Theories of ordinal strength $\varphi 20$ and $\varphi 2 \varepsilon_0$

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Marc Heissenbüttel

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Leiter der Arbeit:

Prof. Dr. Gerhard Jäger
Institut für Informatik und angewandte Mathematik

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1 Introduction

In this thesis we will study different principles in the context of the well-known second order theories ACA_0 and ACA and we will give a proof-theoretic analysis of the resulting theories. These two fragments of the formal system of second order arithmetic comprise the usual number-theoretic axioms as well as the defining equations for all primitive-recursive functions, comprehension for arithmetical formulas as well as induction on the natural numbers, for ACA_0 this is formulated for sets as for ACA this is the full second order induction scheme. Further we have the following principles:

1. The axiom of ω model reflection in second order arithmetic (RFN) basically is the axiom that for every set X, there exists a set Y, which models ACA_0 and contains X.
2. The (ω-Jump) axiom states that for any given set X, there exists a set (a hierarchy) Y, satisfying $(Y)_0 = X$ and $(\forall x)((Y)_{x+1} = TJ((Y)_x))$, where TJ denotes the Turing jump.
3. Finally we have the Bar Rule (BR), which permits one to infer the scheme of transfinite induction on a primitive recursive relation $\prec$ when it has been proven that $\prec$ is well-ordered.

The following results extend and refine previous similar results of Rathjen, who showed in [7] that $|\mathsf{ACA}_0 + (\mathsf{BR})| = |\mathsf{ACA}_0 + (\omega\text{-Jump})| = \varphi 20$. In Section 3 we will show that the principle of ω model reflection (RFN) is equivalent over ACA_0, and hence ACA, to the (ω-Jump) axiom. As an immediate consequence we obtain that the proof-theoretic ordinal of $\mathsf{ACA}_0+(\mathsf{RFN})$ is $\varphi 20$ as it was announced in Jäger and Strahm [4]. An upper bound for the proof strength of $\mathsf{ACA}_0 + (\mathsf{BR})$ is obtained in Section 4 by embedding this theory into the theory $\mathsf{ACA}_0+(\mathsf{RFN})$. Further we will see in Section 6 that in fact this bound is sharp, when we will give a well-ordering proof of $\mathsf{ACA}_0 + (\mathsf{BR})$. As an extension of Section 3 we will determine in Section 5 the proof strength of the theory ACA augmented first with (ω-Jump) and secondly with (RFN). We will give a well-ordering proof of $\mathsf{ACA}+(\mathsf{RFN})$ and therefore obtain a lower bound for the proof-theoretic ordinal, and by making use of Schütte's semi-formal system RA^* of [9], we will see that in fact this is the greatest lower bound.

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2 Introduction of $\mathcal{L}_2$ and second order theories

2.1 The syntax of second order theories

For technical reasons we choose a Tait-style formulation of the language $\mathcal{L}_2$ of second order arithmetic. More precisely, $\mathcal{L}_2$ contains the following basic symbols:

1. Countably many free number variables $a, b, c, \ldots, u, v, w$ and bound number variables $x, y, z, \ldots$ as well as free set variables $U, V, W, \ldots$ and bound set variables $X, Y, Z, \ldots$ (all four sorts possibly with subscripts).

2. Symbols for all primitive recursive functions and relations.

3. The symbol $\sim$ for forming negative literals.

4. The symbols $\in$ for the membership relation between numbers and sets.

5. The propositional connectives $\vee$ and $\wedge$ and the quantifiers $\exists$ and $\forall$.

As auxiliary symbols we have parentheses and commas. Observe that there is no propositional connective $\neg$ for negation. Further we exhibit the relation symbols $<$ and $=$ which denote the standard less and equality relation on the natural number, respectively. If $\mathcal{Z}$ and $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are finite strings and $u_1, \ldots, u_n$ is a sequence of pairwise disjoint free variables, then

$$\mathcal{Z}[\mathbf{a}_1, \ldots, \mathbf{a}_n / u_1, \ldots, u_n]$$

is the $\mathcal{L}_2$-formula which is obtained from $\mathcal{Z}$ by simultaneously replacing all free occurrences of the variables $u_1, \ldots, u_n$ by $\mathbf{a}_1, \ldots, \mathbf{a}_n$. We often simply write $\mathcal{Z}(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ instead of $\mathcal{Z}[\mathbf{a}_1, \ldots, \mathbf{a}_n / u_1, \ldots, u_n]$. The number terms $r, s, t, \ldots$ of $\mathcal{L}_2$ are defined as usual, the set terms are just the set variables. Positive literals of $\mathcal{L}_2$ are all expressions $R(s_1, \ldots, s_n)$ and $(s \in U)$ for R an n-ary relation symbol. The negative literals of $\mathcal{L}_2$ are all expressions $\sim E$ so that E is a positive literal of $\mathcal{L}_2$.

The formulas of $\mathcal{L}_2$ $A, B, F, \ldots$ are defined inductively as follows:

1. All literals of $\mathcal{L}_2$ are $\mathcal{L}_2$-formulas.

2. If A and B are $\mathcal{L}_2$-formulas, then $(A \vee B)$ and $(A \wedge B)$ are $\mathcal{L}_2$-formulas.

3. If A is an $\mathcal{L}_2$-formula in which x does not occur, then $(\exists x)A[x/u]$ and $(\forall x)A[x/u]$ are $\mathcal{L}_2$-formulas.

4. If A is an $\mathcal{L}_2$-formula in which X does not occur, then $(\exists X)A[X/U]$ and $(\forall X)A[X/U]$ are $\mathcal{L}_2$-formulas.

We often write $(s \neq t)$ and $(s \notin U)$ instead of $\sim(s = t)$ and $\sim(s \in U)$, respectively. The notation $\vec{e}$ is a shorthand for a finite string $e_1, \ldots, e_n$ of expressions whose length will be specified by the context. We also write $A[\vec{u}]$ to indicate that $\vec{u}$ comprises all free number variables occuring in A and $A[\vec{U}]$ if $\vec{U}$ comprises all free set variables occuring in A.

The negation $\neg A$ of an arbitrary $\mathcal{L}_2$-formula A is inductively defined as usual by making use of the law of double negation and the De Morgan's laws. This means in particular that $\neg A$ is $\sim A$, if A is a positive literal, and $\neg A$ is B, if A is $\sim B$ for some positive literal. The remaining logical connectives are abbreviated as usual.

An $\mathcal{L}_2$-formula is called $\triangle_0$, if it contains no bound set variables and, in addition, every number quantifier is bounded. An $\mathcal{L}_2$ formula is called Π^0_n if it has the form

$$(\forall x_1)(\exists x_2)\ldots(Q_n x_n)A$$

with A being $\triangle_0$. A formula is called Σ^0_n if its negation is a Π^0_n formula. A formula is called arithmetic if it does not contain bound set variables; we write Π^1_0 or Π^0_∞ for the collection of these formulas. Analogously, an $\mathcal{L}_2$ formula is called Π^1_n if it has the form

$$(\forall X_1)(\exists X_2)\ldots(Q_n X_n)A$$

with A arithmetic. A formula is called Σ^1_n if its negation is a Π^1_n formula.

For brevity we often omit brackets in formulas, when there is no risk of confusion. Sometimes we will also use the following abbreviations:

$$\begin{array}{rcl} a \leq b & := & a < b \vee a = b \\ U(t) & := & t \in U \\ U \subseteq V & := & (\forall x)(x \in U \rightarrow x \in V) \\ U = V & := & (\forall x)(x \in U \leftrightarrow x \in V) \\ (\exists x < t)A(x) & := & (\exists x)(x < t \wedge A(x)) \\ (\forall x < t)A(x) & := & (\forall x)(x < t \rightarrow A(x)) \end{array}$$

We presuppose standard notation for coding sequences of natural numbers: $\langle \ldots \rangle$ is a primitive recursive function for forming n tuples $\langle t_0, \ldots, t_{n-1} \rangle$; Seq denotes the primitive recursive set of sequence numbers; $lh(t)$ gives the length of the sequence coded by t, i.e. if $t = \langle t_0, \ldots, t_{n-1} \rangle$ then $lh(t) = n$; $(t)_i$ denotes the ith component of the sequence coded by t if $i < lh(t)$.

In the following we make use of the usual way of coding a finite or infinite sequence of sets of natural numbers into a single one by writing $s \in (U)_t$ instead of $\langle s, t \rangle \in U$.

Accordingly, we have for each $\mathcal{L}_2$ formula A its relativization to the set U, denoted by A^U, which is obtained from A by replacing all quantifiers $(\forall X)(\ldots X\ldots)$ and $(\exists X)(\ldots X\ldots)$ in A by $(\forall x)(\ldots(U)_x\ldots)$ and $(\exists x)(\ldots(U)_x\ldots)$, respectively. Note that A^U is always arithmetic. Finally, element-hood $U\dot{\in}V$ between sets has to be read in the obvious way as $(\exists x)(U=(V)_x)$. Therefore we also denote the relativization of an $\mathcal{L}_2$ formula A to a set U sometimes as $(\exists X\dot{\in}U)(\ldots X\ldots)$ and $(\forall X\dot{\in}U)(\ldots X\ldots)$.

In the sequel we let $\mathsf{LO}(\lhd)$ denote that $\lhd$ is a linear ordering relation, that is for all a,b,c:

$$\neg(a\lhd a) \qquad a\lhd b\wedge b\lhd c\rightarrow a\lhd c \qquad a\lhd b\vee a=b\vee a\lhd b$$

Furthermore we set for all primitive recursive relations $\sqsubset$,

$$\begin{array}{rcl}
\mathsf{PROG}(\sqsubset,U) & := & (\forall x)((\forall y)(y\sqsubset x\rightarrow(y\in U))\rightarrow(x\in U))\\
\mathsf{TI}(\sqsubset,a,U) & := & \mathsf{PROG}(\sqsubset,U)\rightarrow(\forall x\sqsubset a)(x\in U)\\
\mathsf{TI}(\sqsubset,U) & := & \mathsf{PROG}(\sqsubset,U)\rightarrow(\forall x)(x\in U)\\
\mathsf{WF}(\sqsubset) & := & (\forall X)\mathsf{TI}(\sqsubset,X)\\
\mathsf{WO}(\sqsubset) & := & \mathsf{LO}(\sqsubset)\wedge\mathsf{WF}(\sqsubset)
\end{array}$$

2.2 The theories ACA_0^+, RFN_0, and $\mathsf{ACA}_0+(\mathsf{BR})$

It is the purpose of this section to introduce the theories ACA_0^+, ACA^+, RFN_0, RFN and $\mathsf{ACA}_0+(\mathsf{BR})$. All of these subsystems of second order arithmetic will be assumed to contain the rules and axioms of the classical two-sorted Hilbert calculus with equality for numbers, further they consist of the axioms of the theory ACA_0 (or ACA) and some further set existence axioms or rules of inference. Therefore we shortly state here the non-logical axioms of the theory ACA_0 and ACA. They consist of the following $\mathcal{L}_2$-formulas:

I. Number-theoretic Axioms
These comprise the defining equations for the primitive-recursive functions and relations as well as the following axiom for the successor function $\mathcal{S}$:

$$\mathcal{S}(0)\neq 0$$

II. Set Induction Axiom

$$0\in U\wedge(\forall z)(z\in U\rightarrow z+1\in U)\rightarrow(\forall z)(z\in U) \qquad (\mathsf{IND}_0)$$

III. Arithmetical Comprehension Scheme

$$(\exists X)(\forall z)(z \in X \leftrightarrow A(z)) \qquad (\mathsf{ACS})$$

where $A(u)$ is an arithmetical formula of $\mathcal{L}_2$.

The theory ACA_0 comprises the axioms of **I, II** and **III**.

IV. Formula Induction Scheme

$$A(0) \wedge (\forall z)(A(z) \rightarrow A(z+1)) \rightarrow (\forall z)A(z) \qquad (\mathsf{IND})$$

where $A(u)$ is an arbitrary formula of $\mathcal{L}_2$.

Adding scheme **IV** to ACA_0 gives theory ACA.

In general, for any theory Th the subscript 0 denotes restricted induction. This means that Th_0 does not include the full second order induction scheme (IND). Notice that ACA_0 is finitely axiomatizable by a Π^1_2 sentence, see for example Simpson [10] Lemma VIII.1.5. We will denote these sentences from now on by F_{ACA_0}.

We continue with the following definitions.

Definition 2.2.1 *Let $F_\pi(e, \vec{b}, \vec{U})$ be a Π^0_1 formula of $\mathcal{L}_2$ with exactly the displayed free variables. We say that F_π is a universal lightface Π^0_1 if for all Π^0_1 formulas $F_{\pi'}$ of $\mathcal{L}_2$ with the same free variables as F_π we have*

$$(\forall x)(\exists y)(\forall \vec{z})(\forall \vec{Z})(F_\pi(y, \vec{z}, \vec{Z}) \leftrightarrow F_{\pi'}(x, \vec{z}, \vec{Z}))$$

It is well-known that for all numbers of variables there exists a universal lightface Π^0_1 formula.

Definition 2.2.2 *Let $F_\pi(e, b, U) \in \Pi^0_1$ denote a fixed universal lightface formula in $\mathcal{L}_2$ with exactly the displayed free variables. That is e.g. $F_\pi(e, b, U) := \neg(\exists z)\mathcal{T}^U(e, b, z)$, where $\mathcal{T}^U$ is Kleene's T-predicate, relativized to U. Note that $\mathcal{T}^U$ is primitive recursive in U.*

Definition 2.2.3 *The Turing jump of any set U, denoted by $TJ(U)$, is defined as*

$$TJ(U) := \{\langle e, b \rangle :\ (\exists z)T^U(e, b, z)\}$$

This means that $TJ(U)$ is the set of all $\langle e, b\rangle$ such that $\neg F_\pi(e, b, U)$ holds. In a theory comprising ACA_0 the existence of these sets becomes provable. For more details concerning Kleene's T-predicate, fixed universal lightface formulas and recursion theory in general see Rogers [8].

The stage is now set in order to introduce our theories.

V. ω-Jump Hierarchy

The axiom of ω-Jump hierarchy denotes the following formula.

$$(\forall X)(\exists Y)((Y)_0 = X \land (\forall z)((Y)_{z+1} = TJ((Y)_z))) \qquad (\omega - \mathsf{Jump})$$

$TJ(U)$ denotes the Turing jump of a set U as defined in 2.2.3. That is the complete recursively enumerable set relative to U. The theory ACA_0^+ stands for ACA_0 plus the axiom $(\omega - \mathsf{Jump})$. Often we will abbreviate this formula by $(\forall X)(\exists Y)\mathcal{J}_\omega(X, Y)$.

VI. ω Model Reflection

Let us now turn to the axiom of ω model reflection. For any set U, this reflection principle guarantees the existence of a countable coded ω model of ACA_0 which contains U. More formally, we have:

$$(\forall X)(\exists Y)(X \dot{\in} Y \land F^Y_{\mathsf{ACA}_0}) \qquad (\mathsf{RFN})$$

Accordingly, RFN_0 denotes the theory ACA_0 augmented with (RFN).

VII. Bar Rule

The bar rule (BR) is the rule of inference, which permits to infer the scheme of transfinite induction for arbitrary $\mathcal{L}_2$-formulas F on a primitive recursive relation $\prec$ once it has been *proved* that $\prec$ is well-ordered.

$$\frac{\mathsf{WO}(\prec)}{\mathsf{TI}(\prec, F)} \qquad (\mathsf{BR})$$

Obviously the theory $\mathsf{ACA}_0 + (\mathsf{BR})$ extends ACA_0 by each instance of (BR). Rathjen showed in [7] that the bar rule does not permit us to profit from parameters occurring in the relation $\prec$. Therefore we just suppose from now on that there occurs no parameters at all in the relation $\prec$.

Definition 2.2.4 *Let* Th *be a theory formulated in a language containing* $\mathcal{L}_2$*.*

1. *We say that the ordinal* α *is provable in* Th *if there exists a primitive recursive well-ordering* $\prec$ *of order type* α *so that* $\mathsf{Th} \vdash (\forall X)\mathsf{TI}(\prec, X)$*.*

2. *The proof-theoretic ordinal of* Th*, denoted* $|\mathsf{Th}|$*, is the least ordinal which is not provable in* Th*.*

It is well known that $|\mathsf{ACA_0}| = \varepsilon_0$ and $|\mathsf{ACA}| = \varphi 1 \varepsilon_0$ (see for example Schütte [9] Theorem 23.3, Theorem 23.4 and Pohlers [6] Corollary 15.9). Further Rathjen proved in [7] Theorem 3.5 that $|\mathsf{ACA_0} + (\mathsf{BR})| = |\mathsf{ACA}_0^+| = \varphi 20$.

Notice that the (RFN) axiom is a special case of the general scheme of ω model reflection in second order arithmetic as it has been introduced in Friedman [1] and basically states that for every true formula A of second order arithmetic, possibly with parameters, there exists a countable coded ω model of the theory $\mathsf{ACA_0}$, containing these parameters so that A is true is this model. More formally we have for any $\mathcal{L}_2$-formula $A(U)$

$$A(U) \to (\exists X)(U \dot{\in} X \wedge F^X_{\mathsf{ACA}} \wedge A^X(U))$$

The ω model reflection considered in this work (RFN) is the general scheme of ω model reflection, as stated above, restricted to Π^1_1 formulas. Since if $A \in \Pi^1_1$ and $A(U)$ holds, then clearly also $A^V(U)$.

We will prove in the next section that the proof-theoretic ordinal of $\mathsf{RFN_0}$ is also $\varphi 20$ as it was announced by Jäger and Strahm in [4]. As a further result we have from Section 5 that $|\mathsf{RFN}| = |\mathsf{ACA}^+| = \varphi 2 \varepsilon_0$.

3 Equivalence of (RFN) and $(\omega - \mathsf{Jump})$ over ACA_0

The purpose of this section is to show the equivalence of two set existence axioms over a certain theory. Namely that (RFN) is equivalent to $(\omega - \mathsf{Jump})$ over the theory ACA_0.

3.1 RFN_0 proves the $(\omega - \mathsf{Jump})$ axiom

The following is provable in ACA_0.

Lemma 3.1.1 *The Turing jump hierarchy is unique, that is*

$$\mathcal{J}_\omega(U, V_1) \wedge \mathcal{J}_\omega(U, V_2) \rightarrow V_1 = V_2$$

PROOF: Given any set U and any two V_i, with $i \in \{1, 2\}$, such that we have $(V_i)_0 = U \wedge (\forall z)((V_i)_{z+1} = TJ((V_i)_z))$, we show with arithmetical induction (IND_0) on u that $V_1 = V_2$. In the base-case, where $u = 0$, we have $(V_1)_0 = (V_2)_0$. According to the definition of $(\omega - \mathsf{Jump})$ and hence $(V_1)_1 = (V_2)_1$ because the Turing jump is unique. In the case $u \rightarrow u+1$ we have by the induction hypothesis that $(V_1)_u = (V_2)_u$, which is $(\forall x)(x \in (V_1)_u \leftrightarrow x \in (V_2)_u)$. Again, since the Turing jump is unique, we have $(\forall x)(x \in TJ((V_1)_u) \leftrightarrow x \in TJ((V_2)_u))$, which is $(\forall x)(x \in (V_1)_{u+1} \leftrightarrow x \in (V_2)_{u+1})$. □

Compare also Simpson [10] Lemma V.2.3.

Theorem 3.1.2

$$\mathsf{ACA}_0 \vdash (\forall X)(\exists Z)(X \dot{\in} Z \wedge F^Z_{\mathsf{ACA}_0}) \rightarrow (\forall X)(\exists Y)\mathcal{J}_\omega(X, Y)$$

PROOF: Let $Hier(U, V, u)$ denote the $\mathcal{L}_2$-formula given by

$$Hier(U, V, u) := (V)_0 = U \wedge (\forall y < u)((V)_{y+1} = TJ((V)_y))$$

Further, in the following M will denote a set which is a model of ACA_0 and which comprises U. First we want to show that $(\forall x)(\exists Y \dot{\in} M) Hier(U, Y, x)$ holds. Notice that this is an arithmetical formula with set parameter M, and since M is a model of ACA_0 it is closed under arithmetical comprehension. We prove the claim by arithmetical induction (IND_0) on x.

$\underline{u = 0:}$

We have to show that $(\exists Y \dot{\in} M)((Y)_0 = U)$. We obtain a set V by arithmetical comprehension as follows,

$$(\forall y)(y \in V \leftrightarrow (\exists v)(y = \langle v, 0\rangle \wedge v \in U))$$

Note, this is clearly an element of M and hence we have $(\exists Y \dot{\in} M)((Y)_0 = U)$.

$\underline{u \to u+1:}$

We have to prove that

$$(\exists Y_N \dot{\in} M) Hier(U, Y_N, u+1)$$

holds under the induction hypothesis

$$(\exists Y_A \dot{\in} M) Hier(U, Y_A, u)$$

Now we construct the set Y_N by arithmetical comprehension that satisfies the claim.

$$Y_N = Y_A \cup \{\langle x, u+1\rangle : x \in TJ((Y_A)_u))\}$$

This set Y_N is clearly also an element of M, since M is a model of ACA_0 by supposition. We are done because we know by Lemma 3.1.1 that the Turing jump hierarchy is unique.

It remains to prove that there exists a set Z in ACA_0 such that

$$(\forall x) Hier(U, Z, x)$$

We construct this set Z by arithmetical comprehension as follows, where M is again the set from the supposition.

$$(\forall z)(z \in Z \leftrightarrow (\exists x)(\exists y)(\exists Y \dot{\in} M)(z = \langle x, y\rangle \wedge x \in (Y)_y \wedge Hier(U, Y, y)))$$

□

3.2 ACA_0^+ proves the (RFN) axiom

First we have to give some definitions and prove some general properties.

Definition 3.2.1 *For any two sets U and V, we say that V is many-one reducible to U ,denoted by $V \leq_m U$, if there exists a recursive function f such that* $\forall x(x \in V \leftrightarrow f(x) \in U)$

The following is provable in ACA_0.

Lemma 3.2.2 *For any sets U, V we have*

$$V \text{ is recursively enumerable in } U \quad \Longleftrightarrow \quad V \leq_m TJ(U)$$

PROOF: Suppose that V is recursively enumerable in U, then there exists an index $\tilde{g}$ of a partial recursive function g such that

$$b \in V \leftrightarrow (\exists z)\{\tilde{g}\}^U(b) = z \leftrightarrow \langle \tilde{g}, b\rangle \in TJ(U)$$

and thus $V \leq_m TJ(U)$. On the other hand, $TJ(U)$ is recursively enumerable in U by its definition and together with the supposition that V is many-one reducible to $TJ(U)$ we immediately obtain the claim. For more details cf. e.g. Hinman [3] $\square$

Definition 3.2.3 *For any two sets U and V, we say that V is recursive in U, written $V \leq_T U$, if there exists e_0, e_1 such that for all b the following holds.*

$$(b \in V \leftrightarrow F_\pi(e_1, b, U)) \wedge (b \notin V \leftrightarrow F_\pi(e_0, b, U))$$

In this case we say that $e = \langle e_0, e_1\rangle$ is the U-recursive index of V and we also often say that V is Turing reducible to U. Here $F_\pi(e, b, U)$ is a fixed universal lightface Π^0_1 formula as in definition 2.2.2.

Lemma 3.2.4 *The following are provable in ACA_0.*

1. *The relation $\leq_T$ is transitive, i.e. $U \leq_T V \wedge V \leq_T W \rightarrow U \leq_T W$.*
2. $U \leq_m V \rightarrow U \leq_T V$
3. $U \leq_T TJ(U)$
4. $\mathcal{J}_\omega(U, V) \wedge W \leq_T (V)_j \wedge j < i \rightarrow W \leq_T (V)_i$

PROOF: 1.) and 2.) cf. for example Simpson [10] Lemma VIII.1.2 and Rogers [8] Theorem 9.XII and Theorem 9.XIII.
3.)It is immediate from the definitions of $\leq_T$ and TJ.
4.)It is immediate from 3.) and 1.) $\square$

For more details concerning recursion theory see Rogers [8].

We shortly state here the well-known theory RCA_0, which will play an important role in the proof of this section.

The theory RCA_0 is the formal system in the language $\mathcal{L}_2$ whose axioms consist of the number-theoretic axioms plus the schemes of Σ^0_1 induction as well as $\triangle^0_1$ comprehension as stated below.

Definition 3.2.5 *For each $k < \omega$, the scheme of Σ^0_k induction consists of all axioms of the form*

$$A(0) \wedge (\forall z)(A(z) \to A(z+1)) \to (\forall z)A(z)$$

where $A(u)$ is any Σ^0_k formula of $\mathcal{L}_2$.

Notice that $\mathsf{ACA_0}$ proves the Σ^0_k induction scheme for all $k < \omega$, i.e. $\mathsf{ACA_0}$ proves all instances of arithmetical induction (cf. Simpson [10] Lemma IX.1.1)

Definition 3.2.6 *The scheme of $\triangle^0_1$ comprehension consists of all axioms of the form*

$$(\forall z)(A(z) \leftrightarrow B(z)) \to (\exists X)(\forall z)(z \in X \leftrightarrow A(z))$$

where $A(u)$ is a Σ^0_1 and $B(u)$ a Π^0_1 formula.

We have the following important Lemma.

Lemma 3.2.7 *Σ^0_1-comprehension is equivalent to arithmetical comprehension over $\mathsf{RCA_0}$.*

PROOF: cf. for example Simpson [10] Lemma III.1.3. □

Theorem 3.2.8

$$\mathsf{ACA_0} \vdash (\forall X)(\exists Y)\mathcal{J}_\omega(X, Y) \to (\forall X)(\exists Z)(X \dot{\in} Z \wedge F^Z_{\mathsf{ACA_0}})$$

PROOF: We have to show that under the supposition $\mathcal{J}_\omega(U, V)$ we can construct in $\mathsf{ACA_0}$ a set M which comprises U and models $\mathsf{ACA_0}$. We obtain this model M explicitly with arithmetical comprehension as follows.

$$\begin{aligned}(\forall z)(z \in M \leftrightarrow \quad & (\exists x_0)(\exists x_1)(\exists y_0)(\exists y_1)(F_\pi(x_1, y_0, (V)_{y_1}) \wedge \neg F_\pi(x_0, y_0, (V)_{y_1}) \\ & \wedge z = \langle y_0, \langle y_1, x_0, x_1 \rangle\rangle))\end{aligned}$$

This means that in M we collect all sets recursive in any $(V)_j$. Clearly $U \dot{\in} M$, i.e. $(\exists z)(U = (M)_z)$, since $(V)_0 = U$ and every set is recursive in itself and hence there exists $u = \langle v, e_0, e_1 \rangle$ with $U = (M)_u$.

What remains to show is that M is a model of $\mathsf{ACA_0}$, i.e. is closed under arithmetical comprehension. By Lemma 3.2.7 it is sufficient to prove that M is a model of $\mathsf{RCA_0}$ and is closed under Σ^0_1-comprehension. To show that M is a model of $\mathsf{RCA_0}$ we have to prove that M is closed under Δ^0_1-comprehension. Since Σ^0_1-comprehension comprises Δ^0_1-comprehension we are done, if we can prove the M is closed under

Σ^0_1-comprehension.

We claim that M is a model of Σ^0_1-comprehension. Define an arbitrary set $W \in \Sigma^0_1$ with parameters $\vec{U} = U_1, U_2, \ldots, U_n$ from M, which exists in $\mathsf{ACA_0}$ by arithmetical comprehension, with $U_1 \dot{\in} (M)_{i_1}, U_2 \dot{\in} (M)_{i_2}, \ldots, U_n \dot{\in} (M)_{i_n}$, that is

$$(\forall z)(z \in W \leftrightarrow A(z, \vec{u}, \vec{U}))$$

where $A \in \Sigma^0_1$. We have to show that $W \dot{\in} M$.
Notice that $U_1 \leq_T (V)_{i_1}, \ldots, U_n \leq_T (V)_{i_n}$ and by Lemma 3.2.4.4 we have $U_1 \leq_T (V)_i, \ldots, U_n \leq_T (V)_i$ with $i = max(i_1, \ldots, i_n)$. Therefore W is already Σ^0_1-definable in $(V)_i$. By Lemma 3.2.2 we conclude that $W \leq_m TJ((V)_i)$, that is $W \leq_m (V)_{i+1}$. By Lemma 3.2.4.2 we obtain $W \leq_T (V)_{i+1}$. Since W is recursive in $(V)_{i+1}$ we clearly have by the construction of M that $W \dot{\in} M$. □

Corollary 3.2.9 *The theory* $\mathsf{RFN_0}$ *has the proof-theoretic ordinal* $\varphi 20$.

Proof: This is an immediate consequence of Theorem 3.1.2 and Theorem 3.2.8, since $\mathsf{RFN_0}$ and $\mathsf{ACA_0^+}$ are equivalent and we know by Rathjen [7], Theorem 3.5 that $|\mathsf{ACA_0^+}| = \varphi 20$. □

Corollary 3.2.10 *The theories* $\mathsf{ACA^+}$ *and* RFN *are equivalent.*

In Section 5 we will determine the proof-theoretic ordinal of $\mathsf{ACA^+}$, respectively RFN.

4 Upper bound for proof strength of $\mathsf{ACA}_0 + (\mathsf{BR})$

4.1 Embedding of $\mathsf{ACA}_0 + (\mathsf{BR})$ into RFN_0

According to Rathjen [7] Theorem 3.5, ACA_0^+ proves the same Π^1_1 sentences as $\mathsf{ACA}_0 + (\mathsf{BR})$. We know from Section 3 that the theories ACA_0^+ and RFN_0 are equivalent, therefore it's an immediate consequence that RFN_0 also proves the same Π^1_1 sentences. In this section we will give a direct way of embedding $\mathsf{ACA}_0 + (\mathsf{BR})$ into RFN_0 and we will show that RFN_0 even proves all Π^1_2 sentences of $\mathsf{ACA}_0 + (\mathsf{BR})$. In the following $M, N, \ldots$ will denote sets of $\mathcal{L}_2$ which model ACA_0.

Lemma 4.1.1 *The following is provable in* RFN_0 *with* $\vec{V} = V_1, \ldots, V_n$.

$$(\exists Y)(\vec{V} \dot{\in} Y \wedge F^Y_{\mathsf{ACA}_0})$$

Proof: First we code the sets V_i of $\vec{V}$ into a single one and obtain by the (RFN) axiom a set from which we obtain again by arithmetical comprehension a set M, which clearly has the desired properties.

□

Lemma 4.1.2 *We have for all* $\mathcal{L}_2$*-formulas* $C[\vec{V}]$

$$\mathsf{ACA}_0 + (\mathsf{BR}) \vdash C[\vec{V}] \qquad \Rightarrow \qquad \mathsf{RFN}_0 \vdash F^M_{\mathsf{ACA}_0} \wedge \vec{V} \dot{\in} M \rightarrow C^M[\vec{V}]$$

Proof: We prove that by induction on the length n of the derivation in $\mathsf{ACA}_0 + (\mathsf{BR})$.

<u>$n = 0$:</u>

- If $C[\vec{V}]$ is a number theoretical axiom of $\mathsf{ACA}_0 + (\mathsf{BR})$ or a tautology, then obviously $C^M[\vec{V}]$ is an axiom of the same kind of RFN_0 and therefore $\mathsf{RFN}_0 \vdash F^M_{\mathsf{ACA}_0} \wedge \vec{V} \dot{\in} M \rightarrow C^M[\vec{V}]$.
- For the case that $C[\vec{V}] = A(t) \rightarrow (\exists x)A(x)$ or $C[\vec{V}] = (\forall x)A(x) \rightarrow A(t)$ it is easy to see that the claim holds.
- If $C[\vec{V}]$ is one of the remaining logical axioms of the Hilbert calculus, then $U \in \vec{V}$, and $C[\vec{V}] = A(U) \rightarrow (\exists X)A(X)$ or $C[\vec{V}] = (\forall X)A(X) \rightarrow A(U)$. Thus obviously $\mathsf{RFN}_0 \vdash F^M_{\mathsf{ACA}_0} \wedge U \dot{\in} M \rightarrow (A(U) \rightarrow (\exists X \dot{\in} M)A(X))$, and respectively $\mathsf{RFN}_0 \vdash F^M_{\mathsf{ACA}_0} \wedge U \dot{\in} M \rightarrow ((\forall X \dot{\in} M)A(X) \rightarrow A(U))$.
- If $C[\vec{V}]$ is of the form $(0 \in U \wedge (\forall z)(z \in U \rightarrow z+1 \in U)) \rightarrow (\forall z)(z \in U)$, where $U = \vec{V}$, then clearly $\mathsf{RFN}_0 \vdash F^M_{\mathsf{ACA}_0} \wedge U \dot{\in} M \rightarrow C^M[U]$.

- Let $C[\vec{V}]$ be an instance of the arithmetical comprehension scheme $(\exists X)(\forall z)(z \in X \leftrightarrow A(x, \vec{V}))$. Then $C^M[\vec{V}]$ is of the form

$$(\exists X \dot{\in} M)(\forall z)(z \in X \leftrightarrow A(z, \vec{V}))$$

  where $A(u, \vec{V})$ is an $\mathcal{L}_2$ formula which is arithmetical in $\vec{V}$. We reason in RFN_0 and have to show that $\mathsf{RFN}_0 \vdash F^M_{\mathsf{ACA}_0} \wedge \vec{V} \dot{\in} M \rightarrow (\exists x)(\forall z)(z \in (M)_x \leftrightarrow A(z, \vec{V}))$.
  By Lemma 4.1.1 we obtain a set M, which is a model of ACA_0 and comprises $\vec{V}$. Since models of ACA_0 are closed under arithmetical comprehension, the claim follows easily.

$\underline{n > 0}$:

- If the last inference was Modus Ponens, then $\mathsf{ACA}_0 + (\mathsf{BR}) \overset{n}{\vdash} C[\vec{V}]$, i.e. there exist $n_0 < n, n_1 < n$ such that $\mathsf{ACA}_0 + (\mathsf{BR}) \overset{n_0}{\vdash} A[\vec{U}, \vec{V}] \rightarrow C[\vec{V}]$ and $\mathsf{ACA}_0 + (\mathsf{BR}) \overset{n_1}{\vdash} A[\vec{U}, \vec{V}]$, where we assume the elements of $\vec{U}$ to be pairwise disjoint from the elements of $\vec{V}$. By the induction hypothesis, we have

$$\mathsf{RFN}_0 \vdash F^M_{\mathsf{ACA}_0} \wedge \vec{U}, \vec{V} \dot{\in} M \rightarrow (A^M[\vec{U}, \vec{V}] \rightarrow C^M[\vec{V}]) \quad (1)$$
$$\mathsf{RFN}_0 \vdash F^M_{\mathsf{ACA}_0} \wedge \vec{U} \dot{\in} M \rightarrow A^M[\vec{U}, \vec{V}] \quad (2)$$

  Hence we can infer from (1) and (2) that

$$\mathsf{RFN}_0 \vdash F^M_{\mathsf{ACA}_0} \wedge \vec{U}, \vec{V} \dot{\in} M \rightarrow C^M[\vec{V}]$$

  Because $\vec{U}$ does not occur in $C^M[\vec{V}]$ and surely the empty set ϕ is an element of every model of ACA_0, we set $\vec{U} = \phi$ and obtain

$$\mathsf{RFN}_0 \vdash F^M_{\mathsf{ACA}_0} \wedge \vec{V} \dot{\in} M \rightarrow C^M[\vec{V}]$$

- If the last inference was an existential or universal number quantification, then we have the induction hypothesis and apply just the same rule of inference again.

- If the last inference was an existential set quantification, then there exists $n_0 < n$ such that

$$\mathsf{ACA}_0 + (\mathsf{BR}) \overset{n_0}{\vdash} A[U, \vec{V}] \rightarrow B[\vec{V}]$$

  where U is different from all V_i in $\vec{V}$. Then $C[\vec{V}] = (\exists X)A[X, \vec{V}] \rightarrow B[\vec{V}]$. With the induction hypothesis we obtain

$$\mathsf{RFN}_0 \vdash F^M_{\mathsf{ACA}_0} \wedge U, \vec{V} \dot{\in} M \rightarrow (A^M[U, \vec{V}] \rightarrow B^M[\vec{V}])$$

By means of propositional logic this is equivalent to

$$\mathsf{RFN_0} \vdash A^M[U, \vec{V}] \land U \dot{\in} M \to ((F^M_{\mathsf{ACA_0}} \land \vec{V} \dot{\in} M) \to B^M[\vec{V}])$$

By applying the existential set quantification inference we obtain

$$\mathsf{RFN_0} \vdash (\exists X)(A^M[X, \vec{V}] \land X \dot{\in} M) \to ((F^M_{\mathsf{ACA_0}} \land \vec{V} \dot{\in} M) \to B^M[\vec{V}])$$

and again by means of propositional logic we finally obtain

$$\mathsf{RFN_0} \vdash F^M_{\mathsf{ACA_0}} \land \vec{V} \dot{\in} M \to ((\exists X \dot{\in} M) A^M[X, \vec{V}] \to B^M[\vec{V}])$$

- For the case that the last inference was an universal set quantification the proof is similar.
- If the last inference was the bar rule (BR) then

$$\mathsf{ACA_0} + (\mathsf{BR}) \overset{n}{\vdash} \mathsf{TI}(\prec, F[\vec{V}]) \qquad \text{for an } F \in \mathcal{L}_2$$

Thus there exists $n_0 < n$ and

$$\mathsf{ACA_0} + (\mathsf{BR}) \overset{n_0}{\vdash} \mathsf{WO}(\prec)$$

from the definition of WO we have

$$\mathsf{ACA_0} + (\mathsf{BR}) \overset{n_0}{\vdash} (\forall X)\mathsf{TI}(\prec, X) \land \mathsf{LO}(\prec)$$

Since $(\forall X)\mathsf{TI}(\prec, X) \land \mathsf{LO}(\prec)$ contains no free set variables we obtain with the induction hypothesis that the following holds in $\mathsf{RFN_0}$

$$F^M_{\mathsf{ACA_0}} \to (\forall X \dot{\in} M)\mathsf{TI}(\prec, X) \land \mathsf{LO}(\prec) \tag{3}$$

and we have to show that if $F[\vec{V}]$ is an $\mathcal{L}_2$-formula and N is a model of $\mathsf{ACA_0}$, that comprises $\vec{V}$, the following holds in $\mathsf{RFN_0}$

$$\mathsf{TI}(\prec, F^N[\vec{V}]) \land \mathsf{LO}(\prec)$$

First notice that the formula $F^N[\vec{V}]$ is arithmetical. Therefore we have by arithmetical comprehension a set Z,

$$Z = \{x : F^N(x, \vec{V})\} \tag{4}$$

Now by the (RFN) axiom we obtain a set O, which models $\mathsf{ACA_0}$ and contains Z. Together with (3) we can conclude that

$$(\forall X \dot{\in} O)\mathsf{TI}(\prec, X) \land \mathsf{LO}(\prec)$$

and since O comprises the set Z, we clearly have $\mathsf{TI}(\prec, Z) \wedge \mathsf{LO}(\prec)$ and therefore obtain with (4)

$$\mathsf{TI}(\prec, F^M[\vec{V}]) \wedge \mathsf{LO}(\prec)$$

□

Theorem 4.1.3 *Let C be a Π^1_2 formula of $\mathcal{L}_2$ such that $\mathsf{ACA_0} + (\mathsf{BR}) \vdash C[\vec{V}]$, then also $\mathsf{RFN_0} \vdash C[\vec{V}]$.*

Proof: $C[\vec{V}]$ is of the form $(\forall Y)(\exists Z)B[\vec{V}, Y, Z]$ with $B \in \Pi^0_\infty$. From Lemma 4.1.2, we obtain

$$\mathsf{RFN_0} \vdash F^M_{\mathsf{ACA_0}} \wedge \vec{V} \in M \to (\forall y)(\exists z)B[\vec{V}, (M)_y, (M)_z]$$

and therefore we can infer that

$$\mathsf{RFN_0} \vdash F^M_{\mathsf{ACA_0}} \wedge \vec{V} \in M \to (\exists z)B[\vec{V}, (M)_u, (M)_z]$$

Since the class of the models of $\mathsf{ACA_0}$, which contain $\vec{V}$, clearly is a super class of the models of $\mathsf{ACA_0}$, which contain $\vec{V}$ and any U, we can conclude that

$$\mathsf{RFN_0} \vdash F^M_{\mathsf{ACA_0}} \wedge \vec{V}, U \in M \to (\exists z)B[\vec{V}, U, (M)_z]$$

but then it immediately follows that

$$\mathsf{RFN_0} \vdash F^M_{\mathsf{ACA_0}} \wedge \vec{V}, U \in M \to (\exists Z)B[\vec{V}, U, Z]$$

and obtain by a existential set quantification

$$\mathsf{RFN_0} \vdash (\exists Y)(F^Y_{\mathsf{ACA_0}} \wedge \vec{V}, U \in Y) \to (\exists Z)B[\vec{V}, U, Z]$$

By (RFN) we have

$$\mathsf{RFN_0} \vdash \exists Y(\vec{V}, U \dot{\in} Y \wedge F^Y_{\mathsf{ACA_0}})$$

and hence we finally obtain

$$\mathsf{RFN_0} \vdash (\exists Z)B[\vec{V}, U, Z]$$

which is in fact

$$\mathsf{RFN_0} \vdash (\forall Y)(\exists Z)B[\vec{V}, Y, Z]$$

□

Corollary 4.1.4 *The proof-theoretic ordinal of $\mathsf{ACA_0} + (\mathsf{BR})$ is $\leq \varphi 20$.*

Proof: By Theorem 4.1.3 we know that any Π^1_2 sentence provable in $\mathsf{ACA_0} + (\mathsf{BR})$ is also provable in $\mathsf{RFN_0}$. By Corollary 3.2.9 that $|\mathsf{RFN_0}| = \varphi 20$ and together with the fact that $(\forall X)\mathsf{TI}(\prec, X) \in \Pi^1_1$, the result follows easily. □

In Section 6.1 we will see that Corollary 4.1.4 in fact determines the least upper bound for the proof-strength of $\mathsf{ACA_0} + (\mathsf{BR})$.

5 The proof-theoretic strength of ACA^+ and RFN

5.1 The wellordering proof of RFN_0 and RFN

In this section we will analyze the proof-theoretic strength of RFN. Since we know from Section 3 that RFN and ACA^+ are equivalent, the result therefore also holds for RFN. First we will determine a lower bound for the proof-theoretic ordinal by showing that any ordinal $\alpha < \varphi 2\varepsilon_0$ is provable in RFN. To show that this bound is strict, we embed ACA^+ into the semi-formal system RA^* of Schütte [9].

Small Greek letter $\alpha, \beta, \gamma, \delta, \ldots$, possibly with subscripts, will denote ordinals $< \Gamma_0$ We fix a primitive recursive standard wellordering $\prec$ of order-type Γ_0. Without loss of generality we may assume that the field of $\prec$ is the set of all natural numbers (and that 0 is the least element with respect to $\prec$). Hence each natural number a codes an ordinal, say $ord(a)$, less then Γ_0, and each ordinal $\alpha < \Gamma_0$ is represented in our theories by an unique natural number, say $nr(\alpha)$ (we will denote this natural number often as $\overline{\alpha}$). The reader is assumed to be familiar with the Veblen functions φ_α, cf. Pohlers [6] or Schütte [9].

Further, we inductively define $\omega_n(\alpha)$ and $\varepsilon_n(\alpha)$.

Definition 5.1.1 *For all ordinals α and natural numbers n, we define*

$$\omega_0(\alpha) := \alpha \qquad \varepsilon_0(\alpha) := \alpha$$
$$\omega_{n+1}(\alpha) := \omega^{\omega_n(\alpha)} \qquad \varepsilon_{n+1}(\alpha) := \varepsilon_{\varepsilon_n(\alpha)}$$

Moreover, there exist binary primitive recursive functions $\hat{+}, \hat{\omega}, \tilde{\omega}, \hat{\varepsilon}, \tilde{\varepsilon}, \hat{\varphi}$, which model the usual ordinal operations on these codes, i.e. for m and n natural numbers we have :

1. $\hat{+}(m,n) := nr(ord(m) + ord(n))$
2. $\hat{\omega}(m,n) := nr(\omega^{ord(m)} \cdot n)$
3. $\tilde{\omega}(m,n) := nr(\omega_m(ord(n)))$
4. $\tilde{\varepsilon}(m,n) := nr(\varepsilon_m(ord(n)))$
5. $\hat{\varphi}(m,n) := nr(\varphi(ord(m))(ord(n)))$

We use $\hat{+}, \hat{\omega}, \tilde{\omega}, \tilde{\varepsilon}, \hat{\varphi}$ as primitive symbols of our formal language; and in order to keep notation as simple as possible, we write:

$$\begin{array}{rll} a\hat{+}b & \text{for} & \hat{+}(a,b), \\ \hat{\omega}^a b & \text{for} & \hat{\omega}(a,b), \\ \tilde{\omega}_a(b) & \text{for} & \tilde{\omega}(a,b), \\ \hat{\varphi}ab & \text{for} & \hat{\varphi}(a,b), \end{array} \qquad \begin{array}{rll} \tilde{\varepsilon}_a(b) & \text{for} & \tilde{\varepsilon}(a,b), \\ \hat{\omega}^a & \text{for} & \hat{\omega}(a,1), \\ \hat{\varepsilon}_a & \text{for} & \hat{\varphi}(\overline{1},a), \end{array}$$

In the sequel we also often write $\mathsf{PROG}(U)$ and $\mathsf{TI}(a,U)$ instead of $\mathsf{PROG}(\prec, U)$ and $\mathsf{TI}(\prec, a, U)$.

Further we make the following definitions.

Definition 5.1.2

$$\mathcal{I}(a) := (\forall X)\mathsf{TI}(\prec, a, X)$$

Definition 5.1.3 *For any given set U, we define the set $Sp(U)$, if this set exists, as follows:*

$$Sp(U) := \{a : (\forall y)(y \subset U \to y\hat{+}\hat{\omega}^a\} \subset U)\}$$

where $a \subset U$ abbreviates $(\forall x)(x \prec a \to x \in U)$.

Note that the existence of these sets become provable in a theory comprising $\mathsf{ACA_0}$. This jump operator Sp enables us to jump from α to ω^α.

Lemma 5.1.4

$$\mathsf{ACA_0} \vdash \mathcal{I}(a) \wedge \mathcal{I}(b) \to \mathcal{I}(a\hat{+}b)$$

Lemma 5.1.5

$$\mathsf{ACA_0} \vdash \mathcal{I}(a) \wedge b \prec a \to \mathcal{I}(b)$$

Lemma 5.1.6

$$\mathsf{ACA_0} \vdash (\forall x \prec a)\mathcal{I}(x) \to \mathcal{I}(a)$$

Lemma 5.1.7

$$\mathsf{ACA_0} \vdash \mathsf{PROG}(\prec, V) \to \mathsf{PROG}(\prec, Sp(V))$$

Lemma 5.1.8

$$\mathsf{ACA_0} \vdash \mathsf{TI}(a, Sp(V)) \to \mathsf{TI}(\hat{\omega}^a, V)$$

Proof: For the proofs of these Lemmata compare Schütte [9] Lemma VIII.21.1, Lemma VIII.21.7,($\mathcal{J}4$) and Pohlers [6] Lemma 15.5,Lemma 15.6. □

Lemma 5.1.9 *We have that* $\mathsf{RFN_0}$ *proves the following*

$$\mathcal{I}(a) \to \mathcal{I}(\hat{\varepsilon}_a)$$

Proof: We have the supposition

$$(\forall X)\mathsf{TI}(a, X) \quad (5)$$

Now fix an arbitrary U and show that $\mathsf{TI}(\hat{\varepsilon}_a, U)$ holds. Because of the (RFN) axiom we have that for this fixed set U there exists a set W such that $U \dot{\in} W$ and F^W_{ACA}. Further define a set V by arithmetical comprehension

$$(\forall z)(z \in V \leftrightarrow \mathcal{I}^W(\hat{\varepsilon}_z))$$

where $\mathcal{I}^W$ is the relativization of $\mathcal{I}$ to the set W, that is $\mathcal{I}^W(\hat{\varepsilon}_u) = (\forall X \dot{\in} W)\mathsf{TI}(\hat{\varepsilon}_u, X)$. Under the supposition that

$$\mathsf{PROG}(\prec, V) \quad (6)$$

we have by (5)

$$(\forall x \prec a)(x \in V) \quad (7)$$

But since the following holds in $\mathsf{RFN_0}$

$$\mathsf{PROG}(V) \wedge (\forall x \prec a)(x \in V) \to a \in V$$

we obtain by the supposition and (7) that $(a \in V)$. That is

$$(\forall X \dot{\in} W)\mathsf{TI}(\hat{\varepsilon}_a, X)$$

Since $U \dot{\in} W$ we have

$$\mathsf{TI}(\hat{\varepsilon}_a, U)$$

What remains to be proved is that this set V, from supposition (6), is progressive. In order to establish $\mathsf{PROG}(\prec, V)$ it is equivalent to prove the following:

1. $\overline{0} \in V$
2. $b \in V \to b \hat{+} \overline{1} \in V$

3. $Lim(b) \wedge (\forall x \prec b)(x \in V) \rightarrow b \in V$

where $Lim(b)$ indicates that the natural number b codes a limit ordinal.

1. $\underline{b = 0}$:

   The proof is completely analogous as where we prove (9), but in this case we prove the following formula in $\mathsf{RFN_0}$ with induction on z

   $$(\forall z)(\forall X \dot{\in} W)\mathsf{TI}(\tilde{\omega}_z(\overline{0}), X)$$

   because if $c \prec \hat{\varepsilon}_{\overline{0}}$, then there already exists a natural number u such that $c \prec \tilde{\omega}_u(\overline{0})$, and therefore by Lemma 5.1.5 and Lemma 5.1.6, it is sufficient to prove the above.

2. $\underline{b \text{ codes a successor ordinal:}}$

   We have the supposition $b \in V$, that is

   $$(\forall X \dot{\in} W)\mathsf{TI}(\hat{\varepsilon}_b, X) \tag{8}$$

   and have to show that $(b \hat{+} \overline{1} \in V)$ also holds, which is equivalent to

   $$(\forall X \dot{\in} W)\mathsf{TI}(\hat{\varepsilon}_{(b \hat{+} \overline{1})}, X)$$

   because, if $c \prec \hat{\varepsilon}_{(t \hat{+} \overline{1})}$, then there already exists a natural number u such that $c \prec \tilde{\omega}_u(\hat{\varepsilon}_t \hat{+} \overline{1})$, we only have to show by Lemma 5.1.5 and Lemma 5.1.6 that

   $$(\forall z)(\forall X \dot{\in} W)\mathsf{TI}(\tilde{\omega}_z(\hat{\varepsilon}_b \hat{+} \overline{1}), X) \tag{9}$$

   We prove this by arithmetical induction ($\mathsf{IND_0}$) on z.

   - $\underline{u = 0:}$

     We clearly have that $\mathsf{RFN_0} \vdash \mathcal{I}(\overline{1})$ and together with the supposition (8) and Lemma 5.1.4 we obtain

     $$(\forall X \dot{\in} W)\mathsf{TI}(\hat{\varepsilon}_b \hat{+} \overline{1}, X)$$

- $\underline{u \to u+1:}$

  We have the induction hypothesis

  $$(\forall X \dot{\in} W)\mathsf{TI}(\tilde{\omega}_u(\hat{\varepsilon}_b \hat{+} \bar{1}), X)$$

  and want to prove that

  $$(\forall X \dot{\in} W)\mathsf{TI}(\tilde{\omega}_{u+1}(\hat{\varepsilon}_b \hat{+} \bar{1}), X)$$

  First notice that $Sp(U)$ is an arithmetical formula with set parameter U. Choose an arbitrary set $U \dot{\in} W$. Since W is a model of $\mathsf{ACA_0}$, it is closed under arithmetical comprehension and therefore the set defined by $Sp(U)$ is also an element of W (i.e. $Sp(U) \dot{\in} W$). Hence by the induction hypothesis we have

  $$\mathsf{TI}(\tilde{\omega}_u(\hat{\varepsilon}_b \hat{+} \bar{1}), Sp(U))$$

  Together with Lemma 5.1.8 we obtain

  $$\mathsf{TI}(\hat{\omega}^{\tilde{\omega}_u(\hat{\varepsilon}_b \hat{+} \bar{1})}, U)$$

  Since U was an arbitrary set of W and $\mathsf{RFN_0}$ proves that $\hat{\omega}^{\tilde{\omega}_u(\hat{\varepsilon}_b \hat{+} \bar{1})} = \tilde{\omega}_{u+1}(\hat{\varepsilon}_b \hat{+} \bar{1})$ we finally obtain

  $$(\forall X \dot{\in} W)\mathsf{TI}(\tilde{\omega}_{u+1}(\hat{\varepsilon}_b \hat{+} \bar{1}), X)$$

  completing the induction step.

3. <u>b codes a limit ordinal:</u>

   We have the presupposition $(\forall x \prec b)(x \in V)$, that is

   $$(\forall x \prec b)((\forall X \dot{\in} W)(\mathsf{PROG}(X) \to (\forall y \prec \hat{\varepsilon}_x)(y \in X)))$$

   and want to show $(b \in V)$, which is

   $$(\forall X \dot{\in} W)(\mathsf{PROG}(X) \to (\forall y \prec \hat{\varepsilon}_b)(y \in X))$$

   We suppose that $\mathsf{PROG}(U)$ and have to show that $(\forall y \prec \hat{\varepsilon}_b)(y \in U)$. Since b codes a limit ordinal, we have for all $y \prec \hat{\varepsilon}_b$ there exists a $b_0 \prec b$ such that already $y \prec \hat{\varepsilon}_{b_0}$ and by the supposition we know therefore that $y \in U$.

□

With this Lemma 5.1.9 we immediately obtain the following Theorem.

Theorem 5.1.10 $\mathsf{RFN_0}$ *proves the formula* $\mathcal{I}(\overline{\alpha})$ *for all* $\alpha < \varphi 20$.

PROOF: If $\alpha < \varphi 20$, then there is an $n < \omega$ such that $\alpha < \varepsilon_n(0)$. Since $\mathsf{RFN_0} \vdash (\forall x)(\neg x \prec 0)$, we clearly have

$$\mathsf{RFN_0} \vdash \mathcal{I}(0)$$

n-fold application of Lemma 5.1.9 leads to

$$\mathsf{RFN_0} \vdash \mathcal{I}(\tilde{\varepsilon}_{\overline{n}}(0))$$

which is

$$\mathsf{RFN_0} \vdash (\forall X)(\mathsf{PROG}(\prec, X) \to (\forall x \prec \tilde{\varepsilon}_{\overline{n}}(0))(x \in X)$$

Together with Lemma 5.1.5 this implies

$$\mathsf{RFN_0} \vdash (\forall X)(\mathsf{PROG}(\prec, X) \to (\forall x \prec \overline{\alpha})(x \in X))$$

which is in fact

$$\mathsf{RFN_0} \vdash \mathcal{I}(\overline{\alpha})$$

□

Corollary 5.1.11 *For a lower bound of the proof-theoretic ordinal of* $\mathsf{RFN_0}$ *we have*

$$|\mathsf{RFN_0}| \geq \varphi 20$$

If the theory $\mathsf{RFN_0}$ comprises in addition the full second order induction scheme (IND), we get another lower bound for the proof-theoretic ordinal of this theory RFN.First we want to show that

Lemma 5.1.12 RFN *proves the following*

$$\mathcal{I}(a) \to (\forall x)\mathcal{I}(\tilde{\varepsilon}_x(a))$$

PROOF: Assume that $\mathcal{I}(a)$ holds. We prove $(\forall x)\mathcal{I}(\tilde{\varepsilon}_x(a))$ also holds by induction (IND) on x.

$\underline{u = 0:}$

Clearly

$$\mathsf{RFN} \vdash \mathcal{I}(a) \rightarrow \mathcal{I}(a)$$

and hence it holds for $u = 0$.

$\underline{u \rightarrow u + 1:}$

By induction hypothesis we have

$$\mathcal{I}(\tilde{\varepsilon}_u(a))$$

and obtain by Lemma 5.1.9 that

$$\mathcal{I}(\hat{\varepsilon}_{\tilde{\varepsilon}_u(a)})$$

which is

$$\mathcal{I}(\tilde{\varepsilon}_{u+1}(a))$$

since RFN_0 proves $\hat{\varepsilon}_{\tilde{\varepsilon}_u(a)} = \tilde{\varepsilon}_{u+1}(a)$.

□

Lemma 5.1.13 *We have that* RFN *proves*

$$\mathsf{PROG}(\mathcal{I}(\hat{\varphi}\overline{2}a))$$

PROOF: Again in order to establish $\mathsf{PROG}(\mathcal{I}(\hat{\varphi}\overline{2}a))$ it is equivalent to prove the following.

1. $\mathcal{I}(\hat{\varphi}\overline{2}0)$
2. $\mathcal{I}(\hat{\varphi}\overline{2}a) \rightarrow \mathcal{I}(\hat{\varphi}\overline{2}(a + 1))$
3. $Lim(a) \wedge (\forall x \prec a)\mathcal{I}(\hat{\varphi}\overline{2}x) \rightarrow \mathcal{I}(\hat{\varphi}\overline{2}a)$

1. $\underline{a = 0}$:

   Clearly

   $$\mathsf{RFN} \vdash \mathcal{I}(\overline{0})$$

   Together with Lemma 5.1.12 we get

   $$\mathsf{RFN} \vdash (\forall z)\mathcal{I}(\tilde{\varepsilon}_z(\overline{0}))$$

and if $c \prec \hat{\varphi}\overline{20}$ then there exists a natural number u such that $c \prec \hat{\varepsilon}_u(\overline{0})$.
Using Lemma 5.1.5 and Lemma 5.1.6 we obtain

$$\mathsf{RFN} \vdash \mathcal{I}(\hat{\varphi}\overline{20})$$

2. $\underline{a \text{ codes a successor ordinal:}}$

   We have the supposition

   $$\mathcal{I}(\hat{\varphi}\overline{2}a)$$

   and have to show that

   $$\mathcal{I}(\hat{\varphi}\overline{2}(a\hat{+}\overline{1}))$$

   also holds. Clearly

   $$\mathsf{RFN} \vdash \mathcal{I}(\overline{1}) \tag{10}$$

   and with Lemma 5.1.4 we obtain from (10) and the supposition

   $$\mathcal{I}(\hat{\varphi}\overline{2}a\hat{+}\overline{1}). \tag{11}$$

   From (11) and Lemma 5.1.12 we obtain

   $$(\forall z)\mathcal{I}(\hat{\varepsilon}_z(\hat{\varphi}\overline{2}a\hat{+}\overline{1})).$$

   Further, we have that if $c \prec \hat{\varphi}\overline{2}(a\hat{+}\overline{1})$ then there exists a natural number u such that $c \prec \tilde{\varepsilon}_u(\hat{\varphi}\overline{2}a\hat{+}\overline{1})$.
   Using Lemma 5.1.5 and Lemma 5.1.6 we obtain

   $$\mathcal{I}(\hat{\varphi}\overline{2}(a\hat{+}\overline{1}))$$

3. $\underline{a \text{ codes a limit ordinal:}}$

   We have the supposition $(\forall x \prec a)\mathcal{I}(\hat{\varphi}\overline{2}a)$, that is

   $$(\forall x \prec a)((\forall X)(\mathsf{PROG}(X) \rightarrow (\forall y \prec \hat{\varphi}\overline{2}x)(y \in X)))$$

   and want to show $\mathcal{I}(\hat{\varphi}\overline{2}a)$, which is

   $$(\forall X)(\mathsf{PROG}(X) \rightarrow (\forall y \prec \hat{\varphi}\overline{2}a)(y \in X))$$

   We suppose $\mathsf{PROG}(U)$ and we need to show that $(\forall y \prec \hat{\varphi}\overline{2}a)(y \in U)$. Since a codes a limit ordinal, we have for all $y \prec \hat{\varphi}\overline{2}a$ there exists an $a_0 \prec a$ such that $y \prec \hat{\varphi}\overline{2}a_0$ and by the supposition we know therefore that $y \in U$.

□

Theorem 5.1.14 *If* $\alpha < \varphi 2\varepsilon_0$*, then* $\mathsf{RFN} \vdash \mathcal{I}(\overline{\alpha})$

Proof: Letting $A(t) := \mathcal{I}(\hat{\varphi}\overline{2}t)$ we have by Lemma 5.1.13

$$\mathsf{RFN} \vdash (\forall x)(\forall y \prec x A(y) \to A(x))$$

Since we know by Schütte [9] Lemma VIII.21.6 that

$$\mathsf{ACA} \vdash (\forall x)(\forall y \prec x A(y) \to A(x)) \to (\forall x \prec \overline{\beta})A(x)$$

holds for every ordinal $\beta < \varepsilon_0$

$$\mathsf{RFN} \vdash A(\overline{\beta})$$

follows for all $\beta < \varepsilon_0$. If $\alpha < \varphi 2\varepsilon_0$, there exists $\beta < \varepsilon_0$ such that $\alpha < \varphi 2\beta$. Hence by Lemma 5.1.5 we obtain

$$\mathsf{RFN} \vdash \mathcal{I}(\overline{\alpha})$$

for all $\alpha < \varphi 2\varepsilon_0$. □

Corollary 5.1.15 $|\mathsf{RFN}| \geq \varphi 2\varepsilon_0$ *and* $|\mathsf{ACA}^+| \geq \varphi 2\varepsilon_0$

Proof: It is immediate from Theorem 5.1.14 and Lemma 3.2.10. □

In the next section we will see that Corollary 5.1.15 determines in fact the greatest lower bound for the proof-strength of RFN, respectively ACA^+.

5.2 An upper bound for the proof strength of ACA^+

The purpose of the next subsection is to introduce the semi-formal system RA^* (compare Schütte [9] for more details).

5.2.1 The semi-formal system RA^*

In the language $\mathcal{L}^*$ of RA^* we have countably infinitely many bound number variables $(x, y, z, a, b, c \ldots)$, as well as countably infinitely many free set variables of level α for each ordinal α $(U^\alpha, V^\alpha, W^\alpha, \ldots)$ and countably infinitely many bound set variables of level β for each ordinal $\beta \neq 0$ $(X^\beta, Y^\beta, Z^\beta, \ldots)$. Further, $\mathcal{L}^*$ comprises the same function and relation symbols as $\mathcal{L}_2$. That means there is a symbol for each n-ary primitive recursive function and n-ary primitive recursive relation. Number terms of $\mathcal{L}^*$ $(r, s, t, \ldots)$ are exactly the closed number terms of $\mathcal{L}_2$. The numerals of $\mathcal{L}^*$ are inductively given by $\overline{0} := 0$ and $\overline{n+1} := \overline{n} + 1$. The set terms of $\mathcal{L}^*$ (S, T, R) are defined simultaneously with the formulas of $\mathcal{L}^*$ $(A, B, F, \ldots)$:

Definition 5.2.1

1. *U^α is a set term.*
2. *If F is an $\mathcal{L}^*$-formula, then $\{x : F\}$ is a set term.*
3. *$R(t_1, \ldots, t_n)$ is an $\mathcal{L}^*$-formula for n-ary primitive recursive relation symbols R and number terms $t_1, \ldots, t_n$.*
4. *$(t \in S), (t \notin S)$ are $\mathcal{L}^*$-formulas for number terms t and set terms S.*
5. *Formulas are closed under $\vee, \wedge, (\exists x), (\forall x), (\exists X^\alpha), (\forall X^\alpha)$ for $\alpha \neq 0$.*

The negation $\neg F$ of an $\mathcal{L}^*$-formula F is defined as usual by applying the de Morgan's rules. We say two formulas F and G are numerically equivalent, if they differ only in (closed) number terms which have the same value. Further we use the same abbreviations as in $\mathcal{L}_2$, that is e.g.: $S = T := (\forall x)(x \in S \leftrightarrow x \in T)$ and $(\forall x < r)A(x) := (\forall x)(x < r \rightarrow A(x))$.

The level of a set term S and a formula F of $\mathcal{L}^*$ is defined by

Definition 5.2.2

$$lev(S) = max\{\alpha : a\ set\ variable\ X^\alpha\ or\ U^\alpha\ occurs\ in\ S\}$$

$$lev(F) = max\{\alpha : a\ set\ variable\ X^\alpha\ or\ U^\alpha\ occurs\ in\ F\}$$

otherwise the level of S, or of F respectively, is 0.

Now we define inductively the rank $rk(A)$ for an $\mathcal{L}^*$-formulas A:

Definition 5.2.3 *1. A is of the form $R(t_1, \ldots, t_n)$, then $rk(A) = 0$.*

2. *A is of the form $(F \wedge G)$ or $(F \vee G)$, then $rk(A) = max\{rk(F), rk(G)\} + 1$.*

3. *A is of the form $(\exists x)F(x)$ or $(\forall x)F(x)$, then $rk(A) = rk(F(\overline{0})) + 1$.*

4. *A is of the form $(t \in U^\alpha)$ or $(t \notin U^\alpha)$, then $rk(A) = \omega\alpha$.*

5. *A is of the form $(t \in \{x : F(x)\})$ or $(t \notin \{x : F(x)\})$, then $rk(A) = rk(F(t)) + 1$.*

6. *A is of the form $(\exists X^\alpha)F(X^\alpha)$ or $(\forall X^\alpha)F(X^\alpha)$, then $rk(A) = max\{\omega \cdot lev((\forall X^\alpha)F(X^\alpha)), rk(F(U^0)) + 1\}$.*

Notice that $rk(F) = rk(\neg F)$. We make the following observations:

- If $lev(F) = \alpha$, then $\omega\alpha \leq rk(F) < \omega(\alpha + 1)$.
- If $lev(S) < \alpha$, then $rk(F(S)) < rk((\exists X^\alpha)F(X^\alpha))$.

Derivations in RA^* are denoted in a Tait-style manner: $\Gamma, \triangle, \ldots$ denote a finite set of $\mathcal{L}^*$-formulas.
The axioms of RA^* are given as follows:

I. Number-theoretic Axiom:

$$\Gamma, R(t_1, \ldots, t_n) \qquad (Ax1)$$

if R is primitive recursive relation symbol and $R(t_1, \ldots, t_n)$ is true.

II. Equality Axiom for set variables:

$$\Gamma, s \in U^\alpha, t \notin U^\alpha \qquad (Ax2)$$

for arbitrary set variables of level α and s,t (closed) number terms which have the same value.

The rules of RA^* are divided into four groups:

III. Logical Rules:

$$\frac{\Gamma, A}{\Gamma, A \vee B} \ (\vee 1), \qquad \frac{\Gamma, B}{\Gamma, A \vee B} \ (\vee 2), \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B} \ (\wedge).$$

IV. Set Term Rules:

$$\frac{\Gamma, F(t)}{\Gamma, t \in \{x : F(x)\}} \ (\in 1), \qquad \frac{\Gamma, \neg F(t)}{\Gamma, t \notin \{x : F(x)\}} \ (\in 2)$$

V. Quantifier Rules:

$$\frac{\Gamma, F(s)}{\Gamma, (\exists x)F(x)} \ (\exists^0), \qquad \frac{\ldots, \Gamma, F(s), \ldots \ for \ all \ number \ terms \ s}{\Gamma, (\forall x)F(x)} \ (\forall^0)$$

$$\frac{\Gamma, F(S) \qquad lev(S) < \alpha}{\Gamma, (\exists X^\alpha)F(X^\alpha)} \ (\exists^1), \qquad \frac{\ldots, \Gamma, F(S), \ldots \ for \ all \ S, \ lev(S) < \alpha}{\Gamma, (\forall X^\alpha)F(X^\alpha)} \ (\forall^1)$$

VI. Cut Rule:

$$\frac{\Gamma, F \qquad \Gamma, \neg F}{\Gamma} \ (cut)$$

Definition 5.2.4 *Inductive definition of* $\mathsf{RA}^* \mid\!\frac{\alpha}{\rho}\ F$

1. If F is an axiom of the system RA^* then $\mathsf{RA}^* \mid\!\frac{\alpha}{\rho}\ F$ holds for all ordinals α and ρ.
2. If $\mathsf{RA}^* \mid\!\frac{\alpha_i}{\rho}\ F_i$ and $\alpha_i < \alpha$ for every premise F_i of an inference or a cut of rank $< \rho$ then $\mathsf{RA}^* \mid\!\frac{\alpha}{\rho}\ F$ holds for the conclusion F of that inference.

As an immediate consequence we obtain the following Lemmata by an easy transfinite induction on the length α of the derivation.

Corollary 5.2.5

If $\mathsf{RA}^* \mid\!\frac{\alpha}{\rho}\ \Gamma \quad, \alpha \leq \beta$ *and* $\rho \leq \sigma$, *then* $\mathsf{RA}^* \mid\!\frac{\beta}{\sigma}\ \Gamma$ *also holds.*

Corollary 5.2.6

If $\mathsf{RA}^* \mid\!\frac{\alpha}{\rho}\ \triangle$ *and* $\triangle \subset \Gamma$, *then* $\mathsf{RA}^* \mid\!\frac{\alpha}{\rho}\ \Gamma$.

A formula F is said to be deducible in RA^* with order α and rank ρ if $\mathsf{RA}^* \mid\!\frac{\alpha}{\rho}\ F$ holds. Thus α is an upper bound for the orders of the inference which occur in the deduction of F while ρ says that every cut which occurs in the deduction of F has rank $< \rho$. If $\mathsf{RA}^* \mid\!\frac{\alpha}{0}\ F$ holds then the formula F has a cut-free deduction with order α.

Further we also introduce the following notation. We write $\mathsf{RA}^* \mathrel{\vdash\!\frac{<\alpha}{\rho}} \Gamma$ if there exists an $\alpha_0 < \alpha$ such that $\mathsf{RA}^* \mathrel{\vdash\!\frac{\alpha_0}{\rho}} \Gamma$, and analogously $\mathsf{RA}^* \mathrel{\vdash\!\frac{\alpha}{<\rho}} \Gamma$ if there exists a $\rho_0 < \rho$ such that $\mathsf{RA}^* \mathrel{\vdash\!\frac{\alpha}{\rho_0}} \Gamma$.

In addition we have the following Lemmata for all ordinals α and ρ:

Lemma 5.2.7

$$\mathsf{RA}^* \mathrel{\vdash\!\frac{\alpha}{\rho}} A(\overline{n}) \qquad \Longleftrightarrow \qquad \mathsf{RA}^* \mathrel{\vdash\!\frac{\alpha}{\rho}} A(t) \qquad \textit{for all number terms } t \textit{ with value } n$$

Lemma 5.2.8

$$\mathsf{RA}^* \mathrel{\vdash\!\frac{\alpha}{\rho}} \Gamma, (\forall x)A(x) \qquad \Longrightarrow \qquad \mathsf{RA}^* \mathrel{\vdash\!\frac{\alpha}{\rho}} \Gamma, A(t) \qquad \textit{for all number terms } t$$

Lemma 5.2.9

$$\mathsf{RA}^* \mathrel{\vdash\!\frac{\alpha}{\rho}} \Gamma, A \wedge B \qquad \Longrightarrow \qquad \mathsf{RA}^* \mathrel{\vdash\!\frac{\alpha}{\rho}} \Gamma, A \quad \textit{and} \quad \mathsf{RA}^* \mathrel{\vdash\!\frac{\alpha}{\rho}} \Gamma, B$$

Proof: The proof of these Lemmata is an easy transfinite induction on the length α of the derivation in RA^*. □

Lemma 5.2.10

$$\mathsf{RA}^* \mathrel{\vdash\!\frac{\alpha}{\rho}} \Gamma, A \vee B \qquad \Longrightarrow \qquad \mathsf{RA}^* \mathrel{\vdash\!\frac{\alpha}{\rho}} \Gamma, A, B$$

Proof: The proof is by transfinite induction on α.

- If $A \vee B$ is not the main formula of the last inference, then either Γ is an axiom and so is Γ, A, B or we have the premises $\mathsf{RA}^* \mathrel{\vdash\!\frac{\alpha_i}{\rho}} \Gamma_i, A \vee B$. But then we have $\mathsf{RA}^* \mathrel{\vdash\!\frac{\alpha_i}{\rho}} \Gamma_i, A, B$ by the induction hypothesis and obtain $\mathsf{RA}^* \mathrel{\vdash\!\frac{\alpha}{\rho}} \Gamma, A, B$ by the same inference.
- If $A \vee B$ is the main formula of the last inference, then it is an $(\vee)$-inference whose premise is $\mathsf{RA}^* \mathrel{\vdash\!\frac{\alpha_0}{\rho}} \Gamma, A \vee B, A$ or $\mathsf{RA}^* \mathrel{\vdash\!\frac{\alpha_0}{\rho}} \Gamma, A$ from which we obtain by Corollary 5.2.6 also $\mathsf{RA}^* \mathrel{\vdash\!\frac{\alpha_0}{\rho}} \Gamma, A \vee B, A$. By the induction hypothesis it follows $\mathsf{RA}^* \mathrel{\vdash\!\frac{\alpha_0}{\rho}} \Gamma, A, B, A$ and by Corollary 5.2.5 $\mathsf{RA}^* \mathrel{\vdash\!\frac{\alpha}{\rho}} \Gamma, A, B$, since the set $\{A, B, A\}$ and $\{A, B\}$ are equal.

□

Lemma 5.2.11 *If A_0 and A_1 are two numerically equivalent $\mathcal{L}^*$-formulas with rank α, we have*

$$\mathsf{RA}^* \vdash_0^{\alpha \cdot 2} \neg A_0, A_1$$

PROOF:
We prove the claim by induction on the rank of A_0, resp. A_1.

$\underline{rk(A_0) = 0:}$

Then the set $\{\neg A_0, A_1\}$ clearly is an axiom.

$\underline{rk(A_0) > 0:}$

A_0 is a complex formula, then the claim follows immediately from the induction hypothesis.

- A_0 is of the form $s \in U^\alpha$, then clearly A_1 is of the form $t \in U^\alpha$ and hence the set $\{s \in U^\alpha, t \notin U^\alpha\}$ is an Axiom $(Ax2)$.
- A_0 is of the form $B_0 \vee C_0$. Then we have by the induction hypothesis that

  $$\mathsf{RA}^* \vdash_0^{\alpha_0 \cdot 2} \neg B_0, B_1 \qquad \mathsf{RA}^* \vdash_0^{\alpha_1 \cdot 2} \neg C_0, C_1$$

  for $\alpha_0 = rk(B_0)$ and $\alpha_1 = rk(C_0)$ and where again B_0, B_1 and C_0, C_1 are numerically equivalent formulas.

  With the $(\vee 1)$ and $(\vee 2)$-inference we obtain

  $$\mathsf{RA}^* \vdash_0^{\alpha_0 \cdot 2+1} \neg B_0, B_1 \vee C_1 \qquad \mathsf{RA}^* \vdash_0^{\alpha_1 \cdot 2+1} \neg C_0, B_1 \vee C_1$$

  Because $\alpha_0 < \alpha, \alpha_1 < \alpha$ we have with an $(\wedge)$-inference

  $$\mathsf{RA}^* \vdash_0^{\alpha \cdot 2} \neg B_0 \wedge \neg C_0, B_1 \vee C_1$$

- The case that A_0 is of the form $B_0 \wedge C_0$ can be proven in the same way.
- If A_0 is of the form $(\exists x)B_0(x)$. Then we have by the induction hypothesis an $\alpha_0 = rk(B_0(s)) < \alpha$ such that

  $$\mathsf{RA}^* \vdash_0^{\alpha_0 \cdot 2} \neg B_0(s), B_1(s) \qquad \text{for all number terms } s$$

  where $B_0(s), B_1(s)$ are numerically equivalent formulas.

  With the $(\forall^0)$-inference we obtain

  $$\mathsf{RA}^* \vdash_0^{\alpha_0 \cdot 2+1} (\forall x)\neg B_0(x), B_1(s)$$

and finally with an $(\exists^0)$-inference we have

$$\mathsf{RA}^* \vdash_0^{\alpha\cdot 2} (\forall x)\neg B_0(x), (\exists x)B_1(x)$$

- For the case that A_0 is of the form $(\forall x)B_0(x)$ the proof is analogous.
- A_0 is of the form $(\exists X^\gamma)B_0(X^\gamma)$ then we have by the induction hypothesis that

$$\mathsf{RA}^* \vdash_0^{\alpha_0\cdot 2} \neg B_0(S), B_1(S) \qquad \text{for all set terms } S \text{ of level} < \gamma$$

with $\alpha_0 = rk(B_0(S)) < \alpha$ and where $B_0(S), B_1(S)$ are numerically equivalent formulas.

With the $(\forall^1)$-inference we obtain

$$\mathsf{RA}^* \vdash_0^{\alpha_0\cdot 2+1} (\forall X^\gamma)\neg B_0(X^\gamma), B_1(S)$$

and finally with an $(\exists^1)$-inference we have

$$\mathsf{RA}^* \vdash_0^{\alpha\cdot 2} (\forall X^\gamma)\neg B_0(X^\gamma), (\exists X^\gamma)B_1(X^\gamma)$$

- For the case that A_0 is of the form $(\forall X^\gamma)B_0(X^\gamma)$ the proof is analogous.
- A_0 is of the form $r \in \{x : F_0(x)\}$ then we obtain by the induction hypothesis an $\alpha_0 = rk(F_0(r)) < \alpha$ such that

$$\mathsf{RA}^* \vdash_0^{\alpha_0\cdot 2} \neg F_0(r), F_1(s) \qquad \text{for all terms } r, s, \text{ which have the same value}$$

where $F_0(r), F_1(r)$ are numerically equivalent formulas. Together with $(\in 1)$ and $(\in 2)$ we obtain

$$\mathsf{RA}^* \vdash_0^{\alpha_0\cdot 2+2} r \notin \{x : F_0(x)\}, s \in \{x : F_1(x)\}$$

- The case that A_0 is of the form $r \notin \{x : F_0(x)\}$ can be proven similarly.

□

Further we have that RA^* proves the following equality lemma for arbitrary set terms.

Lemma 5.2.12 *For S, T arbitrary set terms and $A_0(U^0)$ and $A_1(U^0)$ numerically equivalent formulas we have with $\alpha_0 = rk(A_0(S)), \alpha_1 = rk(A_1(T))$ that*

$$\mathsf{RA}^* \vdash_0^{max(\alpha_0,\alpha_1)\cdot 2+3} \neg(S = T), \neg A_0(S), A_1(T)$$

PROOF: We prove the claim by the complexity of the formula $A_0(U^0)$.

- If $A_0(U^0)$ is of the form $R(s_1,\dots,s_n)$ then $A_1(T)$ is also of the form $R(t_1,\dots,t_n)$ where s_i has the same value as t_i for all $i \in \{1,2,\dots,n\}$. Then clearly the set $\{\neg(S=T), \neg R(s_1,\dots,s_n), R(t_1,\dots,t_n)\}$ is an Axiom $(Ax1)$ of RA^*.

- If $A_0(U^0)$ is of the form $s \in U^0$, then $A_1(T)$ is of the form $t \in T$, where s,t have the same value and we have to prove that

$$\mathsf{RA}^* \,\Big|\!\frac{\,max(\alpha_0,\alpha_1)\cdot 2+3\,}{0}\; \neg(S=T), s \notin S, t \in T$$

We obtain from Lemma 5.2.11 that

$$\mathsf{RA}^* \,\Big|\!\frac{\,\alpha_0\cdot 2\,}{0}\; s \notin S, t \in T, t \in S$$

$$\mathsf{RA}^* \,\Big|\!\frac{\,\alpha_1\cdot 2\,}{0}\; s \notin S, t \in T, t \notin T$$

where again s and t have the same value.
By the $(\wedge)$-inference we obtain

$$\mathsf{RA}^* \,\Big|\!\frac{\,max(\alpha_0,\alpha_1)\cdot 2+1\,}{0}\; s \notin S, t \in T, t \in S \wedge t \notin T$$

and therefore by the $(\vee 1)$-inference

$$\mathsf{RA}^* \,\Big|\!\frac{\,max(\alpha_0,\alpha_1)\cdot 2+2\,}{0}\; s \notin S, t \in T, (t \in S \wedge t \notin T) \vee (t \notin S \wedge t \in T)$$

This is

$$\mathsf{RA}^* \,\Big|\!\frac{\,max(\alpha_0,\alpha_1)\cdot 2+2\,}{0}\; s \notin S, t \in T, \neg(t \in S \leftrightarrow t \in T)$$

And finally by the $(\exists^0)$-rule

$$\mathsf{RA}^* \,\Big|\!\frac{\,max(\alpha_0,\alpha_1)\cdot 2+3\,}{0}\; s \notin S, t \in T, \neg(\forall x)(x \in S \leftrightarrow x \in T)$$

- If $A_0(U^0)$ is of the form $s \in \{x : B_0(x,U^0)\}$, then $A_1(T)$ is of the form $t \in \{x : B_1(x,T)\}$, where $B_0(0,U^0)$ and $B_1(0,U^0)$ are numerically equivalent formulas and we obtain by the induction hypothesis $\beta_0 = rk(B_0(0,S)) < \alpha_0$ and $\beta_1 = rk(B_1(0,T)) < \alpha_1$ such that

$$\mathsf{RA}^* \,\Big|\!\frac{\,max(\beta_0,\beta_1)\cdot 2+3\,}{0}\; \neg(S=T), \neg B_0(s,S), B_1(t,T)$$

where s,t have the same value. Together with $(\in 1)$ and $(\in 2)$ we obtain

$$\mathsf{RA}^* \,\Big|\!\frac{\,max(\beta_0,\beta_1)\cdot 2+5\,}{0}\; \neg(S=T), s \notin \{x : B_0(x,S)\}, t \in \{x : B_1(x,T)\}$$

- If $A_0(U^0)$ is of the form $B_0(U^0) \vee C_0(U^0)$, then $A_1(T)$ is of the form $B_1(T) \vee C_1(T)$, where again $B_0(U^0), B_1(U^0)$ and $C_0(U^0), C_1(U^0)$ are numerically equivalent formulas. By the induction hypothesis we obtain

$$\mathsf{RA}^* \mathrel{\substack{max(\beta_0,\beta_1)\cdot 2+3 \\ \vdash \\ 0}} \neg(S = T), \neg B_0(S), B_1(T)$$

$$\mathsf{RA}^* \mathrel{\substack{max(\gamma_0,\gamma_1)\cdot 2+3 \\ \vdash \\ 0}} \neg(S = T), \neg C_0(S), C_1(T)$$

with $\beta_0 = rk(B_0(S)) < \alpha_0, \gamma_0 = rk(C_0(S)) < \alpha_0, \beta_1 = rk(B_1(T)) < \alpha_1$ and $\gamma_1 = rk(C_1(T)) < \alpha_1$.
With the $(\vee 1)$ and $(\vee 2)$-inference we obtain

$$\mathsf{RA}^* \mathrel{\substack{max(\beta_0,\beta_1)\cdot 2+4 \\ \vdash \\ 0}} \neg(S = T), \neg B_0(S), B_1(T) \vee C_1(T)$$

$$\mathsf{RA}^* \mathrel{\substack{max(\gamma_0,\gamma_1)\cdot 2+4 \\ \vdash \\ 0}} \neg(S = T), \neg C_0(S), B_1(T) \vee C_1(T)$$

and have by the $(\wedge)$-inference

$$\mathsf{RA}^* \mathrel{\substack{max(\beta_0,\beta_1,\gamma_0,\gamma_1)\cdot 2+5 \\ \vdash \\ 0}} \neg(S = T), \neg B_0(S) \wedge \neg C_0(S), B_1(T) \vee C_1(T)$$

- The case that $A_0(U^0)$ is of the form $B_0(U^0) \wedge C_0(U^0)$ can be proven in the same way.

- If $A_0(U^0)$ is of the form $(\exists x)B_0(x, U^0)$, then $A_1(T)$ is of the form $(\exists x)B_1(x, T)$, where $B_0(0, U^0), B_1(0, U^0)$ are numerically equivalent formulas. Then we have by the induction hypothesis $\beta_0 = rk(B_0(0, S)) < \alpha_0$ and
$\beta_1 = rk(B_1(0, T)) < \alpha_1$ such that

$$\mathsf{RA}^* \mathrel{\substack{max(\beta_0,\beta_1)\cdot 2+3 \\ \vdash \\ 0}} \neg(S = T), \neg B_0(s, S), B_1(s, T) \qquad \text{for all number terms } s$$

With the $(\exists^0)$-inference we have

$$\mathsf{RA}^* \mathrel{\substack{max(\beta_0,\beta_1)\cdot 2+4 \\ \vdash \\ 0}} \neg(S = T), \neg B_0(s, S), (\exists x)B_1(x, T)$$

and finally with the $(\forall^0)$-inference we obtain

$$\mathsf{RA}^* \mathrel{\substack{max(\beta_0,\beta_1)\cdot 2+5 \\ \vdash \\ 0}} \neg(S = T), (\forall x)\neg B_0(x, S), (\exists x)B_1(x, T)$$

- The case that $A_0(U^0)$ is of the form $(\forall x)B_0(x, U^0)$ can be proven similarly.

- If $A_0(U^0)$ is of the form $(\exists X^\gamma)B_0(X^\gamma, U^0)$, then $A_1(T)$ is of the form $(\exists X^\gamma)B_1(X^\gamma, T)$, where $B_0(V^0, U^0), B_1(V^0, U^0)$ are numerically equivalent formulas. Then we have by the induction hypothesis $\beta_0 = rk(B_0(R, S)) < \alpha_0$ and $\beta_1 = rk(B_1(R, T)) < \alpha_1$ such that for all set terms R of level $< \gamma$

  $$\mathsf{RA}^* \vdash^{max(\beta_0,\beta_1)\cdot 2+3}_{0} \neg(S = T), \neg B_0(R, S), B_1(R, T)$$

  With the $(\exists^1)$-inference we have

  $$\mathsf{RA}^* \vdash^{max(\beta_0,\beta_1)\cdot 2+4}_{0} \neg(S = T), \neg B_0(R, S), (\exists X^\gamma)B_1(X^\gamma, T)$$

  and finally with the $(\forall^1)$-inference we obtain

  $$\mathsf{RA}^* \vdash^{max(\beta_0,\beta_1)\cdot 2+5}_{0} \neg(S = T), (\forall X^\gamma)\neg B_0(X^\gamma, S), (\exists X^\gamma)B_1(X^\gamma, T)$$

$\square$

Further standard proof-theoretic techniques can be applied for the system RA^* to obtain the following cut elimination Theorems, cf. Pohlers [6] Theorem 12.3 and Theorem 18.4 or Schütte [9] Theorem 22.7 and Theorem 22.8

Theorem 5.2.13

$$\mathsf{RA}^* \vdash^{\alpha}_{\rho+1} \Gamma \qquad \Longrightarrow \qquad \mathsf{RA}^* \vdash^{2^\alpha}_{\rho} \Gamma$$

Theorem 5.2.14

$$\mathsf{RA}^* \vdash^{\alpha}_{\beta+\omega^\rho} \Gamma \qquad \Longrightarrow \qquad \mathsf{RA}^* \vdash^{\varphi\rho\alpha}_{\beta} \Gamma$$

5.2.2 Embedding of ACA^+ into RA^*

In the next step we embed ACA^+ into RA^*. For that we have to make the following definition:

Definition 5.2.15 *An $\mathcal{L}^*$-formula F^α is an α-instance of an $\mathcal{L}_2$-formula F if F^α is obtained from F by*

1. *replacing all free number variables by arbitrary closed number terms.*
2. *free set variables are replaced by arbitrary set terms of $\mathcal{L}^*$ with level $< \alpha$.*
3. *bound set variables get the superscript α.*

Notice that if F^α is an α-instance of an $\mathcal{L}_2$-formula, then $rk(F^\alpha) < \omega(\alpha+1)$.

We want to prove that if $\mathsf{ACA}^+ \vdash F$, and F^ω is an ω-instance of F then there exists a natural number n such that $\mathsf{RA}^* \mathrel{\vert\!\!\frac{<\varepsilon_0}{\omega^2+n}} F^\omega$.

Lemma 5.2.16 *Let $A[u_1,\ldots,u_n,U_1,\ldots,U_m]$ be one of the number-theoretic or logical axioms of* ACA*. Then we have for all α-instances A^α of A that*

$$\mathsf{RA}^* \mathrel{\vert\!\!\frac{\omega(\alpha+1)\cdot 2+3}{0}} A^\alpha(r_1,\ldots,r_n,S_1^{\gamma_1},\ldots,S_m^{\gamma_m})$$

(with $\gamma_i < \alpha$).

Proof:

- If $A[u_1,\ldots,u_n,U_1,\ldots,U_m]$ is one of the number-theoretic axioms or the equality axiom $u=u$, then $A^\alpha(r_1,\ldots,r_n,S_1^{\gamma_1},\ldots,S_m^{\gamma_m})$ is an axiom $(Ax1)$ of RA^*.
- If $A[u_1,\ldots,u_n,U_1,\ldots,U_m]$ is of the form $B(t) \rightarrow (\exists x)B(x)$ then we have by Lemma 5.2.11 that $\mathsf{RA}^* \mathrel{\vert\!\!\frac{\omega(\alpha+1)\cdot 2}{0}} B^\alpha(r), \neg B^\alpha(r)$. Applying first the $(\exists^0)$-inference $\mathsf{RA}^* \mathrel{\vert\!\!\frac{\omega(\alpha+1)\cdot 2+1}{0}} (\exists x)B^\alpha(x), \neg B^\alpha(r)$ and then the $(\vee)$-inference twice we obtain $\mathsf{RA}^* \mathrel{\vert\!\!\frac{\omega(\alpha+1)\cdot 2+3}{0}} B^\alpha(r) \rightarrow (\exists x)B^\alpha(x)$.
- If $A[u_1,\ldots,u_n,U_1,\ldots,U_m]$ is of the form $(\forall x)B(x) \rightarrow B(t)$ then we have by Lemma 5.2.11 that $\mathsf{RA}^* \mathrel{\vert\!\!\frac{\omega(\alpha+1)\cdot 2}{0}} B^\alpha(r), \neg B^\alpha(r)$. Applying first the $(\exists^0)$-inference $\mathsf{RA}^* \mathrel{\vert\!\!\frac{\omega(\alpha+1)\cdot 2+1}{0}} (\exists x)\neg B^\alpha(x), B^\alpha(r)$ and then the $(\vee)$-inference we obtain $\mathsf{RA}^* \mathrel{\vert\!\!\frac{\omega(\alpha+1)\cdot 2+3}{0}} (\forall x)B^\alpha(x) \rightarrow B^\alpha(r)$.
- If $A[u_1,\ldots,u_n,U_1,\ldots,U_m]$ is of the form $B(U) \rightarrow (\exists X)B(X)$ then we have by Lemma 5.2.11 that $\mathsf{RA}^* \mathrel{\vert\!\!\frac{\omega(\alpha+1)\cdot 2}{0}} B^\alpha(S^\gamma), \neg B^\alpha(S^\gamma)$ for arbitrary set terms S with level $\gamma < \alpha$.
  Applying the $(\exists^1)$-inference $\mathsf{RA}^* \mathrel{\vert\!\!\frac{\omega(\alpha+1)\cdot 2+1}{0}} (\exists X^\alpha)B^\alpha(X^\alpha), \neg B^\alpha(S^\gamma)$ and then the $(\vee)$-inference twice we obtain $\mathsf{RA}^* \mathrel{\vert\!\!\frac{\omega(\alpha+1)\cdot 2+3}{0}} B^\alpha(S^\gamma) \rightarrow (\exists X^\alpha)B^\alpha(X^\alpha)$.
- If $A[u_1,\ldots,u_n,U_1,\ldots,U_m]$ is of the form $(\forall X)B(X) \rightarrow B(U)$ then we have by Lemma 5.2.11 that $\mathsf{RA}^* \mathrel{\vert\!\!\frac{\omega(\alpha+1)\cdot 2}{0}} B^\alpha(S^\gamma), \neg B^\alpha(S^\gamma)$ for arbitrary set terms S with $\gamma < \alpha$.
  Applying the $(\exists^1)$-inference $\mathsf{RA}^* \mathrel{\vert\!\!\frac{\omega(\alpha+1)\cdot 2+1}{0}} (\exists X^\alpha)\neg B^\alpha(X^\alpha), B^\alpha(S^\gamma)$ and then the $(\vee)$-inference we obtain $\mathsf{RA}^* \mathrel{\vert\!\!\frac{\omega(\alpha+1)\cdot 2+3}{0}} (\forall X^\alpha)B^\alpha(X^\alpha) \rightarrow B^\alpha(S^\gamma)$.

- $A[u_1, \ldots, u_n, U_1, \ldots, U_m]$ is of the form $s(\vec{u}) = t(\vec{u}) \to (B(\vec{u}, s(\vec{u})) \to B(\vec{u}, t(\vec{u})))$ with $\vec{u} = u_1, \ldots, u_n$. We denote $r_1, \ldots, r_n$ as $\vec{r}$ and have to distinguish two cases.

  1. If $s(\vec{r})$ and $t(\vec{r})$ have not the same value. Then the formula $s(\vec{r}) \neq t(\vec{r})$ is an axiom $(Ax1)$. Therefore we have that

  $$\mathsf{RA}^* \mathrel{|\!\frac{0}{0}} s(\vec{r}) \neq t(\vec{r}), \neg B^\alpha(\vec{r}, s(\vec{r})), B^\alpha(\vec{r}, t(\vec{r}))$$

  Applying four times the $(\vee)$-inference we obtain

  $$\mathsf{RA}^* \mathrel{|\!\frac{4}{0}} s(\vec{r}) = t(\vec{r}) \to (B^\alpha(\vec{r}, s(\vec{r})) \to B^\alpha(\vec{r}, t(\vec{r})))$$

  2. $s(\vec{r})$ and $t(\vec{r})$ have the same value. Hence

  $$\mathsf{RA}^* \mathrel{|\!\frac{\omega(\alpha+1)\cdot 2}{0}} \neg B^\alpha(\vec{r}, s(\vec{r})), B^\alpha(\vec{r}, t(\vec{r}))$$

  by Lemma 5.2.11. With applying the $(\vee)$-inference three times we obtain $\mathsf{RA}^* \mathrel{|\!\frac{\omega(\alpha+1)\cdot 2+3}{0}} s(\vec{r}) = t(\vec{r}) \to (B^\alpha(\vec{r}, s(\vec{r})) \to B^\alpha(\vec{r}, t(\vec{r})))$.

Therewith the proof of the Lemma is finished. □

So now let us turn to the full second order induction scheme (IND).

Lemma 5.2.17 *If $A^\alpha(0)$ is an $\mathcal{L}^*$-formula of level α, then we have for all natural numbers n that*

$$\mathsf{RA}^* \mathrel{|\!\frac{\omega(\alpha+1)\cdot 2+2n}{0}} \neg A^\alpha(\overline{0}), \neg(\forall x)(A^\alpha(x) \to A^\alpha(x+1)), A^\alpha(\overline{n})$$

Proof: We prove the claim with induction on n.

$n = 0$:

Holds obviously by Lemma 5.2.11.

$n \to n+1$:

From the induction hypothesis we have

$$\mathsf{RA}^* \mathrel{|\!\frac{\omega(\alpha+1)\cdot 2+2n}{0}} \neg A^\alpha(\overline{0}), \neg(\forall x)(A^\alpha(x) \to A^\alpha(x+1)), A^\alpha(\overline{n})$$

Moreover, we have again from Lemma 5.2.11 that

$$\mathsf{RA}^* \vdash_0^{\omega(\alpha+1)\cdot 2} \neg A^\alpha(\overline{n+1}), A^\alpha(\overline{n+1})$$

By the $(\wedge)$-inference we obtain

$$\mathsf{RA}^* \vdash_0^{\omega(\alpha+1)\cdot 2+2n+1} \neg A^\alpha(\overline{0}), \neg(\forall x)(A^\alpha(x) \to A^\alpha(x{+}1)), A^\alpha(\overline{n}) \wedge \neg A^\alpha(\overline{n+1}), A^\alpha(\overline{n+1})$$

and then by applying the $(\exists^0)$-inference

$$\mathsf{RA}^* \vdash_0^{\omega(\alpha+1)\cdot 2+2n+2} \neg A^\alpha(\overline{0}), \neg(\forall x)(A^\alpha(x) \to A^\alpha(x+1)), A^\alpha(\overline{n+1})$$

□

Lemma 5.2.18 *If F is an instance of the full second order induction scheme $A(0) \wedge (\forall x)(A(x) \to A(x+1)) \to (\forall x)A(x)$, and F^α is an α-instance of F then*

$$\mathsf{RA}^* \vdash_0^{\omega(\alpha+1)\cdot 2+\omega+4} A^\alpha(\overline{0}) \wedge (\forall x)(A^\alpha(x) \to A^\alpha(x+1)) \to (\forall x)A^\alpha(x)$$

PROOF: By 5.2.17 we have

$$\mathsf{RA}^* \vdash_0^{\omega(\alpha+1)\cdot 2+2n} \neg A^\alpha(\overline{0}), \neg(\forall x)(A^\alpha(x) \to A^\alpha(x+1)), A^\alpha(\overline{n})$$

for all natural numbers n. With Lemma 5.2.7 and the $(\forall^0)$-inference of RA^* we obtain

$$\mathsf{RA}^* \vdash_0^{\omega(\alpha+1)\cdot 2+\omega} \neg A^\alpha(\overline{0}), \neg(\forall x)(A^\alpha(x) \to A^\alpha(x+1)), (\forall x)A^\alpha(x)$$

and by applying the $(\vee)$-inference four times we obtain the claim. □

Now let us turn to the arithmetic comprehension scheme.

Lemma 5.2.19 *If F is an instance of the arithmetic comprehension scheme $(\exists X)(\forall z)(z \in X \leftrightarrow A(z))$ and F^α is an α-instance of F (with $\alpha \neq 0$), then*

$$\mathsf{RA}^* \vdash_0^{\omega(\alpha+1)\cdot 2} (\exists X^\alpha)(\forall z)(z \in X^\alpha \leftrightarrow A^\alpha(z))$$

PROOF: Notice that the level $A^\alpha(s)$ is $\alpha_0 < \alpha$ by definition, since A^α does not comprise any bound set variables. By Lemma 5.2.11 we have that for all number terms s that

$$\mathsf{RA}^* \vdash_0^{\omega(\alpha_0+1)\cdot 2} A^\alpha(s), \neg A^\alpha(s)$$

With $(\in 1)$ and $(\in 2)$ respectively we conclude

$$\mathsf{RA}^* \mathrel{\substack{\omega(\alpha_0+1)\cdot 2+1 \\ \vdash \\ 0}} A^\alpha(s), s \notin \{x : A^\alpha(x)\} \qquad \mathsf{RA}^* \mathrel{\substack{\omega(\alpha_0+1)\cdot 2+1 \\ \vdash \\ 0}} \neg A^\alpha(s), s \in \{x : A^\alpha(x)\}$$

Now applying the $(\vee)$-inference twice and then the $(\wedge)$-inference we get

$$\mathsf{RA}^* \mathrel{\substack{\omega(\alpha_0+1)\cdot 2+4 \\ \vdash \\ 0}} s \in \{x : A^\alpha(x)\} \leftrightarrow A^\alpha(s)$$

for all for all number terms s. Therefore we obtain by $(\forall^0)$

$$\mathsf{RA}^* \mathrel{\substack{\omega(\alpha_0+1)\cdot 2+5 \\ \vdash \\ 0}} (\forall z)(z \in \{x : A^\alpha(x)\} \leftrightarrow A^\alpha(z))$$

Because $\{x : A^\alpha(x)\}$ is a set term of level $\alpha_0 < \alpha$, we obviously have by $(\exists^1)$

$$\mathsf{RA}^* \mathrel{\substack{\omega(\alpha_0+1)\cdot 2+6 \\ \vdash \\ 0}} (\exists X^\alpha)(\forall z)(z \in X^\alpha \leftrightarrow A^\alpha(z))$$

and since $\omega(\alpha_0 + 1) \cdot 2 + 6 < \omega(\alpha + 1) \cdot 2$ the claim follows easily. $\square$

Theorem 5.2.20 *For all $\mathcal{L}_2$-formulas $A[u_1, \ldots, u_n, U_1, \ldots, U_m]$ with*

$$\mathsf{ACA} \vdash A[u_1, \ldots, u_n, U_1, \ldots, U_m]$$

there exists an ordinal $\beta < \omega(\alpha+1) \cdot 2 + \omega \cdot 2$ and a natural number m such that for all $\alpha-$instances of A

$$\mathsf{RA}^* \mathrel{\substack{\beta \\ \vdash \\ \omega\cdot\alpha+m}} A^\alpha(r_1, \ldots, r_n, S_1^{\gamma_1}, \ldots, S_m^{\gamma_m})$$

(with $\gamma_i < \alpha$).

Proof: Proof by induction on the length n of the derivation of $A[u_1, \ldots, u_n, U_1, \ldots, U_m]$ in ACA.

<u>$n = 0$:</u>

$A[u_1, \ldots, u_n, U_1, \ldots, U_m]$ is an axiom of ACA. By Lemma 5.2.16, Lemma 5.2.18 and Lemma 5.2.19 we are done.

<u>$n > 0$:</u>

We will concentrate on two cases.

- First if the last inference was Modus Ponens, then we have $n_0 < n, n_1 < n$ such that

$$\mathsf{ACA} \overset{n_0}{\vdash} B[v_1, \ldots, v_p, V_1, \ldots, V_q]$$

$$\mathsf{ACA} \overset{n_1}{\vdash} B[v_1, \ldots, v_p, V_1, \ldots, V_q] \rightarrow A[u_1, \ldots, u_n, U_1, \ldots, U_m]$$

By the induction hypothesis we obtain ordinals $\beta_0 < \omega(\alpha+1) \cdot 2 + \omega \cdot 2$, $\beta_1 < \omega(\alpha+1) \cdot 2 + \omega \cdot 2$ and natural numbers m_0 and m_1 such that

$$\mathsf{RA}^* \vdash^{\beta_0}_{\omega \cdot \alpha + m_0} B^\alpha(s_1, \ldots, s_p, T_1^{\delta_1}, \ldots, T_q^{\delta_q}) \tag{12}$$

$$\mathsf{RA}^* \vdash^{\beta_1}_{\omega \cdot \alpha + m_1} B^\alpha(s_1, \ldots, s_p, T_1^{\delta_1}, \ldots, T_q^{\delta_q}) \rightarrow A^\alpha(r_1, \ldots, r_n, S_1^{\gamma_1}, \ldots, S_m^{\gamma_m}) \tag{13}$$

(with $\gamma_i < \alpha, \delta_i < \alpha$).
By Lemma 5.2.10 it follows from (13) that

$$\mathsf{RA}^* \vdash^{\beta_1}_{\omega \cdot \alpha + m_1} \neg B^\alpha(s_1, \ldots, s_p, T_1^{\delta_1}, \ldots, T_q^{\delta_q}), A^\alpha(r_1, \ldots, r_n, S_1^{\gamma_1}, \ldots, S_m^{\gamma_m})$$

and finally we obtain with (12) by applying the (*cut*)-inference, since $max(\omega \cdot \alpha + m_1, \omega \cdot \alpha + m_2, rk(B^\alpha(\vec{s}, \vec{R}))) < \omega(\alpha+1)$, a natural number m such that

$$\mathsf{RA}^* \vdash^{max(\beta_0, \beta_1)+1}_{\omega \cdot \alpha + m} A^\alpha(r_1, \ldots, r_n, S_1^{\gamma_1}, \ldots, S_m^{\gamma_m})$$

with $max(\beta_0, \beta_1) + 1 < \omega(\alpha+1) \cdot 2 + \omega \cdot 2$.

- If the last inference was an universal number quantification then $A[u_1, \ldots, u_n, U_1, \ldots, U_m]$ is of the form $C(u_1, \ldots, u_n, U_1, \ldots, U_m) \rightarrow (\forall x) B(x, u_1, \ldots, u_n, U_1, \ldots, U_m)$, and we have an $n_0 < n$ such that

$$\mathsf{ACA} \overset{n_0}{\vdash} C(u_1, \ldots, u_n, U_1, \ldots, U_m) \rightarrow B(u, u_1, \ldots, u_n, U_1, \ldots, U_m)$$

where u is pairwise disjoint from all u_i with $i \in \{1, \ldots, n\}$, and obtain by induction hypothesis an ordinal $\beta_0 < \omega(\alpha+1) \cdot 2 + \omega \cdot 2$ and a natural number m such that for number terms t

$$\mathsf{RA}^* \vdash^{\beta_0}_{\omega \cdot \alpha + m} C(r_1, \ldots, r_n, S_1^{\gamma_1}, \ldots, S_m^{\gamma_m}) \rightarrow B^\alpha(t, r_1, \ldots, r_n, S_1^{\gamma_1}, \ldots, S_m^{\gamma_m})$$

(with $\gamma_i < \alpha$).
By Lemma 5.2.10 we obtain

$$\mathsf{RA}^* \vdash^{\beta_0}_{\omega \cdot \alpha + m} \neg C(r_1, \ldots, r_n, S_1^{\gamma_1}, \ldots, S_m^{\gamma_m}), B^\alpha(t, r_1, \ldots, r_n, S_1^{\gamma_1}, \ldots, S_m^{\gamma_m})$$

for all number terms t. By applying the $(\forall^0)$-inference of RA^* we conclude

$$\mathsf{RA}^* \,\big|\!\frac{\beta_0+1}{\omega\cdot\alpha+m} \quad \begin{array}{l}\neg C(r_1,\dots,r_n,S_1^{\gamma_1},\dots,S_m^{\gamma_m}),\\ (\forall x)B^\alpha(x,r_1,\dots,r_n,S_1^{\gamma_1},\dots,S_m^{\gamma_m})\end{array}$$

and finally by applying the $(\vee)$-inference twice

$$\mathsf{RA}^* \,\big|\!\frac{\beta_0+3}{\omega\cdot\alpha+m} \quad \begin{array}{l}C(r_1,\dots,r_n,S_1^{\gamma_1},\dots,S_m^{\gamma_m}) \rightarrow\\ (\forall x)B^\alpha(x,r_1,\dots,r_n,S_1^{\gamma_1},\dots,S_m^{\gamma_m})\end{array}$$

with $\beta_0 + 1 < \omega(\alpha+1)\cdot 2 + \omega \cdot 2$.

□

If we restrict ourselves in Theorem 5.2.20 to $\omega-$instances we immediately obtain the following corollary.

Corollary 5.2.21 *For all $\mathcal{L}_2$-formulas $A[u_1,\dots,u_n,U_1,\dots,U_m]$ with*

$$\mathsf{ACA} \vdash A[u_1,\dots,u_n,U_1,\dots,U_m]$$

there exists, if A^ω is an ω-instance of A, an ordinal $\alpha < \omega^2\cdot 2+\omega\cdot 4$ and and a natural number m such that

$$\mathsf{RA}^* \,\big|\!\frac{\alpha}{\omega^2+m} \; A^\omega(r_1,\dots,r_n,S_1^{l_1},\dots,S_m^{l_m})$$

(with $l_i < \omega$).

Finally let us turn to $(\omega - \mathsf{Jump})$. Notice that we showed in Lemma 3.1.1 that the Turing jump hierarchy is unique, provable in $\mathsf{ACA_0}$. Therefore we know by Corollary 5.2.21, this is also provable in RA^* for every ω-instance, where the length of the derivation is restricted by $\omega^2\cdot 2+\omega\cdot 4$ and every formula has a rank $< \omega^2+\omega$.

We make the following definitions:

Definition 5.2.22 *1. $\mathcal{H}(U,V,a) := (V)_0 = U \wedge (\forall x < a)((V)_{x+1} = TJ((V)_x))$*

2. $(U)^c := \{\langle a,b\rangle : b \le c \wedge \langle a,b\rangle \in U\}$

3. For all natural numbers n, we define set terms $TJ^n(S)$ inductively by $TJ^0(S) := S$ and $TJ^{n+1}(S) := \{\langle e,b\rangle : (\exists z)\mathcal{T}^{TJ^n(S)}(e,b,z)\}$, denoted by $TJ(TJ^n(S))$, where $\mathcal{T}^U$ is Kleene's T-predicate, relativized to U.

4. *For all natural numbers n we define set terms $\mathcal{R}^S_n$ as*
$\mathcal{R}^S_n := \{\langle a, b\rangle : \bigvee_{i=0}^n b = i \wedge a \in TJ^i(S)\}$

Notice that $t \in TJ^n(S)$ is a finite formula of $\mathcal{L}^*$ and $\mathcal{R}^S_n$ is a finite set term of $\mathcal{L}^*$ for all natural numbers n with $\mathbf{lev}(\mathcal{R}^{\mathbf{S}}_{\mathbf{n}}) = \mathbf{lev}(\mathbf{S})$.

First we prove the following Lemma.

Lemma 5.2.23 *For all natural numbers n, we have that*

$$\mathsf{RA}^* \mathrel{\vert\!\frac{\omega(\alpha+1)\cdot 2+n\cdot 10+11}{<\omega(\alpha+1)}} \mathcal{H}(S^\alpha, \mathcal{R}^{S^\alpha}_n, \overline{n})$$

Proof: We show that with an induction on n.

<u>$n = 0$:</u>

We have by the definition of $\mathcal{H}$ that

$$\mathcal{H}(S^\alpha, \mathcal{R}^{S^\alpha}_0, \overline{0}) := (\mathcal{R}^{S^\alpha}_0)_0 = S^\alpha \wedge (\forall x < \overline{0})((\mathcal{R}^{S^\alpha}_0)_{x+1} = TJ((\mathcal{R}^{S^\alpha}_0)_x))$$

Since $(\mathcal{R}^{S^\alpha}_0)_0 = S^\alpha$ is an abbreviation and together with the definition of $\mathcal{R}^S_n$ this is

$$(\forall x)(\langle x, 0\rangle \in \{y : (\exists z_1)(\exists z_2)(y = \langle z_1, z_2\rangle \wedge z_2 = 0 \wedge z_1 \in S^\alpha)\} \leftrightarrow x \in S^\alpha)\wedge$$
$$(\forall x < \overline{0})((\mathcal{R}^{S^\alpha}_0)_{x+1} = TJ((\mathcal{R}^{S^\alpha}_0)_x))$$

It is not difficult to show (though a little bit cumbersome) that this holds (for more details of this proof compare Appendix A1) for a cut-free deduction in RA^*, where the length of the derivation is restricted by $\omega(\alpha + 1) \cdot 2 + 10$.

<u>$n \to n + 1$:</u>

By the induction hypothesis we have that $\mathcal{H}(S^\alpha, \mathcal{R}^{S^\alpha}_n, \overline{n})$ holds. More formally

$$\mathsf{RA}^* \mathrel{\vert\!\frac{\omega(\alpha+1)\cdot 2+n\cdot 10+11}{<\omega(\alpha+1)}} (\mathcal{R}^{S^\alpha}_n)_0 = S^\alpha \wedge (\forall x < \overline{n})((\mathcal{R}^{S^\alpha}_n)_{x+1} = TJ((\mathcal{R}^{S^\alpha}_n)_x))$$

Together with Lemma 5.2.9 we obtain that

$$\mathsf{RA}^* \mathrel{\vert\!\frac{\omega(\alpha+1)\cdot 2+n\cdot 10+11}{<\omega(\alpha+1)}} (\mathcal{R}^{S^\alpha}_n)_0 = S^\alpha \tag{14}$$

$$\mathsf{RA}^* \mathrel{\vert\!\frac{\omega(\alpha+1)\cdot 2+n\cdot 10+11}{<\omega(\alpha+1)}} (\forall x < \overline{n})((\mathcal{R}^{S^\alpha}_n)_{x+1} = TJ((\mathcal{R}^{S^\alpha}_n)_x)) \tag{15}$$

From (15) we obtain with Lemma 5.2.8 and Lemma 5.2.10 for all natural numbers i

$$\mathsf{RA}^* \mathrel{\vert\!\frac{\omega(\alpha+1)\cdot 2+n\cdot 10+11}{<\omega(\alpha+1)}} \neg(\bar{i} < \overline{n}), ((\mathcal{R}^{S^\alpha}_n)_{\bar{i}+1} = TJ((\mathcal{R}^{S^\alpha}_n)_{\bar{i}})) \tag{16}$$

In a first step we want to prove that this also hold for $\mathcal{R}^{S^\alpha}_{n+1}$ instead of $\mathcal{R}^{S^\alpha}_{n}$. We obtain the desired result by making use several times of Lemma 5.2.12.
First we have to distinguish two cases depending on the value of i.

1. By the definition of $\mathcal{R}^{S^\alpha}_{n}$ we have that for all natural numbers $i < n$

$$\mathsf{RA}^* \vdash^{\omega(\alpha+1)\cdot 2+12}_{0} \neg(\bar{i} < \bar{n}), (\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}+1} = (\mathcal{R}^{S^\alpha}_{n})_{\bar{i}+1}$$

   (for more details compare also Appendix A2).

2. Further for all natural numbers $i \geq n$ we have

$$\mathsf{RA}^* \vdash^{0}_{0} \neg(\bar{i} < \bar{n}), (\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}+1} = (\mathcal{R}^{S^\alpha}_{n})_{\bar{i}+1}$$

Therefore we have for all natural numbers i that

$$\mathsf{RA}^* \vdash^{\omega(\alpha+1)\cdot 2+12}_{0} \neg(\bar{i} < \bar{n}), (\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}+1} = (\mathcal{R}^{S^\alpha}_{n})_{\bar{i}+1} \tag{17}$$

Completely analogously we obtain also for all natural numbers i

$$\mathsf{RA}^* \vdash^{\omega(\alpha+1)\cdot 2+12}_{0} \neg(\bar{i} < \bar{n}), (\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}} = (\mathcal{R}^{S^\alpha}_{n})_{\bar{i}} \tag{18}$$

By Lemma 5.2.12 and with $A(U^\alpha) := ((\mathcal{R}^{S^\alpha}_{n})_{\bar{i}+1} = TJ(U^\alpha))$ we have for all natural numbers i

$$\mathsf{RA}^* \vdash^{\omega(\alpha+1)\cdot 2+3}_{0} \neg((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}} = (\mathcal{R}^{S^\alpha}_{n})_{\bar{i}}), \neg A((\mathcal{R}^{S^\alpha}_{n})_{\bar{i}}), A((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}}) \tag{19}$$

Analogously by Lemma 5.2.12 and with $B(U^\alpha) := ((U^\alpha = TJ(\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}}))$ we have for all natural numbers i

$$\mathsf{RA}^* \vdash^{\omega(\alpha+1)\cdot 2+3}_{0} \neg((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}+1} = (\mathcal{R}^{S^\alpha}_{n})_{\bar{i}+1}), \neg B((\mathcal{R}^{S^\alpha}_{n})_{\bar{i}+1}), B((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}+1}) \tag{20}$$

We conclude from (16) and (19) by the (cut)-rule that

$$\mathsf{RA}^* \vdash^{\omega(\alpha+1)\cdot 2+n\cdot 10+12}_{<\omega(\alpha+1)} \neg(\bar{i} < \bar{n}), \neg((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}} = (\mathcal{R}^{S^\alpha}_{n})_{\bar{i}}), A((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}})$$

since $rk(A((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}})) < \omega(\alpha+1)$. By (18) we obtain with a cut for all natural numbers i

$$\mathsf{RA}^* \vdash^{\omega(\alpha+1)\cdot 2+n\cdot 10+13}_{<\omega(\alpha+1)} \neg(\bar{i} < \bar{n}), A((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}})$$

since $rk(\neg((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}} = (\mathcal{R}^{S^\alpha}_{n})_{\bar{i}})) < \omega(\alpha+1)$. With (20) we obtain by a cut that

$$\mathsf{RA}^* \mid\frac{\omega(\alpha+1)\cdot 2+n\cdot 10+14}{<\omega(\alpha+1)} \neg(\bar{i} < \overline{n}), \neg((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}+1} = (\mathcal{R}^{S^\alpha}_{n})_{\bar{i}+1}), B((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}+1})$$

since also $rk(B((\mathcal{R}^{S^\alpha}_{n})_{\bar{i}+1})) < \omega(\alpha+1)$ and with (17) and a cut

$$\mathsf{RA}^* \mid\frac{\omega(\alpha+1)\cdot 2+n\cdot 10+15}{<\omega(\alpha+1)} \neg(\bar{i} < \overline{n}), B((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}+1})$$

and that is for all natural numbers i

$$\mathsf{RA}^* \mid\frac{\omega(\alpha+1)\cdot 2+n\cdot 10+15}{<\omega(\alpha+1)} \neg(\bar{i} < \overline{n}), (\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}+1} = TJ((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}}) \tag{21}$$

In a next step we have to show that (21) also holds for $i = n$.
We can prove that if $i = n$ then (for more details compare Appendix A3 and A4).

$$\mathsf{RA}^* \mid\frac{\omega(\alpha+1)\cdot 2+10}{0} (\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}+1} = TJ^{n+1}(S^\alpha) \tag{22}$$

and

$$\mathsf{RA}^* \mid\frac{\omega(\alpha+1)\cdot 2+10}{0} (\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}} = TJ^{n}(S^\alpha) \tag{23}$$

Further we have to distinguish two cases depending on the value of i.

1. For $i = n$ and Lemma 5.2.12 we have with
   $A(U^\alpha) := (\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}+1} = TJ(U^\alpha)$ that

   $$\mathsf{RA}^* \mid\frac{\omega(\alpha+1)\cdot 2+3}{0} \neg((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}} = TJ^n(S^\alpha)), \neg(A(TJ^n(S^\alpha))), A((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}})$$

   Using (22), (23), and the definition that $TJ^{n+1}(S^\alpha) = TJ(TJ^n(S^\alpha))$ we obtain with applying twice the (cut)-inference for $i = n$ that

   $$\mathsf{RA}^* \mid\frac{\omega(\alpha+1)\cdot 2+12}{<\omega(\alpha+1)} (\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}+1} = TJ((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}})$$

   since the level of $\mathcal{R}^{S^\alpha}_{n+1}$ and $TJ^n(S^\alpha)$ is α, and finally we have by Corollary 5.2.6 for $i = n$ that

   $$\mathsf{RA}^* \mid\frac{\omega(\alpha+1)\cdot 2+12}{<\omega(\alpha+1)} \neg(\bar{i} = \overline{n}), (\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}+1} = TJ((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}})$$

2. Further we also have for all $i \neq n$ that

   $$\mathsf{RA}^* \mid\frac{0}{0} \neg(\bar{i} = \overline{n}), (\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}+1} = TJ((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}})$$

Therefore we have for all natural numbers i that

$$\mathsf{RA}^* \vdash^{\omega(\alpha+1)\cdot 2+12}_{<\omega(\alpha+1)} \neg(\bar{i} = \overline{n}), (\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}+1} = TJ((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}}) \tag{24}$$

From (21) and (24) we obtain by the ($\wedge$)-rule of inference for all natural numbers i that

$$\mathsf{RA}^* \vdash^{\omega(\alpha+1)\cdot 2+n\cdot 10+16}_{<\omega(\alpha+1)} \neg(\bar{i} = \overline{n}) \wedge \neg(\bar{i} < \overline{n}), (\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}+1} = TJ((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}}) \tag{25}$$

With $(Ax1)$ we obtain for all natural numbers i that

$$\mathsf{RA}^* \vdash^{0}_{0} \neg(\bar{i} < \overline{n}+1), (\bar{i} = \overline{n}), (\bar{i} < \overline{n})$$

and applying the ($\vee$)-inference twice we obtain

$$\mathsf{RA}^* \vdash^{2}_{0} \neg(\bar{i} < \overline{n}+1), (\bar{i} = \overline{n}) \vee (\bar{i} < \overline{n})$$

and obtain by Corollary 5.2.6 and by a cut with (25) that for all natural numbers i

$$\mathsf{RA}^* \vdash^{\omega(\alpha+1)\cdot 2+n\cdot 10+17}_{<\omega(\alpha+1)} \neg(\bar{i} < \overline{n}+1), (\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}+1} = TJ((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}})$$

and by applying once more the ($\vee$)-inference twice we conclude by Lemma 5.2.7 and the ($\forall^0$)-inference that

$$\mathsf{RA}^* \vdash^{\omega(\alpha+1)\cdot 2+n\cdot 10+20}_{<\omega(\alpha+1)} (\forall x)(x < \overline{n}+1 \rightarrow (\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}+1} = TJ((\mathcal{R}^{S^\alpha}_{n+1})_{\bar{i}}))$$

and obtain together with (14) by applying the ($\wedge$)-inference

$$\mathsf{RA}^* \vdash^{\omega(\alpha+1)\cdot 2+n\cdot 10+21}_{<\omega(\alpha+1)} (\mathcal{R}^{S^\alpha}_{n})_0 = S^\alpha \wedge (\forall x)(x < \overline{n}+1 \rightarrow (\mathcal{R}^{S^\alpha}_{n+1})_{x+1} = TJ((\mathcal{R}^{S^\alpha}_{n+1})_{x}))$$

and therefore the length of derivation is $\omega(\alpha+1)\cdot 2 + (n+1)\cdot 10 + 11$.
This finishes the induction step. □

Lemma 5.2.24 RA^* *proves all* ω*-instances of the* $(\omega - \mathsf{Jump})$*:*

$$\mathsf{RA}^* \vdash^{\omega^2+2}_{<\omega^2} (\exists X^\omega)\mathcal{H}_\omega(S^l, X^\omega)$$

(with $l < \omega$*).*

PROOF: We define a set term T^{l+1} as

$$T^{l+1} := \{\langle a,b\rangle : (\exists Z^{l+1})(\mathcal{H}(S^l, Z^{l+1}, b) \wedge a \in (Z^{l+1})_b)\}$$

with $lev(T^{l+1}) = lev(Z^{l+1}) = l+1 < \omega$.

We can prove that there exists $\beta_0 < \omega^2$ such that for all natural numbers n

$$\mathsf{RA}^* \mid\frac{\beta_0}{<\omega^2} \mathcal{R}_n^{S^l} = (T^{l+1})^{\overline{n}} \tag{26}$$

(for more details compare Appendix A5).

By Lemma 5.2.12 we have that there exists $\beta_1 < \omega^2$ such that for all natural numbers n

$$\mathsf{RA}^* \mid\frac{\beta_1}{0} \neg(\mathcal{R}_n^{S^l} = (T^{l+1})^{\overline{n}}), \neg\mathcal{H}(S^l, \mathcal{R}_n^{S^l}, \overline{n}), \mathcal{H}(S^l, (T^{l+1})^{\overline{n}}, \overline{n})$$

since $lev(\mathcal{H}(S^l, \mathcal{R}_n^{S^l}, \overline{n})) < \omega$ and $lev(\mathcal{H}(S^l, (T^{l+1})^{\overline{n}}, \overline{n})) < \omega$.
With (26) we obtain by a cut that

$$\mathsf{RA}^* \mid\frac{max(\beta_0,\beta_1)+1}{<\omega^2} \neg\mathcal{H}(S^l, \mathcal{R}_n^{S^l}, \overline{n}), \mathcal{H}(S^l, (T^{l+1})^{\overline{n}}, \overline{n})$$

since $rk(\mathcal{R}_n^{S^l} = (T^{l+1})^{\overline{n}}) < \omega^2$.
By Lemma 5.2.23 (with $\alpha = l$) we conclude by applying the (cut)-inference and with $\delta(n) = max(\beta_0, \beta_1, \omega(l+1)\cdot 2 + n\cdot 10 + 11) + 2 < \omega^2$ that

$$\mathsf{RA}^* \mid\frac{\delta(n)}{<\omega^2} \mathcal{H}(S^l, (T^{l+1})^{\overline{n}}, \overline{n}) \tag{27}$$

for all natural numbers n, since also $rk(\mathcal{H}(S^l, \mathcal{R}_n^{S^l}, \overline{n})) < \omega^2$.

Now we can also prove that (27) implies for all n, there exists $\gamma(n) < \omega^2$ with

$$\mathsf{RA}^* \mid\frac{\gamma(n)}{<\omega^2} \mathcal{H}(S^l, T^{l+1}, \overline{n})$$

(for more details compare Appendix A6)
By Lemma 5.2.7 and the $(\forall^0)$-inference we obtain

$$\mathsf{RA}^* \mid\frac{\omega^2}{<\omega^2} (\forall x)\mathcal{H}(S^l, T^{l+1}, x) \tag{28}$$

Further notice that ACA proves the following

$$(\forall x)\mathcal{H}(U,V,x) \leftrightarrow \mathcal{J}_\omega(U,V)$$

and therefore we obtain by Lemma 5.2.20 for an $l+2$-instance (with $l < \omega$)

$$\mathsf{RA}^* \vdash^{<\omega^2}_{<\omega^2} (\forall x)\mathcal{H}(S^l, T^{l+1}, x) \leftrightarrow \mathcal{J}_\omega(S^l, T^{l+1})$$

and with Lemma 5.2.9 and Lemma 5.2.10 also

$$\mathsf{RA}^* \vdash^{<\omega^2}_{<\omega^2} \neg(\forall x)\mathcal{H}(S^l, T^{l+1}, x), \mathcal{J}_\omega(S^l, T^{l+1})$$

and together with (28) we obtain by a cut that

$$\mathsf{RA}^* \vdash^{\omega^2+1}_{<\omega^2} \mathcal{J}_\omega(S^l, T^{l+1})$$

since also $rk((\forall x)\mathcal{H}(S^l, T^{l+1}, x)) < \omega^2$, and finally by an $(\exists^1)$-inference that

$$\mathsf{RA}^* \vdash^{\omega^2+2}_{<\omega^2} (\exists X^\omega)\mathcal{J}_\omega(S^l, X^\omega)$$

since $lev(T^{l+1}) = l+1 < \omega$. □

Theorem 5.2.25 *For all $\mathcal{L}_2$-formulas $A[u_1, \dots, u_n, U_1, \dots, U_m]$ with*

$$\mathsf{ACA}^+ \vdash A[u_1, \dots, u_n, U_1, \dots, U_m]$$

there exists an ordinal $\alpha < \omega^2 \cdot 2 + \omega \cdot 4$ and and a natural number m such that for all ω-instances of A

$$\mathsf{RA}^* \vdash^{\alpha}_{\omega^2+m} A^\omega(r_1, \dots, r_n, S_1^{l_1}, \dots, S_m^{l_m})$$

(with $l_i < \omega$).

Proof: The proof is completely analogous as for Theorem 5.2.20. The only difference is the case if $A[u_1, \dots, u_n, U_1, \dots, U_m]$ is the $(\omega - Jump)$ axiom. But by Lemma 5.2.24 we know that the claim also holds in this case. □

Let $\sqsubset$ be any primitive recursive wellordering. Then we denote the ordertype of $\sqsubset$ with $|\sqsubset|$.

Further we have the following from Schütte [9] Theorem 23.2 and Pohlers [6] Theorem 13.10.

Theorem 5.2.26 *For all primitive recursive wellorderings $\sqsubset$ we have that*

$$\mathsf{RA}^* \vdash^{\delta}_{0} \mathsf{WF}^\beta(\sqsubset) \text{ where } \beta \neq 0 \text{ implies } |\sqsubset| \leq \omega \cdot \delta.$$

Therefore we obtain the following corollary.

Corollary 5.2.27

$$|\mathsf{ACA}^+| \leq \varphi 2\varepsilon_0$$

Proof: Suppose $\mathsf{ACA}^+ \vdash \mathsf{WF}(\sqsubset)$ holds. By Theorem 5.2.25 ACA^+ can be interpreted in RA^* where every formula has a rank $< \omega^2 + \omega$ and the order of inference is restricted by $\omega^2 \cdot 2 + \omega \cdot 4$. Therefore it follows that the formula $\mathsf{WF}^\omega(\sqsubset)$ has a deduction in RA^* for a natural number n with

$$\mathsf{RA}^* \mathrel{\substack{\omega^2\cdot 2+\omega\cdot 4 \\ \vdash \\ \omega^2+n}} \mathsf{WF}^\omega(\sqsubset)$$

It follows by applying the first cut elimination Theorem 5.2.13 n-times that there exists $\delta < \varepsilon_0$ such that

$$\mathsf{RA}^* \mathrel{\substack{\delta \\ \vdash \\ \omega^2}} \mathsf{WF}^\omega(\sqsubset)$$

and then finally by the second cut elimination Theorem 5.2.14 we obtain

$$\mathsf{RA}^* \mathrel{\substack{\varphi 2\delta \\ \vdash \\ 0}} \mathsf{WF}^\omega(\sqsubset)$$

By Theorem 5.2.26 we have $|\sqsubset| \leq \omega \cdot \varphi 2\delta = \varphi 2\delta < \varphi 2\varepsilon_0$. Consequently ACA^+ has a proof-theoretic ordinal $\leq \varphi 2\varepsilon_0$.

□

As a consequence of Corollary 5.1.15, Corollary 5.2.27, and the result of Section 3 we obtain the following theorem.

Theorem 5.2.28

$$|\mathsf{RFN}| = |\mathsf{ACA}^+| = \varphi 2\varepsilon_0$$

6 Additional Results

The notation follows that of subsection 5.1

6.1 The well-ordering proof of $\mathsf{ACA}_0 + (\mathsf{BR})$

Lemma 6.1.1 *We have for all ordinals α,*

$$\mathsf{ACA}_0 + (\mathsf{BR}) \vdash (\forall X)\mathsf{TI}(\overline{\alpha}, X) \qquad \Longrightarrow \qquad \mathsf{ACA}_0 + (\mathsf{BR}) \vdash (\forall X)\mathsf{TI}(\hat{\varepsilon}_{\overline{\alpha}}, X)$$

Proof: First notice that $\mathsf{ACA}_0 + (\mathsf{BR})$ proves every instance of the full second order induction scheme (IND). Therefore the theories $\mathsf{ACA}_0 + (\mathsf{BR})$ and $\mathsf{ACA} + (\mathsf{BR})$ are equivalent.
We have that $\mathsf{ACA}_0 + (\mathsf{BR}) \vdash (\forall X)\mathsf{TI}(\overline{\alpha}, X)$ and can conclude by (BR) that $\mathsf{ACA}_0 + (\mathsf{BR}) \vdash \mathsf{TI}(\overline{\alpha}, F)$ for all $\mathcal{L}_2$-formulas F. We define $F(a)$ to be the formula $(\forall X)\mathsf{TI}(\hat{\varepsilon}_a, X)$. Hence we have

$$\mathsf{ACA}_0 + (\mathsf{BR}) \vdash \mathsf{PROG}(\prec, F) \to (\forall x \prec \overline{\alpha})F(x) \tag{29}$$

Further $\mathsf{ACA}_0 + (\mathsf{BR})$ proves

$$\mathsf{ACA}_0 + (\mathsf{BR}) \vdash \mathsf{PROG}(\prec, F) \to ((\forall x \prec \overline{\alpha})F(x) \to F(\overline{\alpha})) \tag{30}$$

By Schütte [9] Lemma 21.7. we know that F is progressive and hence we can conclude from (29) and (30) that $\mathsf{ACA}_0 + (\mathsf{BR}) \vdash F(\overline{\alpha})$ which is

$$\mathsf{ACA}_0 + (\mathsf{BR}) \vdash (\forall X)\mathsf{TI}(\hat{\varepsilon}_{\overline{\alpha}}, X)$$

□

Theorem 6.1.2 $\mathsf{ACA}_0 + (\mathsf{BR})$ *proves the formula $\mathcal{I}(\overline{\alpha})$ for all $\alpha < \varphi 20$.*

Proof: The proof is analogous as that of Theorem 5.1.10, only instead of applying Lemma 5.1.9 we use in this case here Lemma 6.1.1. □

Corollary 6.1.3 *For a lower bound of the proof-theoretic ordinal of* $\mathsf{ACA}_0 + (\mathsf{BR})$ *we have*

$$|\mathsf{ACA}_0 + (\mathsf{BR})| \geq \varphi 20$$

7 Appendix

7.1 Details of the proof of Lemma 5.2.23

A1:

We have to show that the following holds.

$$\mathsf{RA}^* \mid\!\frac{\omega(\alpha+1)\cdot 2+10}{0}\quad (\forall x)(\langle x,0\rangle \in \{y:(\exists z_1)(\exists z_2)(y=\langle z_1,z_2\rangle \wedge z_2=0 \wedge z_1 \in S^\alpha)\} \leftrightarrow x \in S^\alpha) \wedge (\forall x<\overline{0})((\mathcal{R}_0^{S^\alpha})_{x+1} = TJ((\mathcal{R}_0^{S^\alpha})_x))$$

By Lemma 5.2.11 and Axiom $(Ax1)$, with $lev(s \in S^\alpha)=\alpha$, and therefore $rk(s \in S^\alpha) < \omega(\alpha+1)$, we have for all number terms s that

$$\mathsf{RA}^* \mid\!\frac{\omega(\alpha+1)\cdot 2}{0}\ s \notin S^\alpha, s \in S^\alpha \qquad \mathsf{RA}^* \mid\!\frac{0}{0}\ \langle s,0\rangle = \langle s,0\rangle \qquad \mathsf{RA}^* \mid\!\frac{0}{0}\ 0=0$$

and obtain by applying the $(\wedge)$-inference twice that

$$\mathsf{RA}^* \mid\!\frac{\omega(\alpha+1)\cdot 2+2}{0}\ s \notin S^\alpha, \langle s,0\rangle = \langle s,0\rangle \wedge 0=0 \wedge s \in S^\alpha$$

By an $(\exists^0)$-inference we have

$$\mathsf{RA}^* \mid\!\frac{\omega(\alpha+1)\cdot 2+3}{0}\ s \notin S^\alpha, (\exists z_2)(\langle s,0\rangle = \langle s,z_2\rangle \wedge z_2=0 \wedge s \in S^\alpha)$$

and obtain, again by an $(\exists^0)$-inference

$$\mathsf{RA}^* \mid\!\frac{\omega(\alpha+1)\cdot 2+4}{0}\ s \notin S^\alpha, (\exists z_1)(\exists z_2)(\langle s,0\rangle = \langle z_1,z_2\rangle \wedge z_2=0 \wedge z_1 \in S^\alpha)$$

By an $(\in 1)$-inference we have

$$\mathsf{RA}^* \mid\!\frac{\omega(\alpha+1)\cdot 2+5}{0}\ s \notin S^\alpha, \langle s,0\rangle \in \{y:(\exists z_1)(\exists z_2)(y=\langle z_1,z_2\rangle \wedge z_2=0 \wedge z_1 \in S^\alpha)\}$$

and by applying the $(\vee)$-inference twice

$$\mathsf{RA}^* \mid\!\frac{\omega(\alpha+1)\cdot 2+7}{0}\ s \notin S^\alpha \rightarrow \langle s,0\rangle \in \{y:(\exists z_1)(\exists z_2)(y=\langle z_1,z_2\rangle \wedge z_2=0 \wedge z_1 \in S^\alpha)\} \tag{31}$$

Further we have to distinguish two cases depending on the values of the (arbitrary) number terms s and t.

1. If s and t have the same value, then we obtain from Lemma 5.2.11 that
$$\mathsf{RA}^* \mathrel{\vdash\!\!\frac{\omega(\alpha+1)\cdot 2}{0}} t \notin S^\alpha, s \in S^\alpha$$
and by applying an $(\vee)$-inference twice we have for all number terms r that
$$\mathsf{RA}^* \mathrel{\vdash\!\!\frac{\omega(\alpha+1)\cdot 2+2}{0}} \langle s,0\rangle \neq \langle t,r\rangle \vee r \neq 0 \vee t \notin S^\alpha, s \in S^\alpha$$

2. If s and t have not the same value then we have from Axiom $(Ax1)$ that for all number terms r
$$\mathsf{RA}^* \mathrel{\vdash\!\!\frac{0}{0}} \langle s,0\rangle \neq \langle t,r\rangle, s \in S^\alpha$$
and obtain by applying an $(\vee)$-inference twice that
$$\mathsf{RA}^* \mathrel{\vdash\!\!\frac{2}{0}} \langle s,0\rangle \neq \langle t,r\rangle \vee r \neq 0 \vee t \notin S^\alpha, s \in S^\alpha$$

Hence we conclude with an $(\forall^0)$-inference that
$$\mathsf{RA}^* \mathrel{\vdash\!\!\frac{\omega(\alpha+1)\cdot 2+3}{0}} (\forall z_2)(\langle s,0\rangle \neq \langle t,z_2\rangle \vee z_2 \neq 0 \vee t \notin S^\alpha), s \in S^\alpha$$
and by another $(\forall^0)$-inference
$$\mathsf{RA}^* \mathrel{\vdash\!\!\frac{\omega(\alpha+1)\cdot 2+4}{0}} (\forall z_1)(\forall z_2)(\langle s,0\rangle \neq \langle z_1,z_2\rangle \vee z_2 \neq 0 \vee z_1 \notin S^\alpha), s \in S^\alpha$$
but this is
$$\mathsf{RA}^* \mathrel{\vdash\!\!\frac{\omega(\alpha+1)\cdot 2+4}{0}} \neg(\exists z_1)(\exists z_2)(\langle s,0\rangle = \langle z_1,z_2\rangle \wedge z_2 = 0 \wedge z_1 \in S^\alpha), s \in S^\alpha$$
and so we have by an $(\in 2)$-inference that
$$\mathsf{RA}^* \mathrel{\vdash\!\!\frac{\omega(\alpha+1)\cdot 2+5}{0}} \langle s,0\rangle \notin \{y : (\exists z_1)(\exists z_2)(y = \langle z_1,z_2\rangle \wedge z_2 = 0 \wedge z_1 \in S^\alpha)\}, s \in S^\alpha$$
and finally by applying the $(\vee)$-inference twice
$$\mathsf{RA}^* \mathrel{\vdash\!\!\frac{\omega(\alpha+1)\cdot 2+7}{0}} \langle s,0\rangle \in \{y : (\exists z_1)(\exists z_2)(y = \langle z_1,z_2\rangle \wedge z_2 = 0 \wedge z_1 \in S^\alpha)\} \rightarrow s \in S^\alpha$$

Together with (31) we obtain by an $(\wedge)$-inference
$$\mathsf{RA}^* \mathrel{\vdash\!\!\frac{\omega(\alpha+1)\cdot 2+8}{0}} \langle s,0\rangle \in \{y : (\exists z_1)(\exists z_2)(y = \langle z_1,z_2\rangle \wedge z_2 = 0 \wedge z_1 \in S^\alpha)\} \leftrightarrow s \in S^\alpha$$
and finally obtain by an $(\forall^0)$-inference that
$$\mathsf{RA}^* \mathrel{\vdash\!\!\frac{\omega(\alpha+1)\cdot 2+9}{0}} (\forall x)(\langle x,0\rangle \in \{y : (\exists z_1)(\exists z_2)(y = \langle z_1,z_2\rangle \wedge z_2 = 0 \wedge z_1 \in S^\alpha)\} \leftrightarrow x \in S^\alpha\}) \tag{32}$$

Moreover we have for all number terms t

$$\mathsf{RA}^* \vdash_0^0 \neg(t < 0), (\mathcal{R}_0^{S^\alpha})_{\overline{n}+1} = TJ((\mathcal{R}_0^{S^\alpha})_{\overline{n}})$$

With ($\vee$1), ($\vee$2) and the ($\forall^0$)-inference we obtain

$$\mathsf{RA}^* \vdash_0^3 (\forall x)(x < 0 \rightarrow (\mathcal{R}_0^{S^\alpha})_{\overline{n}+1} = TJ((\mathcal{R}_0^{S^\alpha})_{\overline{n}}))$$

Together with (32) we obtain the claim by an ($\wedge$)-inference.

A2:

The proof is very similar to A3.

A3:

The proof is also similar to the one of A1. We have to prove that for $\mathbf{i} = \mathbf{n}$

$$\mathsf{RA}^* \vdash_0^{\omega(\alpha+1)\cdot 2+10} (\mathcal{R}_{n+1}^{S^\alpha})_{\overline{i}+1} = TJ^{n+1}(S^\alpha)$$

which is

$$\begin{aligned}(\forall x)(\quad & x \in TJ^{n+1}(S^\alpha) \leftrightarrow \\ & \langle x, \overline{i}+1\rangle \in \{y : (\exists z_1)(\exists z_2)(y = \langle z_1, z_2\rangle \wedge ((z_2 = 0 \wedge z_1 \in S^\alpha) \vee \\ & (z_2 = 1 \wedge z_1 \in TJ(S^\alpha)) \vee \ldots \vee (z_2 = \overline{n}+1 \wedge z_1 \in TJ^{n+1}(S^\alpha))))\})\end{aligned}$$

"$\rightarrow$":

From Lemma 5.2.11 and Axiom $(Ax1)$ respectively we obtain

$$\mathsf{RA}^* \vdash_0^{\omega(\alpha+1)\cdot 2} s \notin TJ^{n+1}(S^\alpha), s \in TJ^{n+1}(S^\alpha) \qquad \mathsf{RA}^* \vdash_0^0 \overline{i}+1 = \overline{n}+1$$

since $lev(TJ^{n+1}(S^\alpha)) = \alpha$ and $i = n$. Hence we obtain by an ($\wedge$)-inference that

$$\mathsf{RA}^* \vdash_0^{\omega(\alpha+1)\cdot 2+1} s \notin TJ^{n+1}(S^\alpha), \overline{i}+1 = \overline{n}+1 \wedge s \in TJ^{n+1}(S^\alpha)$$

and by an ($\vee$)-rule of inference we get

$$\begin{aligned}\mathsf{RA}^* \vdash_0^{\omega(\alpha+1)\cdot 2+2} \quad & s \notin TJ^{n+1}(S^\alpha), \overline{i}+1 = 0 \wedge s \in S^\alpha \vee \\ & \overline{i}+1 = 1 \wedge s \in TJ(S^\alpha) \vee \ldots \vee \overline{i}+1 = \overline{n}+1 \wedge s \in TJ^{n+1}(S^\alpha)\end{aligned}$$

Further we have by Axiom $(Ax1)$

$$\mathsf{RA}^* \vdash_0^0 \langle s, \bar{i}+1\rangle = \langle s, \bar{i}+1\rangle$$

Therefore we conclude by an $(\wedge)$-inference that

$$\begin{array}{ll}\mathsf{RA}^* \vdash_0^{\omega(\alpha+1)\cdot 2+3} & s \notin TJ^{n+1}(S^\alpha), \\ & \langle s, \bar{i}+1\rangle = \langle s, \bar{i}+1\rangle \wedge (\bar{i}+1 = 0 \wedge s \in S^\alpha \vee \\ & \bar{i}+1 = 1 \wedge s \in TJ(S^\alpha) \vee \ldots \vee \bar{i}+1 = \overline{n}+1 \wedge s \in TJ^{n+1}(S^\alpha))\end{array}$$

By applying twice the $(\exists^0)$-inference we get

$$\begin{array}{ll}\mathsf{RA}^* \vdash_0^{\omega(\alpha+1)\cdot 2+5} & s \notin TJ^{n+1}(S^\alpha), \\ & (\exists z_1)(\exists z_2)(\langle s, \bar{i}+1\rangle = \langle z_1, z_2\rangle \wedge (z_2 = 0 \wedge z_1 \in S^\alpha \vee \\ & z_2 = 1 \wedge z_1 \in TJ(S^\alpha) \vee \ldots \vee z_2 = \overline{n}+1 \wedge z_1 \in TJ^{n+1}(S^\alpha)))\end{array}$$

And finally by an $(\in 1)$- and two $(\vee)$-inferences we have

$$\begin{array}{ll}\mathsf{RA}^* \vdash_0^{\omega(\alpha+1)\cdot 2+8} & s \in TJ^{n+1}(S^\alpha) \to \langle s, \bar{i}+1\rangle \in \{y : \\ & (\exists z_1)(\exists z_2)(y = \langle z_1, z_2\rangle \wedge (z_2 = 0 \wedge z_1 \in S^\alpha \vee \\ & z_2 = 1 \wedge z_1 \in TJ(S^\alpha) \vee \ldots \vee z_2 = \overline{n}+1 \wedge z_1 \in TJ^{n+1}(S^\alpha)))\}\end{array} \tag{33}$$

<u>"$\leftarrow$":</u>

We have to distinguish several cases depending on the value of the (arbitrary) number terms s, t and r.

- If s and t have not the same value or the value of r is not $i+1$ then we have the following Axiom $(Ax1)$.

  $$\mathsf{RA}^* \vdash_0^0 \langle s, \bar{i}+1\rangle \neq \langle t, r\rangle, s \in TJ^{n+1}(S^\alpha)$$

  and obtain by an $(\vee)$-inference that

  $$\begin{array}{ll}\mathsf{RA}^* \vdash_0^1 & s \in TJ^{n+1}(S^\alpha), \\ & \langle s, \bar{i}+1\rangle \neq \langle t, r\rangle \vee ((r \neq 0 \vee t \notin S^\alpha) \wedge \\ & (r \neq 1 \vee t \notin TJ(S^\alpha)) \wedge \ldots \wedge (r \neq \overline{n}+1 \vee t \notin TJ^{n+1}(S^\alpha)))\end{array}$$

- If s and t have the same value and the value of r is $i+1$ then we have the following by Axiom $(Ax1)$.

$$\mathsf{RA}^* \vdash_0^0 r \neq 0 \qquad \mathsf{RA}^* \vdash_0^0 r \neq 1 \qquad \dots \qquad \mathsf{RA}^* \vdash_0^0 r \neq \overline{n}$$

and obtain by an $(\vee)$-inference

$$\mathsf{RA}^* \vdash_0^1 r \neq 0 \vee t \notin S^\alpha \qquad \dots \qquad \mathsf{RA}^* \vdash_0^1 r \neq \overline{n} \vee t \notin TJ^n(S^\alpha)$$

and by applying the $(\wedge)$-inference n times

$$\mathsf{RA}^* \vdash_0^{n+1} (r \neq 0 \vee t \notin S^\alpha) \wedge \dots (r \neq \overline{n} \vee t \notin TJ^n(S^\alpha)) \tag{34}$$

Further we have by Lemma 5.2.11

$$\mathsf{RA}^* \vdash_0^{\omega(\alpha+1)\cdot 2} t \notin TJ^{n+1}(S^\alpha), s \in TJ^{n+1}(S^\alpha)$$

since $lev(s \in TJ^{n+1}(S^\alpha)) = \alpha$ and obtain by an $(\vee)$-inference

$$\mathsf{RA}^* \vdash_0^{\omega(\alpha+1)\cdot 2+1} r \neq \overline{n}+1 \vee t \notin TJ^{n+1}(S^\alpha), s \in TJ^{n+1}(S^\alpha)$$

Together with (34) we obtain by an $(\wedge)$-inference

$$\begin{aligned} \mathsf{RA}^* \vdash_0^{\omega(\alpha+1)\cdot 2+2} \quad & s \in TJ^{n+1}(S^\alpha), \\ & (r \neq 0 \vee t \notin S^\alpha) \wedge \quad (r \neq 1 \vee t \notin TJ(S^\alpha)) \wedge \\ & \qquad \dots \\ & \qquad (r \neq \overline{n}+1 \vee t \notin TJ^{n+1}(S^\alpha))) \end{aligned}$$

and by an $(\vee)$-inference this yields

$$\begin{aligned} \mathsf{RA}^* \vdash_0^{\omega(\alpha+1)\cdot 2+3} \quad & s \in TJ^{n+1}(S^\alpha), \\ & \langle s, \overline{i}+1 \rangle \neq \langle t, r \rangle \vee ((r \neq 0 \vee t \notin S^\alpha) \wedge \\ & (r \neq 1 \vee t \notin TJ(S^\alpha)) \wedge \dots \wedge (r \neq \overline{n}+1 \vee t \notin TJ^{n+1}(S^\alpha))) \end{aligned}$$

So we conclude by two $(\forall^0)$-inferences

$$\begin{aligned} \mathsf{RA}^* \vdash_0^{\omega(\alpha+1)\cdot 2+5} \quad & s \in TJ^{n+1}(S^\alpha), \\ & (\forall z_1)(\forall z_2)(\langle s, \overline{i}+1 \rangle \neq \langle z_1, z_2 \rangle \vee ((z_2 \neq 0 \vee z_1 \notin S^\alpha) \wedge \\ & (z_2 \neq 1 \vee z_1 \notin TJ(S^\alpha)) \wedge \dots \wedge (z_2 \neq \overline{n}+1 \vee z_1 \notin TJ^{n+1}(S^\alpha)))) \end{aligned}$$

but this is

$$\mathsf{RA}^* \vdash^{\omega(\alpha+1)\cdot 2+5}_{0} \quad s \in TJ^{n+1}(S^\alpha), \neg(\exists z_1)(\exists z_2)(\langle s, \bar{i}+1\rangle = \langle z_1, z_2\rangle \wedge (z_2 = 0 \wedge z_1 \in S^\alpha \vee z_2 = 1 \wedge z_1 \in TJ(S^\alpha) \vee \ldots \vee z_2 = \overline{n}+1 \wedge z_1 \in TJ^{\overline{n}+1}(S^\alpha)))$$

and obtain by ($\in$ 2) and two ($\vee$)-inferences

$$\mathsf{RA}^* \vdash^{\omega(\alpha+1)\cdot 2+8}_{0} \quad \langle s, \bar{i}+1\rangle \in \{y : (\exists z_1)(\exists z_2)(y = \langle z_1, z_2\rangle \wedge (z_2 = 0 \wedge z_1 \in S^\alpha \vee z_2 = 1 \wedge z_1 \in TJ(S^\alpha) \vee \ldots \vee z_2 = \overline{n}+1 \wedge z_1 \in TJ^{n+1}(S^\alpha)))\} \rightarrow s \in TJ^{n+1}(S^\alpha)$$

So we obtain together with (33) by an ($\wedge$)-inference that

$$\mathsf{RA}^* \vdash^{\omega(\alpha+1)\cdot 2+9}_{0} \quad s \in TJ^{n+1}(S^\alpha) \leftrightarrow \langle s, \bar{i}+1\rangle \in \{y : (\exists z_1)(\exists z_2)(y = \langle z_1, z_2\rangle \wedge (z_2 = 0 \wedge z_1 \in S^\alpha \vee z_2 = 1 \wedge z_1 \in TJ(S^\alpha) \vee \ldots \vee z_2 = \overline{n}+1 \wedge z_1 \in TJ^{n+1}(S^\alpha)))\}$$

and finally by an ($\forall^0$)-inference

$$\mathsf{RA}^* \vdash^{\omega(\alpha+1)\cdot 2+10}_{0} \quad (\forall x)(x \in TJ^{n+1}(S^\alpha) \leftrightarrow \langle x, \bar{i}+1\rangle \in \{y : (\exists z_1)(\exists z_2)(y = \langle z_1, z_2\rangle \wedge (z_2 = 0 \wedge z_1 \in S^\alpha \vee z_2 = 1 \wedge z_1 \in TJ(S^\alpha) \vee \ldots \vee z_2 = \overline{n}+1 \wedge z_1 \in TJ^{n+1}(S^\alpha)))\}$$

A4:

The proof is also completely analogous to A3.

7.2 Details of the proof of Lemma 5.2.24

A5:

First notice that ACA proves the following for all natural numbers n

$$a = \langle b, c\rangle \wedge c \leq \overline{n} \wedge (\exists Z)(\mathcal{H}(U, Z, c) \wedge b \in (Z)_c) \leftrightarrow a = \langle b, c\rangle \wedge (\textstyle\bigvee_{i=0}^{n} c = \bar{i} \wedge b \in TJ^i(U))$$

Therefore using Theorem 5.2.20 we obtain $\eta_0 < \omega^2$ such that for an $l+1$-instance (with $l < \omega$) and all natural numbers n

$$\mathsf{RA}^* \vdash^{\eta_0}_{<\omega^2} \quad r = \langle s, t\rangle \wedge t \leq \overline{n} \wedge (\exists Z^{l+1})((S^l, Z^{l+1}, t) \wedge s \in (Z^{l+1})_t) \leftrightarrow r = \langle s, t\rangle \wedge (\textstyle\bigvee_{i=0}^{n} t = \bar{i} \wedge s \in TJ^i(S^l))$$

Together with Lemma 5.2.9 and Lemma 5.2.10 we obtain that

$$\mathsf{RA}^* \overset{\eta_0}{\underset{<\omega^2}{\vdash}} \neg(r = \langle s,t\rangle \wedge t \le \overline{n} \wedge (\exists Z^{l+1})((S^l, Z^{l+1}, t) \wedge s \in (Z^{l+1})_t)), \quad r = \langle s,t\rangle \wedge (\bigvee_{i=0}^{n} t = \overline{i} \wedge s \in TJ^i(S^l))$$

$$\mathsf{RA}^* \overset{\eta_0}{\underset{<\omega^2}{\vdash}} r = \langle s,t\rangle \wedge t \le \overline{n} \wedge (\exists Z^{l+1})((S^l, Z^{l+1}, t) \wedge s \in (Z^{l+1})_t), \quad \neg(r = \langle s,t\rangle \wedge (\bigvee_{i=0}^{n} t = \overline{i} \wedge s \in TJ^i(S^l)))$$

Hence we obtain by $(\in 1)$ and $(\in 2)$ that for all natural numbers n

$$\mathsf{RA}^* \overset{\eta_0+2}{\underset{<\omega^2}{\vdash}} r \notin \{x : x = \langle s,t\rangle \wedge t \le \overline{n} \wedge (\exists Z^{l+1})((S^l, Z^{l+1}, t) \wedge s \in (Z^{l+1})_t)\}, \quad r \in \{x : x = \langle s,t\rangle \wedge (\bigvee_{i=0}^{n} t = \overline{i} \wedge s \in TJ^i(S^l))\}$$

$$\mathsf{RA}^* \overset{\eta_0+2}{\underset{<\omega^2}{\vdash}} r \in \{x : x = \langle s,t\rangle \wedge t \le \overline{n} \wedge (\exists Z^{l+1})((S^l, Z^{l+1}, t) \wedge s \in (Z^{l+1})_t)\}, \quad r \notin \{x : x = \langle s,t\rangle \wedge (\bigvee_{i=0}^{n} t = \overline{i} \wedge s \in TJ^i(S^l))\}$$

and by the $(\vee 1), (\vee 2)$ and $(\wedge)$-inference

$$\mathsf{RA}^* \overset{\eta_0+5}{\underset{<\omega^2}{\vdash}} r \in \{x : x = \langle s,t\rangle \wedge t \le \overline{n} \wedge (\exists Z^{l+1})((S^l, Z^{l+1}, t) \wedge s \in (Z^{l+1})_t)\} \leftrightarrow r \in \{x : x = \langle s,t\rangle \wedge \bigvee_{t=0}^{n} TJ^t(S)\}$$

Since r is an arbitrary closed number term, we finally obtain by Lemma 5.2.7 and the $(\forall^0)$-inference that

$$\mathsf{RA}^* \overset{\eta_0+6}{\underset{<\omega^2}{\vdash}} (\forall z)(\; z \in \{x : x = \langle s,t\rangle \wedge t \le \overline{n} \wedge (\exists Z^{l+1})((S^l, Z^{l+1}, t) \wedge s \in (Z^{l+1})_t)\} \leftrightarrow z \in \{x : x = \langle s,t\rangle \wedge (\bigvee_{i=0}^{n} t = \overline{i} \wedge s \in TJ^i(S^l))\})$$

and this is the assertion (since $\eta_0 + 6 < \omega^2$).

<u>A6:</u>

With the definition of $\mathcal{H}$ we obtain from (27) that

$$\mathsf{RA}^* \overset{\delta(n)}{\underset{<\omega^2}{\vdash}} ((T^{l+1})^{\overline{n}})_0 = S^l \wedge (\forall x < \overline{n})(((T^{l+1})^{\overline{n}})_{x+1} = TJ(((T^{l+1})^{\overline{n}})_x))$$

By Lemma 5.2.9 we have

$$\mathsf{RA}^* \overset{\delta(n)}{\underset{<\omega^2}{\vdash}} ((T^{l+1})^{\overline{n}})_0 = S^l \tag{35}$$

$$\mathsf{RA}^* \overset{\delta(n)}{\underset{<\omega^2}{\vdash}} (\forall x < \overline{n})(((T^{l+1})^{\overline{n}})_{x+1} = TJ(((T^{l+1})^{\overline{n}})_x))$$

and by Lemma 5.2.8 and Lemma 5.2.10 we obtain for all natural numbers i

$$\mathsf{RA}^* \overset{\delta(n)}{\underset{<\omega^2}{\vdash}} \neg(\overline{i} < \overline{n}), ((T^{l+1})^{\overline{n}})_{\overline{i}+1} = TJ(((T^{l+1})^{\overline{n}})_{\overline{i}}) \tag{36}$$

Using Lemma 5.2.12 we obtain $\eta_0 < \omega^2$ with

$$\mathsf{RA}^* \vdash^{\eta_0}_{0} \neg(((T^{l+1})^{\overline{n}})_0 = (T^{l+1})_0), \neg(((T^{l+1})^{\overline{n}})_0 = S^l), (T^{l+1})_0 = S^l \tag{37}$$

since $lev(((T^{l+1})^{\overline{n}})_0 = S^l) = l < \omega$.
It is not difficult to prove that there exists $\eta_1 < \omega^2$ such that

$$\mathsf{RA}^* \vdash^{\eta_1}_{0} ((T^{l+1})^{\overline{n}})_0 = (T^{l+1})_0$$

(for more details compare Appendix A7).
Together with (35) we obtain from (37) by two cuts that

$$\mathsf{RA}^* \vdash^{max(\eta_0,\eta_1,\delta(n))+1}_{<\omega^2} (T^{l+1})_0 = S^l \tag{38}$$

since clearly $rk((T^{l+1})^{\overline{n}})_0 = (T^{l+1})_0) < \omega^2$ and also $rk(((T^{l+1})^{\overline{n}})_0 = S^l) < \omega^2$.

Again we have to distinguish two cases depending on the value of i.

1. We have for all $i \geq n$ that

$$\mathsf{RA}^* \vdash^{0}_{0} \neg(i < \overline{n}), (T^{l+1})_{\overline{i}+1} = TJ((T^{l+1})_{\overline{i}})$$

2. Using Lemma 5.2.12 we have $\eta_0 < \omega^2$ for all $i < n$, since $lev(((T^{l+1})^{\overline{n}})_{\overline{i}+1} = TJ(((T^{l+1})^{\overline{n}})_{\overline{i}})) = l < \omega$, such that

$$\begin{aligned}\mathsf{RA}^* \vdash^{\eta_0}_{0} \quad & \neg(((T^{l+1})^{\overline{n}})_{\overline{i}} = (T^{l+1})_{\overline{i}}), \neg(((T^{l+1})^{\overline{n}})_{\overline{i}+1} = TJ(((T^{l+1})^{\overline{n}})_{\overline{i}})),\\ & ((T^{l+1})^{\overline{n}})_{\overline{i}+1} = TJ((T^{l+1})_{\overline{i}})\end{aligned}$$

and also again $\eta_1 < \omega^2$ for all $i < n$ such that

$$\mathsf{RA}^* \vdash^{\eta_1}_{0} ((T^{l+1})^{\overline{n}})_{\overline{i}} = (T^{l+1})_{\overline{i}}$$

(again for more details compare Appendix A7).
So we conclude by using the (cut)-inference

$$\begin{aligned}\mathsf{RA}^* \vdash^{max(\eta_0,\eta_1)+1}_{<\omega^2} \quad & \neg(\overline{i} < \overline{n}), \neg(((T^{l+1})^{\overline{n}})_{\overline{i}+1} = TJ(((T^{l+1})^{\overline{n}})_{\overline{i}})),\\ & ((T^{l+1})^{\overline{n}})_{\overline{i}+1} = TJ((T^{l+1})_{\overline{i}})\end{aligned}$$

since $rk(((T^{l+1})^{\overline{n}})_{\overline{i}} = (T^{l+1})_{\overline{i}}) < \omega^2$. and by (36) and another cut we obtain

$$\mathsf{RA}^* \vdash^{max(\eta_0,\eta_1,\delta(n))+2}_{<\omega^2} \neg(\overline{i} < \overline{n}), ((T^{l+1})^{\overline{n}})_{\overline{i}+1} = TJ((T^{l+1})_{\overline{i}}) \tag{39}$$

since also $rk(((T^{l+1})^{\overline{n}})_{\overline{i}+1} = TJ(((T^{l+1})^{\overline{n}})_{\overline{i}})) < \omega^2$.
Using again Lemma 5.2.12 we obtain $\eta_0 < \omega^2$ for all $i < n$, since $lev(((T^{l+1})^{\overline{n}})_{\overline{i+1}} = TJ(((T^{l+1})^{\overline{n}})_{\overline{i}})) = l < \omega$, such that

$$\mathsf{RA}^* \vdash_0^{\eta_0} \begin{array}{l} \neg(((T^{l+1})^{\overline{n}})_{\overline{i}+1} = (T^{l+1})_{\overline{i}+1}), \neg(((T^{l+1})^{\overline{n}})_{\overline{i}+1} = TJ((T^{l+1})_{\overline{i}})), \\ ((T^{l+1})^{\overline{n}})_{\overline{i}+1} = TJ((T^{l+1})_{\overline{i}}) \end{array} \tag{40}$$

and also $\eta_1 < \omega^2$ for all $i < n$ such that

$$\mathsf{RA}^* \vdash_0^{\eta_1} ((T^{l+1})^{\overline{n}})_{\overline{i}+1} = (T^{l+1})_{\overline{i}+1} \tag{41}$$

(again for more details compare Appendix A7).
So we conclude by two cuts from (40) with (39) and (41) that

$$\mathsf{RA}^* \vdash_{<\omega^2}^{max(\eta_0,\eta_1,\delta(n))+4} \neg(\overline{i} < \overline{n}), (T^{l+1})_{\overline{i}+1} = TJ((T^{l+1})_{\overline{i}})$$

So we obtain by two $(\vee)$-inferences that for all natural numbers i

$$\mathsf{RA}^* \vdash_{<\omega^2}^{max(\eta_1,\eta_2,\delta(n))+6} \overline{i} < \overline{n} \rightarrow (T^{l+1})_{\overline{i}+1} = TJ((T^{l+1})_{\overline{i}})$$

and finally by an $(\forall^0)$-inference that yields

$$\mathsf{RA}^* \vdash_{<\omega^2}^{\max(\eta_1,\eta_2,\delta(n))+7} (\forall x < \overline{n})((T^{l+1})_{\overline{x}+1} = TJ((T^{l+1})_{\overline{x}}))$$

and together with (38) and an $(\wedge)$-inference

$$\mathsf{RA}^* \vdash_{<\omega^2}^{max(\beta_1,\beta_2,\delta(n))+8} (T^{l+1})_0 = S^l \wedge (\forall x < \overline{n})((T^{l+1})_{\overline{x}+1} = TJ((T^{l+1})_{\overline{x}}))$$

and hence, this is for all n, with $\gamma(n) = max(\beta_1, \beta_2, \delta(n)) + 8 < \omega^2$

$$\mathsf{RA}^* \vdash_{\omega^2}^{\gamma(n)} \mathcal{H}(S^l, T^{l+1}, \overline{n})$$

<u>A7:</u>

We want to prove that $((T^{l+1})^{\overline{n}})_{\overline{i}} = (T^{l+1})_{\overline{i}}$ holds for all natural numbers n and all natural numbers $i \leq n$. That is

$$\mathsf{RA}^* \vdash_0^{<\omega^2} (\forall x)(\begin{array}{l} \langle x, \overline{i} \rangle \in \{y : (\exists z_1)(\exists z_2)(y = \langle z_1, z_2 \rangle \wedge z_2 \leq \overline{n} \wedge z_1 \in T^{l+1})\} \leftrightarrow \\ \langle x, \overline{i} \rangle \in \{y : (\exists z_1)(\exists z_2)(y = \langle z_1, z_2 \rangle \wedge z_1 \in T^{l+1})\}) \end{array}$$

<u>"$\leftarrow$":</u>

We again have to distinguish several cases depending on the value of the (arbitrary) number terms s, t and r.

- If s and t have not the same value or the value of r is not i then we have the following from Axiom $(Ax1)$.

$$\mathsf{RA}^* \mid\!\frac{0}{0}\ \langle s, \bar{i}\rangle \neq \langle t, r\rangle, \langle s, \bar{i}\rangle = \langle t, r\rangle \wedge r \leq \overline{n} \wedge t \in T^{l+1}$$

and obtain by an $(\vee)$-inference that

$$\mathsf{RA}^* \mid\!\frac{1}{0}\ \langle s, \bar{i}\rangle \neq \langle t, r\rangle \vee t \notin T^{l+1}, \langle s, \bar{i}\rangle = \langle t, r\rangle \wedge r \leq \overline{n} \wedge t \in T^{l+1}$$

- If s and t have the same value and the value of r is i (and $i \leq n$) then we have the following Axioms $(Ax1)$.

$$\mathsf{RA}^* \mid\!\frac{0}{0}\ \langle s, \bar{i}\rangle = \langle t, r\rangle \qquad \mathsf{RA}^* \mid\!\frac{0}{0}\ r \leq \overline{n}$$

and obtain with an $(\wedge)$-inference that

$$\mathsf{RA}^* \mid\!\frac{1}{0}\ \langle s, \bar{i}\rangle = \langle t, r\rangle \wedge r \leq \overline{n} \tag{42}$$

Further we have from Lemma 5.2.11 that there exists $\eta_0 < \omega^2$ such that for all number terms t

$$\mathsf{RA}^* \mid\!\frac{\eta_0}{0}\ t \in T^{l+1}, t \notin T^{l+1}$$

since $lev(t \in T^{l+1}) = l + 1$ and therefore $rk(t \in T^{l+1}) < \omega^2$.

So we obtain by an $(\wedge)$-inference using (42) that

$$\mathsf{RA}^* \mid\!\frac{\eta_0+1}{0}\ \langle s, \bar{i}\rangle = \langle t, r\rangle \wedge r \leq \overline{n} \wedge t \in T^{l+1}, t \notin T^{l+1}$$

and by an $(\vee)$-inference

$$\mathsf{RA}^* \mid\!\frac{\eta_0+2}{0}\ \langle s, \bar{i}\rangle = \langle t, r\rangle \wedge r \leq \overline{n} \wedge t \in T^{l+1}, \langle s, \bar{i}\rangle \neq \langle t, r\rangle \vee t \notin T^{l+1}$$

So we conclude by using the $(\exists^0)$-inference twice that

$$\mathsf{RA}^* \mid\!\frac{\eta_0+4}{0}\ (\exists z_1)(\exists z_2)(\langle s, \bar{i}\rangle = \langle z_1, z_2\rangle \wedge z_2 \leq \overline{n} \wedge z_1 \in T^{l+1}), \langle s, \bar{i}\rangle \neq \langle t, r\rangle \vee t \notin T^{l+1}$$

and by two $(\forall^0)$-inferences

$$\begin{aligned}\mathsf{RA}^* \mid\!\frac{\eta_0+6}{0}\quad & (\exists z_1)(\exists z_2)(\langle s, \bar{i}\rangle = \langle z_1, z_2\rangle \wedge z_2 \leq \overline{n} \wedge z_1 \in T^{l+1}),\\ & (\forall z_1)(\forall z_2)(\langle s, \bar{i}\rangle \neq \langle z_1, z_2\rangle \vee z_1 \notin T^{l+1})\end{aligned}$$

and this is

$$\begin{aligned}\mathsf{RA}^* \mid\!\frac{\eta_0+6}{0}\quad & (\exists z_1)(\exists z_2)(\langle s, \bar{i}\rangle = \langle z_1, z_2\rangle \wedge z_2 \leq \overline{n} \wedge z_1 \in T^{l+1}),\\ & \neg(\exists z_1)(\exists z_2)(\langle s, \bar{i}\rangle = \langle z_1, z_2\rangle \wedge z_1 \in T^{l+1})\end{aligned}$$

and hence we obtain by ($\in 1$) and ($\in 2$)

$$\mathsf{RA}^* \vdash^{\eta_0+8}_{0} \begin{array}{l}\langle s,\bar{i}\rangle \in \{y : (\exists z_1)(\exists z_2)(y = \langle z_1, z_2\rangle \wedge z_2 \leq \overline{n} \wedge z_1 \in T^{l+1})\} \\ \langle s,\bar{i}\rangle \notin \{y : (\exists z_1)(\exists z_2)(y = \langle z_1, z_2\rangle \wedge z_1 \in T^{l+1})\}\end{array}$$

and by two ($\vee$)-inferences we get

$$\mathsf{RA}^* \vdash^{\eta_0+10}_{0} \begin{array}{l}\langle s,\bar{i}\rangle \in \{y : (\exists z_1)(\exists z_2)(y = \langle z_1, z_2\rangle \wedge z_1 \in T^{l+1})\} \to \\ \langle s,\bar{i}\rangle \in \{y : (\exists z_1)(\exists z_2)(y = \langle z_1, z_2\rangle \wedge z_2 \leq \overline{n} \wedge z_1 \in T^{l+1})\}\end{array} \tag{43}$$

<u>"$\to$":</u>

We again have to distinguish several cases depending on the value of the (arbitrary) number terms s, t and r.

- If s and t have not the same value or the value of r is not i then we have the following from Axiom ($Ax1$).

  $$\mathsf{RA}^* \vdash^{0}_{0} \langle s,\bar{i}\rangle \neq \langle t,r\rangle, \langle s,\bar{i}\rangle = \langle t,r\rangle \wedge t \in T^{l+1}$$

  and obtain by an ($\vee$)-inference that

  $$\mathsf{RA}^* \vdash^{1}_{0} \langle s,\bar{i}\rangle \neq \langle t,r\rangle \vee \neg(r \leq \overline{n}) \vee t \notin T^{l+1}, \langle s,\bar{i}\rangle = \langle t,r\rangle \wedge t \in T^{l+1}$$

- If s and t have the same value and the value of r is i then we have the following Axiom ($Ax1$).

  $$\mathsf{RA}^* \vdash^{0}_{0} \langle s,\bar{i}\rangle = \langle t,r\rangle$$

  Further we obtain from Lemma 5.2.11 again $\eta_0 < \omega^2$ such that for all number terms t

  $$\mathsf{RA}^* \vdash^{\eta_0}_{0} t \in T^{l+1}, t \notin T^{l+1}$$

  since $lev(t \in T^{l+1}) = l + 1$ and therefore $rk(t \in T^{l+1}) < \omega^2$.

  So we obtain by an ($\wedge$)-inference

  $$\mathsf{RA}^* \vdash^{\eta_0+1}_{0} \langle s,\bar{i}\rangle = \langle t,r\rangle \wedge t \in T^{l+1}, t \notin T^{l+1}$$

  and by an ($\vee$)-inference

  $$\mathsf{RA}^* \vdash^{\eta_0+2}_{0} \langle s,\bar{i}\rangle = \langle t,r\rangle \wedge t \in T^{l+1}, \langle s,\bar{i}\rangle \neq \langle t,r\rangle \vee \neg(r \leq \overline{n}) \vee t \notin T^{l+1}$$

So we conclude by using the $(\exists^0)$-inference twice that

$$\mathsf{RA}^* \mid\!\frac{\eta_0+4}{0}\; (\exists z_1)(\exists z_2)(\langle s,\bar{i}\rangle = \langle z_1,z_2\rangle \wedge z_1 \in T^{l+1}), \langle s,\bar{i}\rangle \neq \langle t,r\rangle \vee \neg(r \leq \overline{n}) \vee t \notin T^{l+1}$$

and by two $(\forall^0)$-inferences

$$\begin{array}{ll}\mathsf{RA}^* \mid\!\frac{\eta_0+6}{0} & (\exists z_1)(\exists z_2)(\langle s,\bar{i}\rangle = \langle z_1,z_2\rangle \wedge z_1 \in T^{l+1}), \\ & (\forall z_1)(\forall z_2)(\langle s,\bar{i}\rangle \neq \langle z_1,z_2\rangle \vee \neg(z_2 \leq \overline{n}) \vee z_1 \notin T^{l+1})\end{array}$$

and this is

$$\begin{array}{ll}\mathsf{RA}^* \mid\!\frac{\eta_0+6}{0} & (\exists z_1)(\exists z_2)(\langle s,\bar{i}\rangle = \langle z_1,z_2\rangle \wedge z_1 \in T^{l+1}), \\ & \neg(\exists z_1)(\exists z_2)(\langle s,\bar{i}\rangle = \langle z_1,z_2\rangle \wedge z_2 \leq \overline{n} \wedge z_1 \in T^{l+1})\end{array}$$

and hence we obtain by $(\in 1)$ and $(\in 2)$

$$\begin{array}{ll}\mathsf{RA}^* \mid\!\frac{\eta_0+8}{0} & \langle s,\bar{i}\rangle \in \{y : (\exists z_1)(\exists z_2)(y = \langle z_1,z_2\rangle \wedge z_1 \in T^{l+1})\} \\ & \langle s,\bar{i}\rangle \notin \{y : (\exists z_1)(\exists z_2)(y = \langle z_1,z_2\rangle \wedge z_2 \leq \overline{n} \wedge z_1 \in T^{l+1})\}\end{array}$$

and by two $(\vee)$-inferences we get

$$\begin{array}{ll}\mathsf{RA}^* \mid\!\frac{\eta_0+10}{0} & \langle s,\bar{i}\rangle \in \{y : (\exists z_1)(\exists z_2)(y = \langle z_1,z_2\rangle \wedge z_2 \leq \overline{n} \wedge z_1 \in T^{l+1})\} \rightarrow \\ & \langle s,\bar{i}\rangle \in \{y : (\exists z_1)(\exists z_2)(y = \langle z_1,z_2\rangle \wedge z_1 \in T^{l+1})\}\end{array}$$

Together with (43) we obtain by an $(\wedge)$-inference

$$\begin{array}{ll}\mathsf{RA}^* \mid\!\frac{\eta_0+11}{0} & \langle s,\bar{i}\rangle \in \{y : (\exists z_1)(\exists z_2)(y = \langle z_1,z_2\rangle \wedge z_2 \leq \overline{n} \wedge z_1 \in T^{l+1})\} \leftrightarrow \\ & \langle s,\bar{i}\rangle \in \{y : (\exists z_1)(\exists z_2)(y = \langle z_1,z_2\rangle \wedge z_1 \in T^{l+1})\} \rightarrow\end{array}$$

and finally by an $(\forall^0)$-inference

$$\begin{array}{ll}\mathsf{RA}^* \mid\!\frac{\eta_0+12}{0} & (\forall x)(\langle x,\bar{i}\rangle \in \{y : (\exists z_1)(\exists z_2)(y = \langle z_1,z_2\rangle \wedge z_2 \leq \overline{n} \wedge z_1 \in T^{l+1})\} \leftrightarrow \\ & \langle x,\bar{i}\rangle \in \{y : (\exists z_1)(\exists z_2)(y = \langle z_1,z_2\rangle \wedge z_1 \in T^{l+1})\}) \rightarrow\end{array}$$

(with $\eta_0 + 12 < \omega^2$).

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Address

Marc Heissenbüttel, Institut für Informatik und angewandte Mathematik, Universität Bern, Neubrückstrasse 10, CH-3012 Bern, Switzerland, `heissen@iam.unibe.ch`