Gamma_{\vec{z}}[u, f(u,\mathsf{S}(v))], v \leq f(u,v)[\exists x \leq f(u,v) A(x,v) \to \\ \exists x \leq f(u,\mathsf{S}(v)) A(x,\mathsf{S}(v))]\end{aligned} \tag{1.4.31}$$

Using persistence ($f(u,v) \leq f(u,\mathsf{S}(v))$) on (1.4.29) we get

$$\Gamma_{\vec{z}}[u, f(u,\mathsf{S}(v))], \exists x \leq f(u,\mathsf{S}(v)) A(x,\mathsf{S}(v)) \tag{1.4.32}$$

Hence we have proved (1.4.29). Now let $t = t(\vec{z})$ be monotonically majorized by a term $t'(\vec{z})$—which exists by the same argument as in the ($\exists^0$)-case. Then we have $t(\vec{z}) \leq t'(\vec{z}) \leq t'(u_1,\ldots,u_n)$. We define $g(u) = f(u, t'(u_1,\ldots,u_n))$ and by persistence, under the assumption $\vec{z} \leq u$ we obtain

$$\Gamma_{\vec{z}}[u, g(u)], \exists x \leq g(u) A(x,t) \tag{1.4.33}$$

<u>(cut)</u>: For some $k_0, k_1 < k$ we get the following two premises of (cut)

$$\mathsf{s\text{-}RCA}_0 \vdash_1^{k_0} \Gamma_{\vec{z}}, A \tag{1.4.34}$$

$$\mathsf{s\text{-}RCA}_0 \vdash_1^{k_1} \Gamma_{\vec{z}}, \neg A \tag{1.4.35}$$

where $\mathrm{rk}(A) = 0$ and hence A is $\mathsf{s}\text{-}\Pi_1^1$, $\neg A$ is $\mathsf{s}\text{-}\Sigma_1^1$. By induction hypothesis, there exist terms $g_0(u)$, $g_1(u)$ such that $u \leq g_i(u)$ ($i = 0,1$) and under the assumption $\vec{z} \leq u$ we have provably in Δ_0-CA:

$$\Gamma_{\vec{z}}[u, g_0(u)], A^{\leq g_0(u)} \tag{1.4.36}$$

$$\Gamma_{\vec{z}}[u, g_1(u)], \neg A^{\leq u} \tag{1.4.37}$$

Define $g(u) := g_1(g_0(u))$ and we have provably in Δ_0-CA, $u \leq g(u)$. Let $u := g_0(u)$, then from (1.4.37) we obtain

$$\Gamma_{\vec{z}}[g_0(u), g(u)], \neg A^{\leq g_0(u)} \tag{1.4.38}$$

Now using persistence, (1.4.36) and (1.4.38) yield

$$\Gamma_{\vec{z}}[u, g(u)], A^{\leq g_0(u)} \tag{1.4.39}$$

$$\Gamma_{\vec{z}}[u, g(u)], \neg A^{\leq g_0(u)} \tag{1.4.40}$$

A cut between these last two sequents gives the desired result. □

Using the above theorem we will almost immediately get the Π^0_2-conservation result of WKL_0 over Δ_0-CA.

Corollary 1.4.3 (Conservation). WKL_0 *is a conservative extension of* Δ_0-CA *for* Π^0_2*-sentences, i.e. if* A *is* Π^0_2 *and* $\mathsf{WKL}_0 \vdash A$ *then* Δ_0-$\mathsf{CA} \vdash A$*.*

Proof. Let $A(u, v)$ be Δ^0_0 and $\mathsf{WKL}_0 \vdash \forall x\, \exists y A(x, y)$. By proposition 1.4.1 and the asymmetric interpretation theorem Δ_0-$\mathsf{CA} \vdash (\forall x \leq u)(\exists y \leq f(u))A(x, y)$ for an appropriate term f. By $(\forall^0)$- and $(\vee_i)$-inversion we conclude

$$\neg u \leq u, \exists y \leq f(u)A(u, y). \tag{1.4.41}$$

A cut between (1.4.41) and the axiom $u \leq u, \exists y \leq f(u)A(u, y)$ yields

$$\exists y \leq f(u)A(u, y)$$

By logic we deduce $\exists y A(u, y)$ from $\exists y \leq f(u)A(u, y)$ and an application of $(\forall^0)$ finally yields $\forall x \exists y A(x, y)$. □

Using the asymmetric interpretation of WKL_0 in Δ_0-CA it does not seem to be possible to get a conservation result for a bigger collection of formulas than Π^0_2-formulas. It remains an open question if there exists a purely proof-theoretic method to obtain full conservation of WKL_0 over Δ_0-CA.

1.5 Π^0_2-conservativity of WKL_0 over PRA

Theoretically we could have interpreted WKL_0 directly in RCA_0 and then proved the conservativity of RCA_0 over PRA. In this thesis we have already defined the weaker theory Δ_0-CA in which we have asymmetrically interpreted WKL_0 via s-RCA_0 (cf. theorem 1.4.2, corollary 1.4.3). Using model-theoretic arguments we will show full conservativity of Δ_0-CA over PRA. We mention that there exist proof-theoretic methods to obtain full conservation of Δ_0-CA over PRA.

First we define the first order language in which PRA is formulated; $\mathcal{L}_1$-terms and formulas are defined similarly to $\mathcal{L}_2$.

Definition 1.5.1. Let $\mathcal{L}_1$ denote the language of first order arithmetic which contains the following symbols:

(1) countably many free number variables $u1, u2, \ldots$

(2) countably many bound number variables $x1, x2, \ldots$

(5) the functions symbols are defined inductively by:

  (i) 0 is a 0-ary function symbol and S is a unary function symbol,

  (ii) for all natural numbers n, m and i with $0 \leq i \leq n$ Cs^n_m and Pr^n_i are n-ary function symbols,

  (iii) if f is an m-ary function symbol and $g_1, \ldots, g_m$ are n-ary function symbols, then $\mathsf{Comp}^n(f, g_0, \ldots, g_m)$ is an n-ary function symbol,

  (iv) if f is an n-ary function symbol and g an $(n+2)$-ary function symbol, then $\mathsf{Rec}^{n+1}(f, g)$ is an $(n+1)$-ary function symbol,

(6) the binary relation symbols $\leq$ and $=$,

(7) the symbol $\sim$ to express complementary propositions,

(8) the logical connectives $\vee, \wedge, \forall, \exists$,

(9) auxiliary symbols.

Definition 1.5.2 (PRA). The axioms of PRA consist of the substitution closure of the following sets:

(A.1) Logical Axioms.

$\Gamma, u = u$
$\Gamma, \neg v = v, \neg A(v), A(v) \quad (A \text{ atomic})$
$\Gamma, \neg A, A \quad (A \text{ atomic})$.

(A.2) Axioms for primitive recursion.

$$
\begin{array}{ll}
\Gamma, \neg \mathsf{S}(u) = 0 & \Gamma, \neg \mathsf{S}(u) = \mathsf{S}(v), u = v \\
\Gamma, \neg u < 0 & \Gamma, \neg u < \mathsf{S}(v), u < v, u = v \\
\Gamma, u < v, u < \mathsf{S}(v), u = v & \Gamma, \neg u < v, u < \mathsf{S}(v) \\
\Gamma \neg u = v, u < v & \Gamma, u < v, u = v, v < u \\
\Gamma, \mathsf{Cs}^n_m(u_1, \ldots, u_n) = m & \Gamma, \mathsf{Pr}^n_i(u_1, \ldots, u_n) = u_i
\end{array}
$$

$$
\begin{array}{l}
\Gamma, \mathsf{Comp}^n(f, g_1, \ldots, g_n)(\vec{u}) = f(g_1(\vec{u}), \ldots, g_n(\vec{u})) \\
\Gamma, \mathsf{Rec}^{n+1}(f, g)(\vec{u}, 0) = f(\vec{u}) \\
\Gamma, \mathsf{Rec}^{n+1}(f, g)(\vec{u}, \mathsf{S}(v)) = g(u, v, \mathsf{Rec}^{n+1}(f, g)(\vec{u}, v))
\end{array}
$$

The logical *rules of inference* are given by.

(R.1) Logical Rules

$$
\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \ (\wedge) \qquad \frac{\Gamma, A}{\Gamma, A \vee B} \ (\vee_1) \qquad \frac{\Gamma, B}{\Gamma, A \vee B} \ (\vee_2)
$$

$$
\frac{\Gamma, A(u)}{\Gamma, \forall x A(x)} \ (\forall), \text{ provided } u \text{ is not a free variable in } \Gamma, \forall x A(x)
$$

$$
\frac{\Gamma, A(t)}{\Gamma, \exists x A(x)} \ (\exists), \text{ where } t \text{ is an arbitrary term.}
$$

$$
\frac{\Gamma, A \quad \Gamma, \neg A}{\Gamma} \ (\mathsf{cut})
$$

(R.2) Mathematical Rules.

For any quantifier-free formula $A(u)$:

$$
\frac{\Gamma, A(0) \quad \Gamma, \forall x(A(x) \to A(\mathsf{S}(x))}{\Gamma, A(t)} \ (\mathsf{QF\text{-}IND}), \quad t \text{ any term}
$$

An arithmetical hierarchy can be defined analogously for $\mathcal{L}_1$; A is Δ_0 if it is generated from literals by means of conjunctions, disjunction and bounded quantification. $\exists x A$ is Σ_{n+1} if A is Π_n and $\forall x A$ is Π_{n+1} if A is Σ_n.

It is a well-known result that PRA proves induction for the bigger collection of bounded formulas. This is, for every bounded formula A there exists a quantifier-free formula B such that $\mathsf{PRA} \vdash A \leftrightarrow B$ and hence $\mathsf{PRA} \vdash (\Delta_0\text{-}\mathsf{IND})$.

To prove conservativity of Δ_0-CA over PRA it is sufficient to show that we can extend every first-order model $\mathcal{M}$ of PRA to a second-order model $\mathcal{M}^\star$ of Δ_0-CA such that for any formula A in the language $\mathcal{L}_1$, $\mathcal{M} \models A \Leftrightarrow \mathcal{M}^\star \models A$. This is because if we assume $\Delta_0\text{-}\mathsf{CA} \vdash A$ and $\mathsf{PRA} \not\vdash A$ for any A in the language $\mathcal{L}_1$ of PRA. By Gödel completeness there exists a model $\mathcal{M} \models \mathsf{PRA}$

such that $\mathcal{M} \not\models A$. As $\mathcal{M}^\star \models \Delta_0\text{-}\mathsf{CA}$ we have consequently $\mathcal{M}^\star \not\models A$ and hence by soundness $\Delta_0\text{-}\mathsf{CA} \nvdash A$, which contradicts our initial assumption.

Consider a model $\mathcal{M}$ of PRA

$$\mathcal{M} = (M, +^{\mathcal{M}}, \cdot^{\mathcal{M}}, \mathsf{S}^{\mathcal{M}}, \leq^{\mathcal{M}}, \dots)$$

In order to extend $\mathcal{M}$ to a second-order model $\mathcal{M}^\star$ of $\Delta_0\text{-}\mathsf{CA}$ we have to define the universe over which set variables will run:
$S \in \mathcal{S}_{\mathcal{M}}$ if there exists a Δ_0 $\mathcal{L}_1$-formula $A(m, \vec{n})$ with parameters $\vec{n} \in M$ such that $S = \{m \in M : \mathcal{M} \models A(m, \vec{n})\}$.

We claim that $\mathcal{M}^\star = (M, \mathcal{S}_{\mathcal{M}}, +^{\mathcal{M}}, \cdot^{\mathcal{M}}, \mathsf{S}^{\mathcal{M}}, \leq^{\mathcal{M}}, \dots)$ is a model of $\Delta_0\text{-}\mathsf{CA}$. The axioms ($\mathsf{A.1}$) and ($\mathsf{A.2}$) from definition 1.1.9 are clearly satisfied as they coincide; we have to verify that $\mathcal{M}^\star$ satisfies $\Delta^0_0\text{-}\mathsf{CA}$ and $\Delta^0_0\text{-}\mathsf{IND}$.

Let A be Δ^0_0 in the language $\mathcal{L}_2$ with parameters from $|\mathcal{M}| \cup \mathcal{S}_{\mathcal{M}^\star}$. Exhibiting the parameters $A \equiv A(u; X_1, \dots, X_m, a_1, \dots, a_k)$. By definition of $\mathcal{S}_{\mathcal{M}}$ we can find a formula $B_i(t_j)$ for every literal $t_j \in X_i$ such that $t_j \in X_i$ holds iff $B_i(t_j)$ holds. As the collection of Δ^0_0 formula is closed under Δ^0_0, we can replace every literal $t_j \in X_i$ in A by the corresponding formula $B_i(t_j)$ and obtain a formula $\tilde{A}$ such that $\mathcal{M}^\star \models A \leftrightarrow \tilde{A}$. $\tilde{A}$ is Δ_0 and formulated in $\mathcal{L}_1$.

Assume $A(u; X_1, \dots, X_m, a_1, \dots, a_k)$ is Δ^0_0, so we can find $\tilde{A}(u)$ with the only free variable u such that $\mathcal{M} \models A(u; X_1, \dots, X_m, a_1, \dots, a_k) \leftrightarrow \tilde{A}(u)$. As $\tilde{A}$ is Δ_0, it defines a set $X := \{m \in M : \tilde{A}(m)\}$ which is then in $\mathcal{S}_{\mathcal{M}}$. So $\mathcal{M}^\star \models \exists X \forall x (x \in X \leftrightarrow \tilde{A}(x))$, and thus $\mathcal{M}^\star \models \exists X \forall x (x \in X \leftrightarrow A(x; X_1, \dots, X_m, a_1, \dots, a_k))$.

Let $\mathcal{M}^\star \models A(0) \wedge \forall x (A(x) \to A(\mathsf{S}(x)))$ by replacing all set parameters in A by appropriate literals $t \in X_i$, we obtain an equivalent $\mathcal{L}_1$-formula $\tilde{A}$ and hence $\mathcal{M} \models \tilde{A}(0) \wedge \forall x (\tilde{A}(x) \to \tilde{A}(\mathsf{S}(x)))$. As $\mathsf{PRA} \vdash \Delta_0\text{-}\mathsf{IND}$ we have $\mathcal{M} \models \tilde{A}(v)$ and thus $\mathcal{M}^\star \models \tilde{A}(v)$ and eventually $\mathcal{M}^\star \models A(v)$.

So far we have just proved:

Lemma 1.5.3. $\Delta_0\text{-}\mathsf{CA}$ *is conservative over* PRA*.*

Composing corollary 1.4.3 and lemma 1.5.3 together we get our main result in this section:

Theorem 1.5.4. WKL_0 *is* Π^0_2*-conservative over* PRA*.*

Using model-theoretic methods or the recursion-theoretic approach as presented in the next part, we can prove the full conservation result of WKL_0 over Σ_1-PA. On the other hand, the proof-theoretic approach seems to me much more intuitive than the recursive one.

The axiomatic method has many advantages
over honest work.

— BERTRAND RUSSEL

Recursion-theoretic Approach

In this chapter we will prove the full conservation result of WKL_0 over Σ_1-PA, following closely Hájek–Pudlák [6] resp. Hájek–Kučera [5]. The necessary definitions will be given, but some rather technical results will be cited only.

2.1 Logical Framework

In this part of the thesis we will work only with first order theories, and we will use them in a Hilbert-style context. $\mathcal{L}_0$ is only the basic language. But in order to prove the main result we will have to extend $\mathcal{L}_0$ twice.

2.1.1 Language $\mathcal{L}_0$ of first order arithmetic

Definition 2.1.1. Let $\mathcal{L}_0$ denote the language of first order arithmetic which contains the following symbols:

(1) countably many variable symbols $u, v, w, x, y, z, \ldots$,

(2) 0 is a 0-ary function symbol, S is a unary function symbol, $+, \cdot$ are binary function symbols,

(3) the binary relation symbols $\leq$, $=$,

(4) the logical connectives $\neg, \wedge, \forall$,

(5) auxiliary symbols.

Terms, literals, formulas and the negation are defined analogously to the definitions in the first chapter. $\rightarrow, \leftrightarrow, \vee$ and $\exists$ are understood as abbreviations—nevertheless we will use them to formulate the logical rules of inference.

If we compare $\mathcal{L}_0$ with $\mathcal{L}_1$ from the first part of the thesis, we see, that $\mathcal{L}_0$ lacks the symbols for all primitive recursive functions (except $\mathsf{S}, +, \cdot$), a reason will be given later.

2.1.2 Hilbert-style calculus

As our main goal does not lie in "the proofs" themselves, we will work in a Hilbert-style context where we formulate the necessary theories as sets of axioms and axiom-schemes and keep the rules of inference identical in all used theories.

Definition 2.1.2. The logical axioms of first-order theories T are all instances of propositional tautologies. The rules of inference are given by

$$\frac{A \to B(t)}{A \to \exists x B(x)} \ (\exists r), \qquad \frac{A(u) \to B}{\exists x A(x) \to B} \ (\exists l),$$

$$\frac{A \to B(u)}{A \to \forall x B(x)} \ (\forall r), \qquad \frac{A(t) \to B}{\forall x A(x) \to B} \ (\forall l),$$

where the free variable u in the rules $(\exists l), (\forall r)$ may not occur in the conclusion of the respective rule.

$$\frac{A \quad A \to B}{B} \ (\mathrm{MP})$$

An axiomatic theory in $\mathcal{L}_0$ is given by a set T of $\mathcal{L}_0$-formulas—the so-called "axioms of T".

Definition 2.1.3 (Provability). By $T \vdash A$ we denote the provability relation in the Hilbert-style calculus. $T \vdash A$ if there exists a finite sequence $A_1, \dots, A_n$ such that

(1) $A \equiv A_n$

(2) for all $k < n$ either

  (i) A_k is a logical axiom or
  (ii) A_k is an axiom of T or
  (iii) A_k is the conclusion of a rule of inference with premises A_i for $i < k$.

2.1.3 $\mathcal{L}_0$-structures and Tarski's truth conditions

In this section we define the terms "model" and "truth condition" which will be used to express satisfiability for $\mathcal{L}_0$-formulas within theories formulated in $\mathcal{L}_0$.

A model $\mathcal{M}$ for the first-order language $\mathcal{L}_0$ consists of a non-empty domain M and for every n-ary predicate P of $\mathcal{L}_0$, an n-ary relation $P^{\mathcal{M}} \subseteq M^n$; for every n-ary function symbol f an n-ary mapping $f^{\mathcal{M}} : M^n \to M$; for every constant c an element $c^{\mathcal{M}} \in M$.

An evaluation e of a term t is a finite mapping whose domain consists of variables, among them at least all variables occurring in t, and whose range is a subset of M.

The value of a term t in a model $\mathcal{M}$ given by an evaluation e is inductively defined by:

$$t[e] := \begin{cases} t^{\mathcal{M}} & \text{if } t \text{ is a constant} \\ e(t) & \text{if } t \text{ is a variable} \\ f^{\mathcal{M}}(t_1[e], \ldots, t_n[e]) & \text{if } t \equiv f(t_1, \ldots, t_n) \end{cases}$$

The following definition is Tarski's truth condition; $\mathcal{M} \models A[e]$ is read as "e satisfies A in $\mathcal{M}$" or "A is true in $\mathcal{M}$ under the evaluation e".

Definition 2.1.4. Let $\mathcal{M}$ be a model, P an n-ary predicate and $t_1, \ldots, t_n$ terms.

(1) If $A \equiv P(t_1, \ldots, t_n)$, then $\mathcal{M} \models A[e]$ if $(t_1[e], \ldots, t_n[e]) \in P^{\mathcal{M}}$.

(2) $\mathcal{M} \models \neg A[e]$ if $\mathcal{M} \not\models A[e]$

(3) $\mathcal{M} \models (A \wedge B)[e]$ if $\mathcal{M} \models A[e]$ and $\mathcal{M} \models B[e]$

(4) $\mathcal{M} \models \forall x A[e]$ if $\mathcal{M} \models A[e']$ for all evaluations e' coinciding with e on all variables except x.

A formula A is true in $\mathcal{M}$ if $\mathcal{M} \models A[e]$ for every possible evaluation e.

If a formula A has only one free variable, say x, and $a \in M$, we will write $\mathcal{M} \models A(a)$ or $\mathcal{M} \models A[a]$ instead of $\mathcal{M} \models A[e]$ where e evaluates x to $a \in M$.

$\mathbb{N}$ will be the standard model of theories formulated in $\mathcal{L}_0$. To be able to work with natural numbers within $\mathcal{L}_0$ we assign a variable free term $\bar{n}$ to every $n \in \mathbb{N}$, $\bar{n} := \mathsf{S}(\mathsf{S}(\ldots\ \mathsf{S}(0)\ldots))$, S occurring n-times. $\bar{n}$ is called *n-th numeral*.

As a convention, we will use infix notation for binary functions and predicates; superfluous parenthesis will be omitted, whenever they do not lead to confusion and provide better readability.

2.1.4 Arithmetical hierarchy

As we did in the proof-theoretic approach, it is convenient to define useful collections of formulas (naturally restricted to first order variables) which build the arithmetical hierarchy.

We make use of the following abbreviations:

(1) $(\exists x \leq y)A \equiv \exists x(x \leq y \wedge A)$

(2) $(\forall x \leq y)A \equiv \forall x(x \leq y \rightarrow A)$

Quantifiers of the form $(\forall x \leq y)$ and $(\exists x \leq y)$ are called bounded. An $\mathcal{L}_0$-formula is called bounded if every quantifier occurring in it is bounded.

Definition 2.1.5 (Arithmetical hierarchy). The arithmetical hierarchy is defined inductively by:

(1) The collection of Σ_0-formulas = Π_0-formulas consists of all bounded $\mathcal{L}_0$-formulas.

(2) A is Σ_{n+1} if $A \equiv \exists x B$ where B is Π_n

(3) A is Π_{n+1} if $A \equiv \forall x B$ where B is Σ_n.

A set $M \subset \mathbb{N}$ is defined by a formula A if $M = \{n \in \mathbb{N} : \mathbb{N} \models A(n)\}$. To be able to talk about the "complexity" of functions and relations in terms of the arithmetical hierarchy, we give the following definition.

Definition 2.1.6. An m-ary relation $R \subset \mathbb{N}^m$ is Σ_n (resp. Π_n) if it is defined by a Σ_n (resp. Π_n) formula with *exactly* m free variables.
A function $f : \mathbb{N}^m \rightarrow \mathbb{N}$ is Σ_n (resp. Π_n) if its graph is Σ_n (resp. Π_n).

A relation R is Δ_n if it is Σ_n and Π_n.

Note that Σ_n relations are complements of Π_n relations and vice versa.

The following definition somewhat widens the class of Σ_n- resp. Π_n-formulas, as we do not only characterize formulas by their syntactical properties; but we will also take into account that some theories T may prove the equivalence between a Σ_n resp. Π_n formulas to an arbitrary one, which we will then call Σ_n resp. Π_n *in* T.

Definition 2.1.7. A formula A is Σ_n resp. Π_n *in a theory* T if there exists a Σ_n resp. Π_n formula B with the same free variables such that $T \vdash A \leftrightarrow B$.

Lemma 2.1.8 (Pairing function). *There is a Σ_0 pairing function, i.e., a one-one mapping $(\cdot,\cdot) : \mathbb{N}^2 \to \mathbb{N}$ increasing in both arguments*

Proof. Define $(m, n) := \frac{1}{2}(m + n + 1)(m + n) + m$ □

As we need to encode finite sequences of natural numbers by natural numbers, we state the following definition

Definition 2.1.9. A coding of finite sequences of natural numbers by natural numbers consists of a primitive-recursive set $\mathsf{Seq} \subset \mathbb{N}$ and three primitive-recursive functions

- lh — $\mathsf{lh}(s)$ is the length of s
- $(\cdot)_i$ — $(s)_i$ is the i-th element of s; $(\langle s_1, \ldots, s_i, \ldots, s_n\rangle)_i = s_i$
- $\star$ — $\star$ is concatenation of sequence-numbers; $\langle s_1, \ldots, s_n\rangle \star \langle t_1, \ldots, t_m\rangle = \langle s_1, \ldots, s_n, t_1, \ldots, t_m\rangle$

such that the following requirements hold:

(1) $\mathsf{lh}(s) < s$ and for every $i < \mathsf{lh}(s)$ we have $(s)_i < s$

(2) there is an empty sequence $\varnothing$ with $\mathsf{lh}(\varnothing) = 0$

(3) monotonicity: if $\mathsf{lh}(s) \leq \mathsf{lh}(s')$ and for each $i < \mathsf{lh}(s)$ we have $(s)_i \leq (s')_i$ then $s \leq s'$.

(4) the set $\mathbb{N} \setminus \mathsf{Seq}$ is infinite.

2.2 Robinson Arithmetic and the theory Σ_1-PA

The theory Σ_1-PA is formalized in the language $\mathcal{L}_0$ and presented in a Hilbert-style calculus containing the axioms for equality. Σ_1-PA is sometimes also referred to as $\mathrm{I}\Sigma_1$.

We start first defining the Robinson arithmetic Q, which is the underlying theory, then we extend it to Σ_1-PA by adding the induction scheme for Σ_1-formulas.

Definition 2.2.1. Robinson arithmetic Q is the theory in $\mathcal{L}_0$ which satisfies the following axioms plus the equality axioms:

(Q.1) $\mathsf{S}(x) \neq \bar{0}$
(Q.2) $\mathsf{S}(x) = \mathsf{S}(y) \rightarrow x = y$
(Q.3) $x \neq \bar{0} \rightarrow (\exists y)(x = \mathsf{S}(y))$
(Q.4) $x + \bar{0} = x$
(Q.5) $x + \mathsf{S}(y) = \mathsf{S}(x + y)$
(Q.6) $x \cdot \bar{0} = \bar{0}$
(Q.7) $x \cdot \mathsf{S}(y) = (x \cdot y) + x$
(Q.8) $x \leq y \leftrightarrow (\exists z)(z + x = y)$

Σ_1-PA is defined from Q by adding the induction scheme

(I.Σ_1) $A(\bar{0}) \wedge (\forall x)(A(x) \rightarrow A(\mathsf{S}(x))) \rightarrow \forall x A(x)$

for every Σ_1-formula $A(x)$.

We remark that it is possible to build up all primitive-recursive functions in Σ_1-PA and prove within Σ_1-PA their properties (e.g., totality). In Σ_1-PA we can develop exponentiation in the usual manner; this is $\exp(0) = \bar{1}$ and for every x, $\exp(\mathsf{S}(x)) = \exp(x) \cdot 2$ are provable.

2.2.1 Gödel numbering of arithmetic

As lh, $(\cdot)_i$ and $\star$ are primitive-recursive we can, based on them define the Gödel numbering of the language $\mathcal{L}_0$. By a Gödel numbering we mean an encoding of $\mathcal{L}_0$-terms t and formulas A by natural numbers $\ulcorner t \urcorner$ and $\ulcorner A \urcorner$. We have to assure that we can reconstruct t and A from $\ulcorner t \urcorner$ and $\ulcorner A \urcorner$. A possible definition of a Gödel numbering may be found in Girard [4] (definition 1.2.22). Furthermore there are three primitive recursive predicates Tm, Fml, Fr and Val such that $\mathsf{Tm}(a)$ holds iff a is the Gödel number of a term in $\mathcal{L}_0$, $\mathsf{Fml}(a)$ holds iff a is the Gödel number of a formula in $\mathcal{L}_0$, $\mathsf{Fr}(a, b)$ holds iff b is the Gödel number of a variable occurring freely in the expression encoded by b and $\mathsf{Val}(t, e)$ is the value of the term t under the evaluation e.

We mention there is a formula $\Sigma_0(x)$ which is Δ_1 in Σ_1-PA such that $\Sigma_0(x)$ holds iff x is the Gödel number of a Σ_0-formula. This result can even be extended to Σ_n- and Π_n-formulas (for a fixed $n \in \mathbb{N}$). We will use them in the definition of the various satisfaction predicates that we are going to define.

Due to the Gödel numbering of $\mathcal{L}_0$ we are able to express within $\mathcal{L}_0$ what it means that a formula of restricted complexity (e.g., Σ_n, Π_n) is true. We are going to develop this by defining a partial satisfaction formula at first, from which we will be able to give a definition of satisfaction for Gödel numbers representing Σ_0-formulas.

Definition 2.2.2 (Partial Satisfaction). q is a *partial satisfaction* for Σ_0-formulas $\leq p$ and their evaluations by elements $\leq r$, written as $\mathsf{PSat}_0(q, p, r)$ if q is a finite mapping whose domain consists of all pairs (z, e), where z is a Σ_0-formula $z \leq p$ and e an evaluation of free variables of z by elements $\leq r$, $\mathrm{ran}(q)$ is a subset of $\{0, 1\}$ and Tarski's truth conditions are satisfied, i.e.,

(1) if z is atomic of the form $u = v$ then $q(z, e) = 1$ if $\mathsf{Val}(u, e) = \mathsf{Val}(v, e)$; if z has the form $u \leq v$ then $q(z, e) = 1$ if $\mathsf{Val}(u, e) \leq \mathsf{Val}(v, e)$,

(2) if z has the form $\neg u$ then $q(z, e) = 1$ if $q(u, e) = 0$.

(3) if z has the form $v \wedge u$ then $q(z, e) = 1$ if $q(u, e) = 1$ and $q(v, e) = 1$,

(4) if z has the form $(\forall x \leq y)u$ then $q(z, e) = 1$ if for every extension e' of e such that $e'(x)$ is defined and $e'(x) \leq e'(y)$ we have $q(u, e') = 1$.

This leads to the following definition of satisfaction for Σ_0 formulas:

Definition 2.2.3. $\mathsf{Sat}_0(z, e) \equiv (\exists q, p, r)[\mathsf{PSat}_0(q, p, r) \wedge q(z, e) = 1] \wedge \Sigma_0(z)$.

And of course we get what we were aiming at: satisfaction for Σ_0-formulas such that Tarski's truth conditions hold.

Lemma 2.2.4. Sat_0 *is* Δ_1 *in* Σ_1-PA *and satisfies Tarski's truth conditions for Gödel numbers representing* Σ_0*-formulas, i.e.,*

(1) if z is atomic of the form $u = v$ then $\mathsf{Sat}_0(z, e)$ iff $\mathsf{Val}(u, e) = \mathsf{Val}(v, e)$; if z has the form $u \leq v$ then $\mathsf{Sat}_0(z, e)$ iff $\mathsf{Val}(u, e) \leq \mathsf{Val}(v, e)$,

(2) if z has the form $\neg u$ then $\mathsf{Sat}_0(z, e)$ iff $\neg \mathsf{Sat}_0(u, e)$,

(3) if z has the form $v \wedge u$ then $\mathsf{Sat}_0(z, e)$ iff $\mathsf{Sat}_0(v, e) \wedge \mathsf{Sat}_0(u, e)$,

(4) if z has the form $(\forall x \leq y)u$ then $\mathsf{Sat}_0(z, e)$ iff for every evaluation e' of u coinciding on all free variables of z except x and such that $e'(x)$ is defined and $e'(x) \leq e'(y)$, $\mathsf{Sat}_0(u, e')$ holds.

A proof of this lemma may be found in Hájek–Pudlák [6], where it is formulated as theorem 1.70.

One might want to have a full satisfaction predicate such that it applies to all $\mathcal{L}_0$-formulas and is true if A is satisfied. This is not possible since it would lead to a contradiction with Gödel's famous incompleteness theorem. Nevertheless it is still possible to define a satisfaction predicate for the collection of Σ_n- resp. Π_n-formulas.

Definition 2.2.5 (Satisfaction for Σ_n/Π_n). For every $n \geq 0$ we define in Σ_1-PA satisfaction for Σ_n-formulas $\mathsf{Sat}_{\Sigma,n}(z,e)$ and Π_n-formulas $\mathsf{Sat}_{\Pi,n}(z,e)$ inductively as follows:

$$\mathsf{Sat}_{\Sigma,0}(z,e) = \mathsf{Sat}_{\Pi,0}(z,e) = \mathsf{Sat}_0(z,e)$$

given $\mathsf{Sat}_{\Sigma,n}(z,e)$ resp. $\mathsf{Sat}_{\Pi,n}(z,e)$ we define

$\mathsf{Sat}_{\Pi,n+1}(z,e) \equiv$ z has the form $(\forall x)u$ where u is number of a Σ_n-formula e evaluates free variables of z, and for each evaluation e' of free variables of u extending e we have $\mathsf{Sat}_{\Sigma,n}(u,e')$.

$\mathsf{Sat}_{\Sigma,n+1}(z,e) \equiv$ z has the form $(\exists x)u$ where u is number of a Π_n-formula, e evaluates free variables of z, and there exists an evaluation e' of free variables of u extending e we have $\mathsf{Sat}_{\Pi,n}(u,e')$

And once again, we can prove what we intended $\mathsf{Sat}_{\Pi,n}(z,e)$ resp. $\mathsf{Sat}_{\Sigma,n}(z,e)$ to be, as the following theorem shows.

Theorem 2.2.6. *The predicates $\mathsf{Sat}_{\Sigma,n}$ and $\mathsf{Sat}_{\Pi,n}$ obey Tarski's truth conditions for Σ_n resp. Π_n formulas, i.e., they obey (1)–(4) from lemma 2.2.4 and additionally (5) for $\mathsf{Sat}_{\Pi,n}$ and (5') for $\mathsf{Sat}_{\Sigma,n}$*

(5) if $m \leq n$, z is Π_m and z has the form $(\forall x)u$ then $\mathsf{Sat}_{\Pi,n}(z,e)$ iff for all evaluations e' of u coinciding with e on the free variables of z such that $\mathsf{Sat}_{\Pi,n}(u,e')$ holds.

(5') if $m \leq n$, z is Σ_m and z has the form $(\forall x)u$ then $\mathsf{Sat}_{\Sigma,n}(z,e)$ iff for all evaluations e' of u coinciding with e on the free variables of z such that $\mathsf{Sat}_{\Sigma,n}(u,e')$ holds.

Remark. $\mathsf{Sat}_{\Sigma,n}$ is Σ_n and $\mathsf{Sat}_{\Pi,n}$ is Π_n for $n \geq 1$.

If $\mathsf{Sat}_\Gamma(z,e)$ is a formula which expresses satisfaction for a collection Γ of formulas (e.g., Σ_n or Π_n) then we may take Γ-formulas with exactly one free variable for codes of Γ-sets. We may also introduce new variables for Γ-sets and quantify over such sets. Whenever we exhibit satisfaction for a collection Γ of formulas we speak may of Γ-definable sets.

2.2.2 Extending the language $\mathcal{L}_0$ to $\mathcal{L}_{0,X}$

By $\mathcal{L}_{0,X}$ we denote the language of first-order arithmetic extended by a new relation symbol X. We let the new atomic formulas consists of the ones from $\mathcal{L}_0$ plus $t \in X$ for any term t. $\mathcal{L}_{0,X}$-formulas are built up from the new atomic ones by closing against the logical connectives and quantifiers. Hence, $\Sigma_0(X)$-formulas are generated from the atomic ones using the logical connectives and bounded quantifiers. $\Sigma_n(X)$ and $\Pi_n(X)$ are defined from Σ_0 in the same way as did for $\mathcal{L}_0$. Of course we can define a Gödel numbering of $\mathcal{L}_{0,X}$ and find a unary predicate $\Sigma_0(X)(z)$ which is true if z is the Gödel number of a $\Sigma_0(X)$-formula.

Definition 2.2.7. A definable set X is *piecewise coded* (p. c.) if for each x there is a sequence s of 0's and 1's of length x such that $(\forall i < x)(i \in X \leftrightarrow (s)_i = 1)$. Each such string s is called piece of X.

The notion of piecewise coded sets is due to Clote [3]. We will be using the fact that Σ_1-PA proves every Σ_1-set to be piecewise coded; this result can even be improved to $\Sigma_0(\Sigma_1)$-sets.

Definition 2.2.8 (Partial Satisfaction). Let X be piecewise coded. q is a partial satisfaction for $\Sigma_0(X)$-formulas $\leq p$, their evaluation by numbers $\leq r$ and partial interpretation of X by a string s of 0's and 1's, written as $\mathsf{PSat}_{0,X}(q, p, r, s)$ if q is a finite mapping whose domain consists of all pairs (z, e) where z is a $\Sigma_0(X)$-formula $z \leq p$, e is an evaluation of free variables of z by elements $\leq r$ with $\mathrm{ran}(q) \subseteq \{0, 1\}$, $\mathsf{lh}(s) > r^p$ and Tarski's truth conditions hold, i.e., for every $(z, e) \in \mathrm{dom}(q)$

(1) if z is atomic of the form $u = v$ then $q(z, e) = 1$ if $\mathsf{Val}(u, e) = \mathsf{Val}(v, e)$; if z has the form $u \leq v$ then $q(z, e) = 1$ if $\mathsf{Val}(u, e) \leq \mathsf{Val}(v, e)$

(2) if z has the form $\neg u$ then $q(z, e) = 1$ if $q(u, e) = 0$,

(3) if z has the form $v \wedge u$ then $q(z, e) = 1$ if $q(u, e) = 1$ and $q(v, e) = 1$,

(4) if z has the form $(\forall x \leq y)u$ then $q(z, e) = 1$ if for every extension e' of e such that $e'(x)$ is defined and $e'(x) \leq e'(y)$ we have $q(u, e') = 1$,

(5) if z has the form $t \in X$ then $q(z, e) = 1$ if $(s)_{\mathsf{Val}(t,e)} = 1$.

The somewhat unusual condition $\mathsf{lh}(s) \geq r^p$ needs a further explanation; it is needed to ensure that $(s)_{\mathsf{Val}(t,e)}$ is defined, since if $t \leq p$ is a Gödel number of a term, and e an evaluation of the latter by numbers $\leq r$, then $\mathsf{Val}(t, e) \leq r^p$ which follows from the fact that $\mathsf{Val}(t, e) \leq r^{\mathsf{lh}(t)}$.

Lemma 2.2.9. $\mathsf{PSat}_{0,X}$ *is* Δ_1 *in* Σ_1-PA.

Proof. The reader may confer lemma 2.59 in Hájek–Pudlák [6] □

Definition 2.2.10. For a piecewise-coded set X we define $\mathsf{Sat}_{0,X}(z,e) \equiv (\exists q,p,r,s)[s \text{ piece of } X \wedge \mathsf{PSat}_{0,X}(q,p,r,s) \wedge q(z,e) = 1] \wedge \Sigma_0(X)(z)$

We can also define $\mathsf{Sat}_{\Sigma,n,X}$ and $\mathsf{Sat}_{\Pi,n,X}$ from the $\mathsf{Sat}_{0,X}$ in much the same way as we did it in definition 2.2.5, under the assumption that X is piecewise coded.

Lemma 2.2.11. *There is a formula* $\mathsf{WSat}_{\Sigma,1}$ *which is* Δ_1 *in* Σ_1-PA *such that* Σ_1-PA *proves that for every piecewise-coded set* X*, every Gödel number* z *of a* $\Sigma_1(X)$*-formula with exactly one free variable and every* x *the following are equivalent:*

(1) $\mathsf{Sat}_{\Sigma,1,X}(z,[x])$ *(*$[x]$ *being the evaluation of the only free variable in* z*).*

(2) $(\exists s \text{ piece of } X)\, \mathsf{WSat}_{\Sigma,1}(z,x,s)$

(3) $(\exists w)(\forall s \text{ piece of } X)[w \leq \mathsf{lh}(s) \rightarrow \mathsf{WSat}_{\Sigma,1}(z,x,s)]$

$\mathsf{WSat}_{\Sigma,1}$ expresses that s witnesses the satisfaction of z by x. In a slightly different formulation $\mathsf{WSat}_{\Sigma,1}$ will be involved in the proof of the low basis theorem. A proof of this lemma is given in Hájek–Pudlák [6] as lemma 2.62.

2.2.3 Extending the language $\mathcal{L}_{0,X}$ to $\mathcal{L}_{0,X,H}$

Let X be a new variable, H a new unary function symbol and add $t \in X$ (t a term) to the atomic formulas. $\Sigma_0^H(X)$-formulas result from the new atomic formulas using logical connectives and bounded quantifiers of the form $\forall x \leq y$, $\forall x \leq H(y)$ and $\exists x \leq y$, $\exists x \leq H(y)$, where $H(y)$ is a Δ_1 total strictly increasing function. The resulting language will be denoted $\mathcal{L}_{0,X,H}$.

As with the extension of $\mathcal{L}_0$ to $\mathcal{L}_{0,X}$, we can define an appropriate Gödel numbering of $\mathcal{L}_{0,X,H}$, define the arithmetical hierarchy and find a unary predicate $\Sigma_0^H(X)(z)$ which is true if z is the Gödel number of a $\Sigma_0^H(X)$-formula. Not very surprisingly, satisfaction for $\Sigma_0^H(X)$-formulas can also be expressed with a $\Delta_1(X)$ formula $\mathsf{Sat}_{0,X}^H$:

Theorem 2.2.12. *There is a formula* $\mathsf{Sat}_{0,X}^H(z,e)$ *obeying Tarski's truth conditions for Gödel numbers of* $\Sigma_0^H(X)$*-formulas and* $\mathsf{Sat}_{0,X}^H(z,e)$ *is* $\Delta_1(X)$ *in* Σ_1-PA.

A proof of this theorem can be found in Hájek–Pudlák [6] (theorem 2.74). There may also be defined a partial satisfaction predicate $\mathsf{PSat}_{0,X}^H(q,u,v,s)$ in the language $\mathcal{L}_{0,X,H}$ as Hájek–Kučera [5] did.

2.2.4 Notion of $\Sigma_0^*(\Sigma_n)$ sets and low $\Sigma_0^*(\Sigma_n)$ sets

The purpose of this section is to introduce two collections of $\mathcal{L}_{0,X,H}$-formulas with specific properties which we will need to prove the low basis theorem. At a first glance, these definitions seem rather technical; but it turns out to be the "right" definition to reach our goal.

Definition 2.2.13 (in Σ_1-PA). X is a $\Sigma_0^*(\Sigma_n)$ set if there exists a total Δ_1 function H and some Σ_n set Y such that X is $\Sigma_0^H(Y)$.

X is low $\Sigma_0^*(\Sigma_n)$ if it is $\Sigma_0^*(\Sigma_n)$ and every $\Sigma_1(X)$ set Y is $\Sigma_0^*(\Sigma_n)$.

We will say X is LL_n to express that X is low $\Sigma_0^*(\Sigma_n)$.

Next we present some useful facts about low $\Sigma_0^*(\Sigma_1)$ sets. We already know that Σ_1-PA proves every Σ_0 and $\Sigma_0(\Sigma_1)$ set to be piecewise coded; this also applies to LL_1 sets. Σ_1-PA proves even more; it can be shown that Σ_1-PA proves induction for LL_1 sets and thus also collection, as the following lemma states.

Lemma 2.2.14. Σ_1-PA *proves induction and collection for* $\Sigma_1(\mathsf{LL}_1)$ *sets.*

A proofs of this lemma is presented in Hájek–Pudlák [6] in detail. And we will also need to know that the LL_1 sets are closed under taking Δ_1.

Theorem 2.2.15. Σ_1-PA *proves, if* Z *is* LL_1 *and* Y *is* $\Delta_1(Z)$ *then* Y *is* LL_1.

Proof. Let Z be LL_1, Y be $\Delta_1(Z)$ and X in $\Sigma_1(Y)$. We are going to show that X is in $\Sigma_1(Z)$.

For appropriate $\Sigma_1(Z)$-formulas A, B (given as Gödel numbers) we have

$$Y = \{y : (\exists s \text{ piece of } Z)\, \mathsf{WSat}_{\Sigma,1}(A, y, s)\}$$
$$-Y = \{y : (\exists s \text{ piece of } Z)\, \mathsf{WSat}_{\Sigma,1}(B, y, s)\}$$

where $-Y$ denotes the complement of Y. Using the previous lemma we get by $\mathrm{B}\Sigma_1(\mathsf{LL}_1)$ (provably in Σ_1-PA) a common bound:

$$(\forall y < a)(\exists s \text{ piece of } Z)[\mathsf{WSat}_{\Sigma,1}(A, y, s) \vee \mathsf{WSat}_{\Sigma,1}(B, y, s)]$$

Thus for some Δ_1-formula D we have

$$t \text{ piece of } Y \leftrightarrow (\exists s \text{ piece of } Z) D(t, s).$$

Now let X be in $\Sigma_1(Y)$. This is for some $\Sigma_1(Y)$-formula C

$$X = \{x : (\exists t \text{ piece of } Y)\, \mathsf{WSat}_{\Sigma,1}(C, x, t)\}$$
$$= \{x : (\exists s \text{ piece of } Z)(\exists t)[D(t, s) \wedge \mathsf{WSat}_{\Sigma,1}(C, x, t)]\}$$

This shows that X is $\Sigma_1(Z)$. Thus we have $\Sigma_1(Y) \subset \Sigma_1(Z) \subset \Sigma_0^\star(\Sigma_1)$ and consequently Y is LL_1. This completes the proof. □

2.3 Low-Basis theorem

The aim of this section is to prove the following theorem, which will be our starting-point towards the ω-interpretation of WKL_0 in Σ_1-PA.

Theorem 2.3.1 (Low-Basis-Theorem). *Every infinite binary Δ_1-tree T has a low $\Sigma_0^\star(\Sigma_1)$ infinite path B (provable in Σ_1-PA).*

Corollary 2.3.2. *Every infinite binary LL_1-tree T has an infinite LL_1-path B (provable in Σ_1-PA).*

Proof. By relativization. Let T be LL_1, thus T is $\Sigma_0^\star(\Sigma_1)$ and every $\Sigma_1(T)$ set is $\Sigma_0^\star(\Sigma_1)$ by definition. By the Low-Basis-Theorem we know that there is a low $\Sigma_0^\star(\Sigma_1(\mathsf{LL}_1))$ path B which is the same as $\Sigma_0^\star(\Sigma_1(\text{low } \Sigma_0^\star(\Sigma_1)))$; since every $\Sigma_1(\mathsf{LL}_1)$ set is by definition $\Sigma_0^\star(\Sigma_1)$ we have B to be $\Sigma_0^\star(\Sigma_0^\star(\Sigma_1))$ which is eventually $\Sigma_0^\star(\Sigma_1)$, hence B is LL_1. □

Before we are going to prove the low-basis theorem we remind the reader of the Δ_1-formula $\mathsf{WSat}_{\Sigma,1}$. Σ_1-PA proves for every piecewise coded set X, every Gödel number z of a $\Sigma_1(X)$ formula with exactly one free variable and every number x the equivalence:

$$\begin{aligned}\mathsf{Sat}_{\Sigma,1,X}(z,[x]) &\leftrightarrow (\exists s \text{ piece of } X)\, \mathsf{WSat}_{\Sigma,1}(z,x,s)\\ &\leftrightarrow (\exists w)(\forall s \text{ piece of } X)[w \le \mathsf{lh}(s) \to \mathsf{WSat}_{\Sigma,1}(z,x,s)]\end{aligned}$$

$\mathsf{WSat}_{\Sigma,1}(z,x,s)$ reads "s witnesses the satisfaction of z by x" (lemma 2.2.11). If X is a binary relation such that for every u the restriction $X \restriction u$ exists as a finite sequence (from which follows that X is p.c.) we have to change the notation of "witnessing" by replacing "piece of" with "restriction of". So we have a new Δ_1-predicate WSat such that:

$$\begin{aligned}\mathsf{Sat}_{\Sigma,1,X}(z,[x]) &\leftrightarrow (\exists x)(\exists s = X \restriction x)\, \mathsf{WSat}(z,x,s)\\ &\leftrightarrow (\exists x)(\forall y \ge x)(\forall s = X \restriction y)\, \mathsf{WSat}(z,x,s)\end{aligned}$$

Later on, we will use this formulation of WSat (and drop the indices to indicate that).

Definition 2.3.3. Let $\mathsf{string}(e)$ be the set of all sequences of 0's and 1's of length e, i.e., $s \in \mathsf{string}(e) \equiv (\forall i < e)[(s)_i = 0 \vee (s)_i = 1] \wedge \mathsf{lh}(s) = e$.
T is unbounded if $(\forall e)(\exists t \in \mathsf{string}(e))(t \in T)$
T is unbounded over t if $(\forall e)(\exists s \in \mathsf{string}(e))(s \in \{t' \in T : t' \supset t\})$

We remark that $s \in \mathsf{string}(e)$ is Δ_0 and $\Sigma_0^\star(\Sigma_1)$ sets are closed under quantification of the form $\forall s \in \mathsf{string}(e)$ and $\exists s \in \mathsf{string}(e)$. In addition "$T$ is unbounded" is Π_1.

Proof of the Low-Basis Theorem. The construction of the path Z happens in steps. We define two strings s_e and c_e of length e in step e; s_e will be a piece of the path Z and c_e holds information about the truncation of T in previous steps. To show that $\Sigma_1(Z)$ is $\Sigma_0^\star(\Sigma_1)$, we need a Δ_1-enumeration (φ_e, a_e) of all pairs consisting of a $\Sigma_1(Z)$-formula with one free variable and of a number. In step e we will decide whether φ_e will be satisfied under the evaluation a_e and Z.

The basic problem of the proof is to show that the path is LL_1. The proof of the lowness requires the infiltration of the $\mathsf{Sat}_{\Sigma,1,X}$-predicate; we will show that in Σ_1-PA $\mathsf{Sat}_{\Sigma,1,X}(\varphi_e, a_e)$ is provably equivalent to a $\Sigma_0^\star(\Sigma_1)$ formula, so $\Sigma_1(X)$ sets are $\Sigma_0^\star(\Sigma_1)$ and thus X is LL_1.

We define two subsets of the Δ_1-tree T. Let s, c be strings of length e:

$$T(e,s,c) := \{t \in T : (t \subseteq s \vee s \subseteq t) \wedge (\forall i < e)[(c)_i = 1 \rightarrow \neg\,\mathsf{WSat}(\varphi_i, a_i, t)\}$$

$$T'(e,s,c) := \{t \in T : t \in T(e,s,c) \wedge \neg\,\mathsf{WSat}(\varphi_e, a_e, t)\}$$

Next, we define a predicate $\mathsf{prolong}$ saying how to prolong the binary strings s, c of length e to s', c' of length $e+1$.

$$\begin{aligned}\mathsf{prolong}(e,s,c,s',c') \equiv\, & s, c \in \mathsf{string}(e) \wedge s', c' \in \mathsf{string}(e+1) \\ & \wedge (s \subseteq s' \wedge t \subset t') \wedge (\mathit{case\_1} \vee \mathit{case\_2})\end{aligned} \tag{2.3.1}$$

where:

case_1: $T'(e,s,c)$ is unbounded, $c' = c \star \langle 1\rangle$, and

$$s' = \begin{cases} s \star \langle 0\rangle & \text{if } T'(e,s,c) \text{ is unbounded over } s \star \langle 0\rangle \\ s \star \langle 1\rangle & \text{otherwise.}\end{cases}$$

case_2: $T'(e,s,c)$ is bounded, $c' = c \star \langle 0\rangle$, and

$$s' = \begin{cases} s \star \langle 0\rangle & \text{if } T(e,s,c) \text{ is unbounded over } s \star \langle 0\rangle \\ s \star \langle 1\rangle & \text{otherwise.}\end{cases}$$

We see that $\mathsf{prolong}$ is $\Sigma_0^\star(\Sigma_1)$, since being unbounded is Π_1. This next lemma gives us the information that we can always prolong s, c in step e if $T(e, s, c)$ is unbounded. This is done by case distinction.

Lemma 2.3.4.

(i) If $T(e, s, c)$ is unbounded then $\exists s', c'\, \mathsf{prolong}(e, s, c, s', c')$,

(ii) If $T(e, s, c)$ is unbounded and $\mathsf{prolong}(e, s, c, s', c')$ holds, so $T(e+1, s', c')$ is unbounded.

We define an *initial* path-predicate $\mathsf{path}(e, s, c)$, from which we will be able to define the path-function Z, in the following way:

$$\begin{aligned}\mathsf{path}(e, s, c) \equiv \quad & s, c \in \mathsf{string}(e) \wedge \\ & (\forall i < e)\, \mathsf{prolong}(i, s \restriction i, c \restriction i, s' \restriction (i+1), c' \restriction (i+1))\end{aligned}$$

As $\mathsf{prolong}$ is a boolean combination of $\Sigma_0^\star(\Sigma_1)$-formulas and $\Sigma_0^\star(\Sigma_1)$ formulas are closed under bounded quantification, path is $\Sigma_0^\star(\Sigma_1)$.

Lemma 2.3.5. *If $T(e, s, c)$ is unbounded then Σ_1-PA proves*

$$(\forall e)[(\exists s, c)\mathsf{path}(e, s, c) \wedge (\forall s, c)(\mathsf{path}(e, s, c) \rightarrow T(e, s, c) \textit{ unbounded})]$$

Proof. This follows from the fact that Σ_1-PA proves induction for $\Sigma_0^\star(\Sigma_1)$-formulas (by lemma 2.2.14); the inductive step follows from the previous lemma. □

Define a function Z (the path-function) as

$$Z(x) = y \equiv \exists s, c \in \mathsf{string}(x+1)[\mathsf{path}(x+1, s, c) \wedge (s)_x = y]$$

Thus Z is $\Sigma_0^\star(\Sigma_1)$ since path is $\Sigma_0^\star(\Sigma_1)$ and $\Sigma_0^\star(\Sigma_1)$-formulas are closed under $\exists s, c \in \mathsf{string}(x+1)$.

This lemma is fundamental for the proof of the lowness of Z

Lemma 2.3.6. $\mathsf{Sat}_{\Sigma,1,Z}(\varphi_e, [a_e])$ *holds iff for the unique s, c satisfying* $\mathsf{path}(e, s, c)$ *we have $T'(e, s, c)$ is bounded.*

Proof. Recall that $\mathsf{Sat}_{\Sigma,1,Z}(\varphi_e, [a_e])$ is equivalent to the existence of a piece t of Z such that $\mathsf{WSat}(\varphi_e, a_e, t)$ holds. We consider the two cases:
Assume we are in case 2: $T'(e, s, c)$ is bounded, and let t be a piece of Z

longer than a bound for $T'(e,s,c)$. Then $t \in T(e,s,c)$ and $\mathsf{WSat}(\varphi_e, a_e, t)$ holds; thus $\mathsf{Sat}_{\Sigma,1,Z}(\varphi_e, a_e)$ holds by lemma 2.2.11.
Now assume case 1, then for $i > e$ we have

$$(\forall s', c' \in \mathsf{string}(i))(\mathsf{path}(i, s', c') \to \neg \mathsf{WSat}(\varphi_e, a_e, s')$$

by induction on i, which is admissible, since the formula is $\Sigma_0^\star(\Sigma_1)$. Hence, there is no piece t of Z such that $\mathsf{WSat}(\varphi_e, a_e, t)$ holds.

□

This lemma yields immediately:

$$\mathsf{Sat}_{\Sigma,1,Z}(\varphi_e, [a_e]) \leftrightarrow \exists s, c \in \mathsf{string}(e+1)[\mathsf{path}(e,s,c) \wedge (c)_e = 0]$$

As $\exists s, c \in \mathsf{string}(e+1)[\mathsf{path}(e,s,c) \wedge (c)_e = 0]$ is $\Sigma_0^\star(\Sigma_1)$ we conclude that $\mathsf{Sat}_{\Sigma,1,Z}(\varphi_e, [a_e])$ is $\Sigma_0^\star(\Sigma_1)$ and thus Z is an LL_1 path. This finishes the proof of the low basis theorem. □

The next problem which arises is concerning recursive comprehension with several parameters. One could think of putting them together into one by taking the disjoint union (Turing join) of them. The Turing join of two classes X, Y is defined as

$$X \oplus Y := \{(x, 0) : x \in X\} \cup \{(0, y) : y \in Y\}$$

Unfortunately, LL_1 classes are probably *not* closed under Turing join. So this last section is devoted to solving the problem with recursive comprehension.

Lemma 2.3.7 (Uniformity). *There is an LL_1-set $B^\star$ such that every infinite binary Δ_1-tree T has a $\Delta_1(B^\star)$ infinite path B.*

Proof. We are going to replace the set of all Δ_1-trees by a single Π_1-tree, such that every Δ_1-tree results from the Π_1-tree by projections.
Assuming that every natural number $m \in \mathbb{N}$ codes a Turing machine (or equivalently: is the index of a recursive function), we define two sets of strings:

$T_m^\Sigma :=$ $\{s$: there exists an accepting computation z with input s on the TM $m.\}$

$T_m^\Pi :=$ $\{s$: every computation z with input s on the TM m is accepting.$\}$

As Turing machines are finite objects, T_m^Σ is Σ_1 and T_m^Π is Π_1, because of the unbounded existential resp. universal quantifier defining these sets. Moreover, $T_m^\Sigma \subseteq T_m^\Pi$ and equality holds iff m defines a total function on the set of all strings.

By (i, j) we denote the usual pairing function (which is Σ_0, cf. lemma 2.1.8). If s is a string of length $x = \mathsf{lh}(s) > (i, j)$, we define s_i to be:

$$s_i := ((s)_{(i,0)}, (s)_{(i,1)}, \ldots, (s)_{(i,k)})$$

where $k = \max\{l : (i, l) < x\}$ and $(s)_{(i,k)}$ denotes the (i, k)-th element of s. By $s_i \upharpoonright j$ we mean the restriction of s_i to j i. e.:

$$s_i \upharpoonright j := ((s)_{(i,0)}, (s)_{(i,1)}, \ldots, (s)_{(i,j)})$$

Furthermore t is a strong element of U, denoted $t \in\in U$ if every initial segment of t is in U (i.e., $\forall s(s \subseteq t \to s \in U)$).

Let us now define the Π_1-tree T

$$s \in T :\equiv \forall (i, j) < \mathsf{lh}(s)[(\exists t \in\in T_i^\Sigma \wedge \mathsf{lh}(t) = j) \to (s_i \upharpoonright j) \in T_i^\Pi]$$

We claim that T is an infinite, binary Π_1-tree. Π_1-ness stems from the fact that Σ_1-PA proves collection. To prove that T is infinite it suffices to show that for every i there is an arbitrarily long s_i in T such that

$$\forall j < \mathsf{lh}(s_i)[(\exists t \in\in T_i^\Sigma \wedge \mathsf{lh}(t) = j) \to (s_i \upharpoonright j) \in T_i^\Pi] \tag{2.3.2}$$

we distinguish two cases: either T_i^Σ has arbitrarily long strong elements, and thus they are all in T_i^Π, too; or there is a maximal length x such that T_i^Σ has a strong element t' of length x, but then any s prolonging t' satisfies (2.3.2).

If we now consider a total function Z with $\mathrm{ran}(Z) \subseteq \{0, 1\}$ which is an infinite path through T and m is a total Turing machine defining a Δ_1 infinite binary tree, then Z_m defined by $Z_m(j) := Z((m, j))$ is a path through T_m. What is left to prove is, that T has an infinite LL_1-path $B^\star$; then the assertion that every infinite binary Δ_1-tree has an infinite LL_1-path B in $\Delta_1(B^\star)$ follows. But this is immediate from the following lemma. □

Lemma 2.3.8. Σ_1-PA *proves that every infinite* Π_1 *tree* T *has an infinite* LL_1 *path* B.

Proof. We must check that the proof of the low basis theorem also works for Π_1-trees. Let T be a Π_1 tree, then the two sets $T(e, s, c)$ and $T'(e, s, c)$ which are used in the proof are also Π_1. As "being unbounded" is Π_1 so is "T is unbounded" for a T which is Π_1. Consequently, prolong will remain $\Sigma_0^\star(\Sigma_1)$ and thus the rest of the proof will be identical. □

Theorem 2.3.9. *For every* LL_1*-set* X *there exists an* LL_1*-set* $B^\star(X)$ *such that every infinite binary* $\Delta_1(X)$*-tree* T *has an infinite* $\Delta_1(B^\star(X))$*-path* B.

Proof. Consequence of the uniformity lemma 2.3.7 by relativization. □

We can interpret the preceding theorem as a definition of an operation $B^\star$ which assigns to every class X the corresponding class $B^\star(X)$. Starting with the empty set $\varnothing$ we can construct a sequence q such that $(q)_0 = \varnothing$, $(q)_i \in \mathsf{LL}_1$ for $i < \mathsf{lh}(q)$ and for $i > 0$ we have $(q)_i = B^\star((q)_i)$. We refer to each such sequence q as a chain.

Let $i \in J$ iff there is a unique chain of length i. Further we define

$$\mathsf{class}(X) \equiv (\exists i \in J)(q = (B_0, \ldots, B_i) \wedge X \in \Delta(B_i))$$

$$\mathsf{number}(x) = (x = x)$$

and we let $\in$ be the membership predicate in the sense of LL_1-classes.

Theorem 2.3.10 (ω-Interpretation). *The predicates* number, class *and* $\in$ *together with* $\mathsf{S}, +, \cdot, \leq, 0$ *define an ω-interpretation of* WKL_0 *in* Σ_1-PA.

Proof. The axioms of WKL_0 are clear, as they coincide with those in Σ_1-PA. We have to verify Δ^0_1-CA, Σ^0_1-IND and WKL.

Σ^0_1-IND: As Σ_1-PA proves Σ_1 induction for LL_1 classes, Σ_1-PA proves Σ^0_1 induction.

Δ^0_1-CA: Let $\mathsf{class}(X_1, \ldots, X_n)$ be the parameters and Z be recursive in $X_1, \ldots, X_n$ i.e., $Z \in \Delta_1(X_1, \ldots, X_n)$. By definition there exists $j \in J$ such that $Z \in \Delta_1(B_j)$, thus $\mathsf{class}(Z)$ and so Δ^0_1-CA holds.

WKL: Let $\mathsf{class}(T)$ be an unbounded binary tree, then for some $i \in J$, $T \in \Delta_1(B_i)$ and by theorem 2.3.9 T has an unbounded path $P \in \Delta_1(B_{i+1})$ and so $\mathsf{class}(P)$ is immediate. □

Corollary 2.3.11 (Harrington's Theorem). WKL_0 *is a conservative extension of* Σ_1-PA.

Rien n'est beau que le vrai

— HERMANN MINKOWSKI

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