

# Autonomous fixed point progressions and fixed point transfinite recursion

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**Abstract.** This paper is a contribution to the area of metapredicative proof theory. It continues recent investigations on the transfinitely iterated fixed point theories  $\widehat{\text{ID}}_\alpha$  (cf. [10]) and addresses the question of autonomy in iterated fixed point theories. An *external* and an *internal* form of autonomous generation of transfinite hierarchies of fixed points of positive arithmetic operators are introduced and proof-theoretically analyzed. This includes the discussion of the principle of so-called *fixed point transfinite recursion*. Connections to theories for iterated inaccessibility in the context of Kripke Platek set theory without foundation are revealed.

## 1 Introduction

The foundational program to study the principles and ordinals which are implicit in a predicative conception of the universe of sets of natural numbers led to the progression of systems of ramified analysis up to the famous Feferman-Schütte ordinal  $\Gamma_0$  in the early sixties. Since then numerous theories have been found which are not *prima facie* predicatively justifiable, but nevertheless have predicative strength in the sense that  $\Gamma_0$  is an upper bound to their proof-theoretic ordinal. It is common to all these predicative theories that their analysis requires methods from predicative proof theory only, in contrast to the present proof-theoretic treatment of stronger impredicative systems. On the other hand, it has long been known that there are natural systems which have proof-theoretic ordinal greater than  $\Gamma_0$  and whose analysis makes use just as well of methods which every proof-theorist would consider to be predicative. Nevertheless, not many theories of the latter kind have been known until recently.

*Metapredicativity* is a new area in proof theory which is concerned with the analysis of formal systems whose proof-theoretic ordinal is beyond the Feferman-Schütte ordinal  $\Gamma_0$ , but which can be given a proof-theoretic analysis that uses *methods* from predicative proof theory only. It has recently been discovered that the world of metapredicativity is extremely rich and that it includes many natural and foundationally interesting formal systems. For previous work in metapredicativity the reader is referred to Jäger, Kahle, Setzer and Strahm [10], Jäger and Strahm [11], Kahle [13], Rathjen [21], and Strahm [27]. A short discussion of this recent research work is given in the last section of this paper.

This paper starts off from the article Jäger, Kahle, Setzer and Strahm [10] on the proof-theoretic analysis of transfinitely iterated fixed point theories  $\widehat{\text{ID}}_\alpha$ . Finitely iterated fixed point theories  $\widehat{\text{ID}}_n$  were introduced and analyzed in Feferman [5] in connection with his proof of Hancock’s conjecture about the strength of Martin-Löf type theory with finitely many universes. It is shown in [5] that the union of the theories  $\widehat{\text{ID}}_n$  for  $n < \omega$  has proof-theoretic ordinal  $\Gamma_0$ . In [10] the proof-theoretic ordinals of  $\widehat{\text{ID}}_\alpha$  for  $\alpha \geq \omega$  are determined by providing a metapredicative ordinal analysis.

The main concern of this article is to study and elucidate various ways of generating hierarchies of fixed points of positive arithmetic operators in an autonomous manner. The simplest form of autonomy is formalized in the theory  $\text{Aut}(\widehat{\text{ID}})$ : the crucial rule of inference of  $\text{Aut}(\widehat{\text{ID}})$  states that whenever we have a *proof* that a specific primitive recursive ordering is wellfounded, then one is allowed to claim the existence of a fixed point hierarchy along that wellordering. We will see that the proof-theoretic ordinal of  $\text{Aut}(\widehat{\text{ID}})$  is  $\varphi_{200}$  for  $\varphi$  a ternary Veblen function. A more general account to autonomy is implemented by the principle of so-called *fixed point transfinite recursion* (FTR), which demands the existence of fixed point hierarchies along arbitrary given wellorderings. (FTR) is more liberal than  $\text{Aut}(\widehat{\text{ID}})$  in the sense that we are no longer dealing with primitive recursive wellorderings only and, moreover, (FTR) does not require a previously recognized proof of wellfoundedness. We introduce two subsystems of analysis  $\text{FTR}_0$  and  $\text{FTR}$  which are based on fixed point transfinite recursion (FTR) and include set and formula induction in the natural numbers, respectively. We show that  $\text{FTR}_0$  and  $\text{FTR}$  have proof-theoretic ordinal  $\varphi_{200}$  and  $\varphi_{20\varepsilon_0}$  respectively. Hence,  $\text{FTR}_0$  has the same proof-theoretic strength as  $\text{Aut}(\widehat{\text{ID}})$ . Upper bounds are obtained by modeling (FTR) in a system of Kripke Platek set theory without foundation which formalizes a hyperinaccessible<sup>1</sup> universe of sets.

The exact plan of this paper is as follows. We start with some ordinal-theoretic preliminaries in Section 2; in particular, we define the ternary Veblen function  $\lambda\alpha, \beta, \gamma. \varphi\alpha\beta\gamma$  which will be relevant in the sequel. In Section 3 we introduce a first order framework for transfinitely iterated fixed point theories. We review the results of [10] about the proof-theoretic ordinal of  $\widehat{\text{ID}}_\alpha$  and see that the theory  $\text{Aut}(\widehat{\text{ID}})$  has ordinal  $\varphi_{200}$ . Section 4 is devoted to the exact definition of fixed point transfinite recursion (FTR) and the corresponding theories  $\text{FTR}_0$  and  $\text{FTR}$ . Moreover, we establish  $\varphi_{200}$  and  $\varphi_{20\varepsilon_0}$  as lower bounds of  $\text{FTR}_0$  and  $\text{FTR}$ , respectively. In particular,  $\text{Aut}(\widehat{\text{ID}})$  is interpretable in  $\text{FTR}_0$ . In Section 5 we first introduce a system of Kripke Platek set theory without foundation for a hyperinaccessible universe of sets, namely the theory  $\text{KPh}^0$ . Then we show that fixed point transfinite recursion (FTR) can be modeled in  $\text{KPh}^0$ , i.e. the theory  $\text{FTR}_0$  is interpretable into  $\text{KPh}^0$ . The full system  $\text{FTR}$  is contained in a slight strengthening of  $\text{KPh}^0$ . Using results of Jäger and Strahm [12] about the proof-theoretic ordinals of these theories for hyperinaccessibility, we find that

<sup>1</sup> Throughout this paper the notions “inaccessible” and “hyperinaccessible” always refer to “recursively inaccessible” and “recursively hyperinaccessible”, respectively.

$\varphi 200$  and  $\varphi 20\varepsilon_0$  are upper bounds for the proof-theoretic ordinal of  $\text{FTR}_0$  and  $\text{FTR}$ , respectively. Finally, in Section 6 of this paper we summarize our results and we discuss various kinds of related metapredicative systems, ranging from subsystems of analysis and systems of Kripke Platek set theory to systems of explicit mathematics with universes.

## 2 The ternary Veblen function

In this section we fix a few ordinal-theoretic facts which will be relevant in the sequel. Namely, we sketch an ordinal notation system which is based on a *ternary* Veblen or  $\varphi$  function. This ordinal function will be sufficient for denoting the proof-theoretic ordinals of the theories considered in this article.

The standard notation system up to the Feferman-Schütte ordinal  $\Gamma_0$  makes use of the usual Veblen hierarchy generated by the *binary* function  $\varphi$ , starting off with the function  $\varphi 0\beta = \omega^\beta$ , cf. Pohlers [20] or Schütte [24]. The *ternary*  $\varphi$  function is obtained as a straightforward generalization of the binary case by defining  $\varphi\alpha\beta\gamma$  inductively as follows:

- (i)  $\varphi 0\beta\gamma$  is just  $\varphi\beta\gamma$ ;
- (ii) if  $\alpha > 0$ , then  $\varphi\alpha 0\gamma$  denotes the  $\gamma$ th ordinal which is strongly critical with respect to all functions  $\lambda\xi, \eta.\varphi\alpha'\xi\eta$  for  $\alpha' < \alpha$ .
- (iii) if  $\alpha > 0$  and  $\beta > 0$ , then  $\varphi\alpha\beta\gamma$  denotes the  $\gamma$ th common fixed point of the functions  $\lambda\xi.\varphi\alpha\beta'\xi$  for  $\beta' < \beta$ .

For example,  $\varphi 10\alpha$  is  $\Gamma_\alpha$ , and more generally,  $\varphi 1\alpha\beta$  denotes a Veblen hierarchy over  $\lambda\alpha.\Gamma_\alpha$ . It is straightforward how to extend these ideas in order to obtain  $\varphi$  functions of all finite arities, and even further to Schütte's Klammersymbole [23].

Let  $A_3$  denote the least ordinal greater than 0 which is closed under the ternary  $\varphi$  function. In the following we confine ourselves to the standard notation system which is based on this function. Since the exact definition of such a system is a straightforward generalization of the notation system for  $\Gamma_0$  (cf. [20, 24]), we do not go into details here. We write  $\prec$  for the corresponding primitive recursive wellordering and assume without loss of generality that the field of  $\prec$  is the set of all natural numbers and 0 is the least element with respect to  $\prec$ .

## 3 The theory $\text{Aut}(\widehat{\text{ID}})$

In this section we first introduce the transfinitely iterated fixed point theories  $\widehat{\text{ID}}_\alpha$  of [10] and we recall the main theorem about their proof-theoretic strength. Then we define the autonomous fixed point theory  $\text{Aut}(\widehat{\text{ID}})$ , which incorporates the most simple form of autonomous generation of fixed point hierarchies. The proof-theoretic ordinal of  $\text{Aut}(\widehat{\text{ID}})$  is  $\varphi 200$ .

In the following we let  $\mathcal{L}$  denote the language of first order arithmetic.  $\mathcal{L}$  includes *number variables*  $(a, b, c, u, v, w, x, y, z, \dots)$ , symbols for all primitive

recursive functions and relations, as well as a unary relation symbol  $U$  whose status will become clear below. The *number terms*  $(r, s, t, \dots)$  and *formulas*  $(A, B, C, \dots)$  of  $\mathcal{L}$  are defined as usual.

If  $P$  and  $Q$  are fresh unary relation symbols, then we let  $\mathcal{L}(P, Q)$  denote the extension of  $\mathcal{L}$  by  $P$  and  $Q$ . We call an  $\mathcal{L}(P, Q)$  formula  $P$  *positive*, if the relation symbol  $P$  has only positive occurrences in it. A  $P$  positive  $\mathcal{L}(P, Q)$  formula which contains at most  $x$  and  $y$  free is called an *inductive operator form*, and we let  $\mathcal{A}(P, Q, x, y)$  range over such forms.

Further, we set for all primitive recursive relations  $\triangleleft$ , all formulas  $A(x)$  and terms  $s$ :

$$\begin{aligned} \text{Prog}(\triangleleft, A) &:= (\forall x)[(\forall y)(y \triangleleft x \rightarrow A(y)) \rightarrow A(x)], \\ \text{TI}(\triangleleft, A) &:= \text{Prog}(\triangleleft, A) \rightarrow (\forall x)A(x), \\ \text{TI}(\triangleleft, s, A) &:= \text{Prog}(\triangleleft, A) \rightarrow (\forall x \triangleleft s)A(x). \end{aligned}$$

We write  $\text{Prog}(A)$  and  $\text{TI}(s, A)$  instead of  $\text{Prog}(\prec, A)$  and  $\text{TI}(\prec, s, A)$ , respectively. If we want to stress the relevant induction variable of the formula  $A$ , we sometimes write  $\text{Prog}(\lambda x.A(x))$  instead of  $\text{Prog}(A)$ .

The stage is now set in order to introduce the theories  $\widehat{\text{ID}}_\alpha$  for each  $\alpha$  less than  $\Lambda_3$ .<sup>2</sup>  $\widehat{\text{ID}}_\alpha$  is formulated in the language  $\mathcal{L}_{\text{fix}}$ , which extends  $\mathcal{L}$  by a new unary relation symbol  $P^A$  for each inductive operator form  $\mathcal{A}(P, Q, x, y)$ . We write  $P_s^A(t)$  for  $P^A(\langle t, s \rangle)$  and  $P_{\prec s}^A(t)$  for  $t = \langle (t)_0, (t)_1 \rangle \wedge (t)_1 \prec s \wedge P^A(t)$ ; here  $\langle \cdot, \cdot \rangle$  denotes a primitive recursive coding function with associated projections  $(\cdot)_0$  and  $(\cdot)_1$ .

The theory  $\widehat{\text{ID}}_\alpha$  for  $\alpha$  *times iterated fixed points* comprises the following axioms: (i) the axioms of Peano arithmetic PA with the scheme of complete induction for all formulas of  $\mathcal{L}_{\text{fix}}$ , (ii) the fixed point axioms

$$(\forall a \prec \alpha)(\forall x)[P_a^A(x) \leftrightarrow \mathcal{A}(P_a^A, P_{\prec a}^A, x, a)]$$

for all inductive operator forms  $\mathcal{A}(X, Y, x, y)$ , as well as (iii) the axioms  $\text{TI}(\alpha, A)$  for all  $\mathcal{L}_{\text{fix}}$  formulas  $A$ . We write  $\widehat{\text{ID}}_{<\alpha}$  for the union of the theories  $\widehat{\text{ID}}_\beta$  for  $\beta$  less than  $\alpha$ .

As usual we call an ordinal  $\alpha$  provable in a theory  $\mathbb{T}$ , if there is a primitive recursive wellordering  $\triangleleft$  of ordertype  $\alpha$  so that  $\mathbb{T} \vdash \text{TI}(\triangleleft, U)$ . The least ordinal which is not provable in  $\mathbb{T}$  is called the *proof-theoretic ordinal* of  $\mathbb{T}$  and is denoted by  $|\mathbb{T}|$ .

The theories  $\widehat{\text{ID}}_\alpha$  provide first paradigmatic examples of metapredicative theories. Their proof-theoretic analysis has been carried through only recently by Jäger, Kahle, Setzer and Strahm in [10]. It turns out that the proof-theoretic ordinal of  $\widehat{\text{ID}}_\alpha$  can be described by means of the function  $\lambda\alpha, \beta. \varphi_1\alpha\beta$ , which forms a Veblen hierarchy starting with the initial function  $\lambda\alpha. \Gamma_\alpha$ .

<sup>2</sup> Of course, the restriction to ordinals less than  $\Lambda_3$  is not essential; its just stems from the choice of our notation system for the purpose of this article.

In order to formulate the main theorem of [10], we let  $\varepsilon(\alpha)$  denote the least  $\varepsilon$  number greater than  $\alpha$ . Moreover, the ordinals  $(\alpha|m)$  are inductively defined by

$$(\alpha|0) := \varepsilon(\alpha), \quad (\alpha|m+1) := \varphi(\alpha|m)0.$$

**Theorem 1.** *Assume that  $\alpha$  is an ordinal less than  $\Lambda_3$  of the form*

$$\alpha = \omega^{1+\alpha_n} + \omega^{1+\alpha_{n-1}} + \dots + \omega^{1+\alpha_1} + m,$$

for ordinals  $\alpha_n \geq \alpha_{n-1} \geq \dots \geq \alpha_1$  and  $m < \omega$ . Then we have

$$|\widehat{\text{ID}}_\alpha| = \varphi 1 \alpha_n (\varphi 1 \alpha_{n-1} (\dots \varphi 1 \alpha_1 (\alpha|m)) \dots).$$

This finishes our short review of the theories  $\widehat{\text{ID}}_\alpha$ . Let us now turn to autonomous fixed point processes. The simplest way to generate fixed point hierarchies autonomously is formalized in the theory  $\text{Aut}(\widehat{\text{ID}})$ . The principal rule of inference of  $\text{Aut}(\widehat{\text{ID}})$  states that whenever we have a *proof* that a specific primitive recursive linear ordering  $\triangleleft$  is wellfounded, then we are allowed to adjoin the axiom which claims the existence of a fixed point hierarchy along  $\triangleleft$  with respect to an operator form  $\mathcal{A}$ .

Since in the theory  $\text{Aut}(\widehat{\text{ID}})$  we are no longer dealing with the fixed primitive recursive wellordering  $\prec$ , but with arbitrary previously recognized primitive recursive wellorderings  $\triangleleft$ , the corresponding fixed point hierarchies depend on these orderings  $\triangleleft$ . Accordingly, for the formulation of  $\text{Aut}(\widehat{\text{ID}})$  we assume that our language  $\mathcal{L}_{\text{fix}}$  includes unary relation symbols  $P^{A, \triangleleft}$  for each operator form  $\mathcal{A}$  and (binary) primitive recursive relation  $\triangleleft$ . The formulas  $P_s^{A, \triangleleft}(t)$  and  $P_{\triangleleft s}^{A, \triangleleft}(t)$  are understood as above.

The theory  $\text{Aut}(\widehat{\text{ID}})$  now extends Peano arithmetic (formulated in the language  $\mathcal{L}_{\text{fix}}$ ) by the *autonomous fixed point hierarchy generation rule* and the *bar rule*, i.e.  $\text{Aut}(\widehat{\text{ID}})$  incorporates the two rules of inference

$$\frac{\text{TI}(\triangleleft, U)}{(\forall a \in \text{field}(\triangleleft))(\forall x)[P_a^{A, \triangleleft}(x) \leftrightarrow \mathcal{A}(P_a^{A, \triangleleft}, P_{\triangleleft a}^{A, \triangleleft}, x, a)]} \quad \text{and} \quad \frac{\text{TI}(\triangleleft, U)}{\text{TI}(\triangleleft, A)},$$

where  $\triangleleft$  denotes a primitive recursive linear ordering (provably say in PA),  $\text{field}(\triangleleft)$  signifies the field of  $\triangleleft$  and  $A$  denotes an arbitrary  $\mathcal{L}_{\text{fix}}$  formula.

The proof-theoretic ordinal of  $\text{Aut}(\widehat{\text{ID}})$  is the ordinal  $\varphi 200$ , i.e. the first ordinal which is strongly critical w.r.t. a Veblen hierarchy above the  $\Gamma$  function. This result essentially follows from Theorem 1. Alternatively, we can say that  $\varphi 200$  is the least ordinal  $\alpha$  such that  $|\widehat{\text{ID}}_{\triangleleft \alpha}| = \alpha$ .

**Theorem 2.**  $|\text{Aut}(\widehat{\text{ID}})| = \varphi 200$ .

*Proof.* We define a canonical fundamental sequence  $(\alpha_n)_{n \in \mathbb{N}}$  for  $\varphi 200$  by setting  $\alpha_0 := \varepsilon_0$  and  $\alpha_{n+1} := \varphi 1 \alpha_n 0$ . Then Theorem 1 immediately yields that each of the theories  $\widehat{\text{ID}}_{\triangleleft \alpha_n}$  is contained in  $\text{Aut}(\widehat{\text{ID}})$ , and consequently  $\varphi 200 \leq |\text{Aut}(\widehat{\text{ID}})|$ . The reverse direction  $|\text{Aut}(\widehat{\text{ID}})| \leq \varphi 200$  is entailed by Theorem 1 as well if one observes that the upper bound arguments given in Section 6 of [10] do not depend on the specific representation of the ordering  $\prec$ .

## 4 The theories $\text{FTR}_0$ and $\text{FTR}$

In this section we introduce the subsystems of analysis  $\text{FTR}_0$  and  $\text{FTR}$ , which incorporate the crucial principle of *fixed point transfinite recursion* ( $\text{FTR}$ ). In analogy to arithmetic transfinite recursion ( $\text{ATR}$ ) (cf. e.g. [6]), ( $\text{FTR}$ ) claims the existence of fixed point hierarchies along arbitrary given wellorderings. ( $\text{FTR}$ ) can be seen as a more liberal account to autonomous fixed point processes in the sense that we are dealing not only with primitive recursive (or arithmetic) wellorderings as in  $\text{Aut}(\widehat{\text{ID}})$  and, moreover, ( $\text{FTR}$ ) does not require a previously recognized *proof* of wellfoundedness. Hence, in a sense, autonomy in  $\text{Aut}(\widehat{\text{ID}})$  could be called *external*, whereas ( $\text{FTR}$ ) formalizes an *internal* form of autonomy. Nevertheless, we will see that these two forms of autonomy have the same proof-theoretic strength as long as induction on the natural numbers in the context of ( $\text{FTR}$ ) is restricted to sets.

Let  $\mathcal{L}_2$  denote the usual language of second order arithmetic, which extends  $\mathcal{L}$  by *set variables*  $X, Y, Z, \dots$  (possibly with subscripts) and the binary relation symbol  $\in$  for elementhood between numbers and sets. Terms and formulas of  $\mathcal{L}_2$  are defined as usual. We write  $s \in (X)_t$  for  $\langle s, t \rangle \in X$ . An  $\mathcal{L}_2$  formula is called *arithmetic*, if it does not contain bound set variables. Similarly as before, we call an arithmetic  $\mathcal{L}_2$  formula  $\mathcal{A}(X, Y, x, y)$  an *inductive operator form* if  $X$  does only occur positively in it; inductive operator forms may contain further free set and number variables.

A set  $X$  of natural numbers can be regarded as a binary relation by stipulating  $s X t$  for  $\langle s, t \rangle \in X$ . In the sequel we let  $\text{LO}(X)$  denote the usual arithmetic formula which expresses that the binary relation  $X$  is a linear ordering of its field,  $\text{field}(X)$ . Moreover, we say that  $X$  is wellfounded if transfinite induction along  $X$  holds w.r.t. all sets  $Z$ , i.e.  $(\forall Z)\text{TI}(X, Z)$ . Finally,  $X$  is a wellordering, in symbols  $\text{WO}(X)$ , if  $X$  is a wellfounded linear ordering.

Our main concern is to build hierarchies of fixed points along arbitrary wellorderings. For that purpose, we introduce the formula  $\text{FHier}_{\mathcal{A}}(X, Y)$  which expresses that  $Y$  is a hierarchy of fixed points along  $X$  w.r.t. the inductive operator form  $\mathcal{A}$ :

$$\text{FHier}_{\mathcal{A}}(X, Y) := (\forall a \in \text{field}(X))(\forall x)[x \in (Y)_a \leftrightarrow \mathcal{A}((Y)_a, (Y)_{Xa}, x, a)].$$

Here  $(Y)_{Xa}$  denotes the set  $\{\langle y, b \rangle : b X a \wedge \langle y, b \rangle \in Y\}$ . Observe that the formula  $\text{FHier}_{\mathcal{A}}(X, Y)$  depends on the additional parameters of the inductive operator form  $\mathcal{A}$ . We are now ready to state the principle of *fixed point transfinite recursion* ( $\text{FTR}$ ), which states for each operator form  $\mathcal{A}$  that an  $\mathcal{A}$  fixed point hierarchy exists along any given wellordering, i.e.

$$(\text{FTR}) \quad (\forall X)[\text{WO}(X) \rightarrow (\exists Y)\text{FHier}_{\mathcal{A}}(X, Y)].$$

In the following we let  $\text{ACA}_0$  denote the standard system of second order arithmetic which includes comprehension for arithmetic formulas and complete induction on the natural numbers for sets. The theory  $\text{FTR}_0$  extends  $\text{ACA}_0$  by each

instance of fixed point transfinite recursion (FTR) and FTR is just  $\text{FTR}_0$  with induction on the natural numbers for arbitrary statements of  $\mathcal{L}_2$ .

In the sequel we will see that the proof-theoretic ordinals of  $\text{FTR}_0$  and FTR are  $\varphi 200$  and  $\varphi 20\varepsilon_0$ , respectively. In order to establish  $\varphi 200$  as a lower bound of  $\text{FTR}_0$  one can either carry through a direct wellordering proof using the methods of [10] or observe that the theory  $\text{Aut}(\widehat{\text{ID}})$  is contained in  $\text{FTR}_0$  in a rather direct manner.

**Theorem 3.**  $\text{Aut}(\widehat{\text{ID}})$  is contained in  $\text{FTR}_0$ .

*Proof.* Rather than giving a global interpretation of the language  $\mathcal{L}_{\text{fix}}$  in  $\mathcal{L}_2$  we inductively translate *each proof* in  $\text{Aut}(\widehat{\text{ID}})$  into  $\text{FTR}_0$ . Thereby, the anonymous relation symbol  $U$  ranges over arbitrary sets in  $\text{FTR}_0$  which means that inductive operator forms of  $\mathcal{L}$  containing  $U$  carry over to operator forms in  $\mathcal{L}_2$  which depend on set parameters. Given a specific proof  $d$  in  $\text{Aut}(\widehat{\text{ID}})$ , the only crucial point is to interpret those relation symbols  $P^{A, \triangleleft}$  which are introduced in  $d$  by the autonomous fixed point hierarchy generation rule. In this case we know by the inductive hypothesis that  $\text{WO}(\triangleleft)$  is provable in  $\text{FTR}_0$  and hence a fixed point hierarchy along  $\triangleleft$  exists by (FTR) giving  $P^{A, \triangleleft}$  its interpretation. Under this interpretation, the bar rule is trivialized and, moreover, complete induction on the natural numbers is only needed for sets in  $\text{FTR}_0$ . This finishes our argument.

**Corollary 1.**  $\varphi 200 \leq |\text{FTR}_0|$ .

In order to see that  $\varphi 20\varepsilon_0$  is a lower bound for FTR, i.e.  $\text{FTR}_0$  with induction on the natural numbers for arbitrary  $\mathcal{L}_2$  formulas, one makes use of the wellordering proofs for the theories  $\widehat{\text{ID}}_\alpha$  given in [10].

**Theorem 4.**  $\varphi 20\varepsilon_0 \leq |\text{FTR}|$ .

*Proof.* (Sketch) Essentially by making use of Main Lemma II in Section 5 of [10], one shows that FTR proves

$$(\forall a)[(\forall X)\text{TI}(X, a) \rightarrow (\forall X)\text{TI}(X, \varphi 1a0)]. \quad (1)$$

Furthermore, using (1) it is immediate to show that FTR derives

$$\text{Prog}(\lambda a.(\forall X)\text{TI}(X, \varphi 20a)). \quad (2)$$

Due to the presence of full formula induction on the natural numbers, transfinite induction with respect to arbitrary  $\mathcal{L}_2$  formulas is available in FTR for fixed initial segments of  $\varepsilon_0$ . From this observation and (2) we immediately obtain our claim, namely that  $(\forall X)\text{TI}(X, \varphi 20\alpha)$  is derivable in FTR for each  $\alpha$  less than  $\varepsilon_0$ .

In the next paragraph we will show that the lower bounds for  $\text{FTR}_0$  and FTR are sharp by modeling fixed point transfinite recursion (FTR) in a system of Kripke Platek set theory without foundation which formalizes a hyperinaccessible universe of sets.

## 5 Hyperinaccessibility without foundation

It is the aim of this section to show that the lower bounds  $\varphi_{200}$  and  $\varphi_{20\varepsilon_0}$  for  $\text{FTR}_0$  and  $\text{FTR}$ , respectively, are sharp. This is done by modeling the schema of fixed point transfinite recursion (FTR) in a universe of sets which forms a limit of inaccessible sets. Below we introduce the theory  $\text{KPh}^0$  which formalizes a hyperinaccessible universe of sets.  $\text{KPh}^0$  includes induction on the natural numbers for sets only and – most importantly – it does not include foundation at all. The corresponding theory with foundation has an enormous proof-theoretic strength which exceeds the strength of  $\Delta_2^1\text{-CA} + (\text{BI})$  by far.

The language  $\mathcal{L}_s$  of  $\text{KPh}^0$  extends the usual language of set theory with  $\in$  and  $=$  by a unary predicate symbol  $\text{Ad}$  to mean that a set is admissible. In addition, we assume that  $\mathcal{L}_s$  includes a constant  $\omega$  for the first infinite ordinal.<sup>3</sup> Variables of  $\mathcal{L}_s$  are denoted by  $a, b, c, x, y, z, u, v, w, f, g, h, \dots$ , and we let  $A, B, C, \dots$  range over the formulas of  $\mathcal{L}_s$ . An  $\mathcal{L}_s$  formula is called  $\Delta_0$  if all its quantifiers are bounded;  $\Sigma_1, \Pi_1, \Sigma, \Pi$  and  $\Delta$  formulas are defined as usual. The formula  $A^a$  is the result of restricting all unbounded quantifiers in  $A$  to  $a$ . We make free use of standard set-theoretic notions and notations, for example  $\text{Tran}(a)$  signifies that  $a$  is a transitive set.

In order to formalize a hyperinaccessible universe of sets we need the notion  $\text{InAcc}(a)$  in order to express that a set  $a$  is inaccessible, i.e. admissible and limit of admissibles:

$$\text{InAcc}(a) := \text{Ad}(a) \wedge (\forall x \in a)(\exists y \in a)(x \in y \wedge \text{Ad}(y)).$$

We are now ready to introduce the theory  $\text{KPh}^0$ . The logical axioms and rules of  $\text{KPh}^0$  are the ones for classical predicate logic with equality. The non-logical axioms of  $\text{KPh}^0$  are divided into the following four groups.

**I. Basic set-theoretic axioms.** For all  $\Delta_0$  formulas  $A(x)$  and  $B(x, y)$ :

(Extensionality)  $(\forall x)(x \in a \leftrightarrow x \in b) \rightarrow a = b$ ,

(Pair)  $(\exists x)(x = \{a, b\})$ ,

(Union)  $(\exists x)(x = \bigcup a)$ ,

( $\Delta_0$  Separation)  $(\exists x)(x = \{y \in a : A(y)\})$ ,

( $\Delta_0$  Collection)  $(\forall x \in a)(\exists y)B(x, y) \rightarrow (\exists z)(\forall x \in a)(\exists y \in z)B(x, y)$ .

**II. Axioms about  $\omega$ .**

(Infinity)  $\emptyset \in \omega \wedge (\forall x \in \omega)(x \cup \{x\} \in \omega)$ ,

( $\omega$  Induction)  $\emptyset \in a \wedge (\forall x \in \omega)[x \in a \rightarrow x \cup \{x\} \in a] \rightarrow (\forall x \in \omega)(x \in a)$ .

**III. Axioms about  $\text{Ad}$ .** For all axioms  $A(\mathbf{x})$  of group I whose free variables belong to  $\mathbf{x}$ :

<sup>3</sup> To be precise, we also presuppose that  $\mathcal{L}_s$  contains the free unary relation symbol  $U$  so that we can use the same definition of proof-theoretic ordinal as before.

(Ad Transitivity)  $\text{Ad}(a) \rightarrow \omega \in a \wedge \text{Tran}(a)$ ,

(Ad Linearity)  $\text{Ad}(a) \wedge \text{Ad}(b) \rightarrow a \in b \vee a = b \vee b \in a$ ,

(Ad Reflection)  $\text{Ad}(a) \rightarrow (\forall \mathbf{x} \in a)A^a(\mathbf{x})$ .

#### IV. Limit of inaccessibles.

(InAcc Limit)  $(\forall x)(\exists y)(x \in y \wedge \text{InAcc}(y))$ .

This finishes the description of  $\text{KPh}^0$ . By  $\text{KPi}^0$  we denote  $\text{KPh}^0$  with axiom IV replaced by

$$(\forall x)(\exists y)(x \in y \wedge \text{Ad}(y)),$$

i.e.,  $\text{KPi}^0$  formalizes an inaccessible universe of sets. Due to Jäger [8], the proof-theoretic ordinal of  $\text{KPi}^0$  is exactly the Feferman-Schütte ordinal  $\Gamma_0$ .

As already mentioned above, we do not give the proof-theoretic analysis of  $\text{KPh}^0$  in this article, since it is contained in Jäger and Strahm [12]. There the exact proof-theoretic strength of the theory  $\text{KPM}^0$  is determined;  $\text{KPM}^0$  formalizes a recursively Mahlo universe of sets without foundation, i.e. it results from the well-known theory  $\text{KPM}$  (cf. Rathjen [22]) by omitting  $\in$  induction completely. The upper bound computation of  $\text{KPM}^0$  goes via a treatment of theories formalizing an  $n$ -hyperinaccessible universe of sets without foundation for each fixed natural number  $n$ . The theory  $\text{KPh}^0$  is just one of these theories, and it is shown in [12] that  $|\text{KPh}^0| \leq \varphi_{200}$ ; indeed, by Strahm [28] this bound is sharp, as we will also see by interpreting  $\text{FTR}_0$  into  $\text{KPh}^0$  below.

**Theorem 5.**  $|\text{KPh}^0| = \varphi_{200}$ .

In our embedding of  $\text{FTR}_0$  into  $\text{KPh}^0$  we will need the important fact that  $\text{KPi}^0$  provides a  $\Sigma_1$  operation which picks an admissible set above any given set. Of course, the natural candidate for an admissible set containing a set  $a$  is the least admissible  $a^+$  above  $a$ , where

$$a^+ := \bigcap \{b : a \in b \wedge \text{Ad}(b)\}.$$

The  $\Sigma_1$  definability of  $a^+$  in  $\text{KPi}^0$  is due to Gerhard Jäger. For completeness, we give a proof of Jäger's theorem; it appears that linearity of admissibles is crucial for his argument.

**Theorem 6.** 1.  $\text{KPi}^0$  proves that  $a^+$  is a set and, in addition,  $\text{Ad}(a^+)$ . Moreover, the function  $a \mapsto a^+$  is  $\Sigma_1$  definable in  $\text{KPi}^0$ .  
2. We have that 1. relativizes to any inaccessible set.

*Proof.* In the following we prove the first part of the theorem only; the second part is immediate by relativization. Let us work informally in  $\text{KPi}^0$ . Given a set  $a$ , the limit axiom of  $\text{KPi}^0$  guarantees the existence of a set  $c$  such that  $\text{Ad}(c)$  and  $a \in c$  and, hence, we have that

$$a^+ = \bigcap \{b \in c \cup \{c\} : a \in b \wedge \text{Ad}(b)\}$$

by linearity of admissibles. This proves that  $a^+$  is indeed a set and one readily sees that the operation  $a \mapsto a^+$  is  $\Sigma_1$  definable. It remains to show that  $a^+$  is admissible, i.e.  $\text{Ad}(a^+)$ . For that purpose we define  $a^{++} := (a^+)^+$  and first convince ourselves that

$$a^+ \neq a^{++}. \quad (3)$$

For a contradiction, assume  $a^+ = a^{++}$ . We have that  $r := \{x \in a^+ : x \notin x\}$  is a set by  $\Delta_0$  separation and, moreover,  $r \in d$  for each admissible set  $d$  such that  $a^+ \in d$ , i.e.  $r \in a^{++}$  by definition. But then  $r \in a^+$  since we have assumed  $a^+ = a^{++}$ . This yields a contradiction since

$$r \in r \leftrightarrow r \in a^+ \wedge r \notin r \leftrightarrow r \notin r.$$

Using (3), there exists a set  $d$  such that  $\text{Ad}(d)$ ,  $a \in d$  and  $a^+ \notin d$ , and indeed we have that  $d = a^+$ . The inclusion  $a^+ \subset d$  is obvious. In order to show that  $d \subset a^+$  we pick an arbitrary set  $b$  with  $\text{Ad}(b)$  and  $a \in b$  and establish  $d \subset b$ . By linearity we have  $d \in b \vee d = b \vee b \in d$ . In case of  $d \in b$  or  $d = b$ ,  $d \subset b$  is obvious. But  $b \in d$  is impossible since this would imply  $a^+ \in d$ , a contradiction to the choice of  $d$ . All together we have shown  $d = a^+$ , which entails  $\text{Ad}(a^+)$  as desired. This finishes our argument. We observe that  $\Delta_0$  collection was not used in this proof.

We now turn to the final preparatory step for our embedding of  $\text{FTR}_0$  into  $\text{KPh}^0$ . Given an inductive operator form  $\mathcal{A}(X, Y, x, y)$  with additional set parameters  $\mathbf{Z}$  and number parameters  $\mathbf{z}$ , we will have to construct an  $\mathcal{A}$  fixed point depending on  $Y, y, \mathbf{Z}, \mathbf{z}$ . Most importantly, the construction of such a fixed point must be *uniform* in these parameters. Due to the above theorem, we know how to pick an admissible set  $(Y, \mathbf{Z})^+$  containing  $Y$  and  $\mathbf{Z}$ . In order to construct an  $\mathcal{A}$  fixed point w.r.t.  $Y, y, \mathbf{Z}, \mathbf{z}$  one can now make use of the *Second Recursion Theorem* of admissible set theory (cf. Barwise [2], p. 157) on the admissible  $(Y, \mathbf{Z})^+$ , thus producing a fixed point  $\text{FP}_{\mathcal{A}}(Y, y, \mathbf{Z}, \mathbf{z})$  uniformly in the given parameters. Note that  $\text{FP}_{\mathcal{A}}(Y, y, \mathbf{Z}, \mathbf{z})$  is  $\Sigma$  on  $(Y, \mathbf{Z})^+$  and, hence, defines a set by  $\Delta_0$  separation. Moreover, the proof of the Second Recursion Theorem does not use foundation. Summing up,  $\text{FP}_{\mathcal{A}}$  is  $\Sigma_1$  definable in  $\text{KPi}^0$  and on any inaccessible set, respectively.

**Theorem 7.**  *$\text{FTR}_0$  is contained in  $\text{KPh}^0$ .*

*Proof.* Of course we work with the standard embedding of the language of analysis  $\mathcal{L}_2$  into the language of set theory  $\mathcal{L}_s$ . Accordingly, we use capital letters also in  $\mathcal{L}_s$  for subsets of the set of natural numbers  $\omega$ . In verifying the axioms of  $\text{FTR}_0$  under this translation, only the axioms about fixed point transfinite recursion (FTR) require special attention. Therefore, let  $\mathcal{A}$  be an inductive operator form with additional parameters  $\mathbf{Z}, \mathbf{z}$ . We work informally in  $\text{KPh}^0$ . First we choose a wellordering  $X$  and observe that transfinite induction along  $X$  is available in  $\text{KPh}^0$  for all  $\Delta_0$  formulas due to the presence of  $\Delta_0$  separation. Using (InAcc Limit), we pick an inaccessible set  $d$  such that  $X, \mathbf{Z}$  belong to  $d$ . Further, we define  $\text{H}_{\mathcal{A}}(X, U, a)$  to be the following  $\mathcal{L}_s$  formula (depending on  $\mathbf{Z}, \mathbf{z}$ ):

$$\text{H}_{\mathcal{A}}(X, U, a) := (\forall b \in \omega)[b = a \vee b \text{ } X \text{ } a \rightarrow (U)_b = \text{FP}_{\mathcal{A}}((U)_{Xb}, b, \mathbf{Z}, \mathbf{z})].$$

Here we use  $\text{FP}_{\mathcal{A}}$  as  $\Sigma_1$  definable on  $d$  and consequently  $\text{H}_{\mathcal{A}}(X, U, a)$  is  $\Delta$  on  $d$ . A straightforward induction along  $X$  yields for each  $a$  in the field of  $X$ :

$$\text{H}_{\mathcal{A}}(X, U, a) \wedge \text{H}_{\mathcal{A}}(X, V, a) \rightarrow (\forall b \in \omega)[b = a \vee b \rightarrow (U)_b = (V)_b]. \quad (4)$$

Moreover, using  $\Sigma$  collection in  $d$  as well as totality of  $\text{FP}_{\mathcal{A}}$  in  $d$ , another induction along  $X$  establishes

$$(\forall a \in \text{field}(X))(\exists U \in d)\text{H}_{\mathcal{A}}(X, U, a). \quad (5)$$

Finally, we can piece together the fixed point hierarchies up to each  $a$  in the field of  $X$  by setting

$$Y := \{\langle y, a \rangle : y \in \omega \wedge a \in \text{field}(X) \wedge (\exists U \in d)[\text{H}_{\mathcal{A}}(X, U, a) \wedge y \in (U)_a]\}.$$

Indeed,  $Y$  exists by  $\Delta_0$  separation and we have  $\text{FHier}_{\mathcal{A}}(X, Y)$  by (4) and (5). This finishes our argument and, hence, the embedding of  $\text{FTR}_0$  into  $\text{KPh}^0$ .

*Remark 1.* We observe that in the above embedding of  $\text{FTR}_0$  into  $\text{KPh}^0$ , we did not make use of *global*  $\Delta_0$  collection. Collection was used only locally in admissible sets. Therefore, global  $\Delta_0$  collection does not contribute to the proof-theoretic strength of  $\text{KPh}^0$ .

We are now in a position to combine Corollary 1, Theorem 5 and Theorem 7.

**Corollary 2.**  $|\text{FTR}_0| = \varphi_{200}$ .

Let us end this section by sketching how one can obtain a sharp upper bound for the theory  $\text{FTR}$ , i.e.,  $\text{FTR}_0$  plus the full schema of formula induction on the natural numbers. As we have noted above (Remark 1), global  $\Delta_0$  collection has not been used in our embedding of  $\text{FTR}_0$  into  $\text{KPh}^0$ . As a consequence, if  $\text{KPh}_-^0$  denotes  $\text{KPh}^0$  *without* global  $\Delta_0$  collection, then  $\text{FTR}_0$  is already contained in  $\text{KPh}_-^0$ . In addition, if  $(\text{F-I}_{\omega})$  denotes the schema of formula induction on the natural numbers in the language  $\mathcal{L}_{\mathfrak{s}}$ , then one readily realizes that Theorem 7 establishes an embedding of  $\text{FTR}$  into  $\text{KPh}_-^0 + (\text{F-I}_{\omega})$ . Moreover, the methods of [12] allow one to show that  $|\text{KPh}_-^0 + (\text{F-I}_{\omega})| \leq \varphi_{20\varepsilon_0}$  and, hence, we obtain together with Theorem 4 an exact calibration of the strength of  $\text{FTR}$ .

**Theorem 8.**  $|\text{FTR}| = \varphi_{20\varepsilon_0}$ .

*Remark 2.* We note that the theory  $\text{KPh}_-^0 + (\text{F-I}_{\omega})$  is stronger than  $\text{KPh}^0 + (\text{F-I}_{\omega})$ . To be precise, we have that  $|\text{KPh}_-^0 + (\text{F-I}_{\omega})| = \varphi_{2\varepsilon_0}$ .

## 6 Conclusion and related systems

In this article we have studied various forms of constructing hierarchies of fixed points of positive arithmetic operators in an autonomous manner. We have seen that the corresponding principles are closely related to systems of Kripke Platek set theory without foundation whose universe of sets forms a limit of inaccessible. We summarize the results of the previous sections in the following theorem.

**Theorem 9.** *We have the following proof-theoretic equivalences:*

1.  $\text{Aut}(\widehat{\text{ID}}) \equiv \text{FTR}_0 \equiv \text{KPh}^0 \equiv \text{KPh}_-^0$ ;
2.  $\text{FTR} \equiv \text{KPh}_-^0 + (\text{F-I}_\omega)$ .

*The theories in the first row have proof-theoretic ordinal  $\varphi_{200}$ , the ones in the second row  $\varphi_{20\varepsilon_0}$ .*

Let us finish this article by mentioning some recent results in metapredicative proof theory which are related to the ones discussed in this paper. There is a broad variety of theories whose proof-theoretic ordinal can be denoted by means of the ternary Veblen function. Among those, theories with a proof-theoretic ordinal that is expressible by the ordinal function  $\lambda_{\alpha, \beta} \cdot \varphi_{1\alpha}\beta$ , i.e., an ordinary Veblen hierarchy above the  $\Gamma$  function, provide first natural examples of metapredicative systems. The theories  $\widehat{\text{ID}}_\alpha$  belong to this family, cf. Theorem 1 of this paper.

Interesting subsystems of second order arithmetic which can be measured against transfinitely iterated fixed point theories are extension of Friedman's  $\text{ATR}_0$  (cf. [6, 9, 25, 26]) by  $\Sigma_1^1$  dependent choice. Let us recall that the schema of *arithmetic transfinite recursion* (ATR) says that arithmetic jump hierarchies exist along any wellordering.  $\text{ATR}_0$  is defined to be  $\text{ACA}_0$  plus all instances of (ATR), and ATR denotes the corresponding system with full formula induction on the natural numbers. Recently, Avigad [1] gave a neat equivalent formulation of (ATR) in terms of a second order fixed point axiom schema. His principle (FP) claims for each positive arithmetic operator form  $\mathcal{A}(X, Y, x, y)$  the existence of an  $\mathcal{A}$  fixed point depending on parameters  $Y, y$ , more precisely:

$$(\exists X)(\forall x)[x \in X \leftrightarrow \mathcal{A}(X, Y, x, y)].$$

It is shown in [1] that (ATR) and (FP) are equivalent over  $\text{ACA}_0$ . The schema of  $\Sigma_1^1$  *dependent choice*, ( $\Sigma_1^1$ -DC), consists of the assertions

$$(\forall X)(\exists Y)A(X, Y) \rightarrow (\forall X)(\exists Z)[(Z)_0 = X \wedge (\forall u)A((Z)_u, (Z)_{u+1})]$$

for each  $\Sigma_1^1$  formula  $A$  of  $\mathcal{L}_2$ . It has long been known that ( $\Sigma_1^1$ -DC) is not provable in  $\text{ATR}_0$ , cf. e.g. Simpson [26]. The exact strength of ( $\Sigma_1^1$ -DC) in the context of (ATR) is determined in Jäger and Strahm [11]; in particular, the following proof-theoretic equivalences are established there:

$$\text{ATR} \equiv \widehat{\text{ID}}_\omega, \quad \text{ATR}_0 + (\Sigma_1^1\text{-DC}) \equiv \widehat{\text{ID}}_{<\omega\omega}, \quad \text{ATR} + (\Sigma_1^1\text{-DC}) \equiv \widehat{\text{ID}}_{<\varepsilon_0}.$$

Thanks to Theorem 1, the corresponding proof-theoretic ordinals are  $\Gamma_{\varepsilon_0}, \varphi_{1\omega}0$  and  $\varphi_{1\varepsilon_0}0$ , respectively. The proof-theoretic ordinal of ATR is previously due to Friedman (cf. Simpson [25]) and Jäger [7]. For connections between the theories  $\widehat{\text{ID}}_\alpha$  and subsystems of analysis based on restricted forms of bar induction the reader is referred to Jäger and Strahm [11].

There are also natural subsystems of  $\text{KPh}^0$  which can be compared to transfinitely iterated fixed point theories. Recall that Jäger's system  $\text{KPi}^0$  (cf. the last

section) has proof-theoretic ordinal exactly  $\Gamma_0$  (cf. Jäger [8]). The system which is obtained from  $\text{KPi}^0$  by omitting *global*  $\Delta_0$  collection is usually denoted by  $\text{KPi}^0$ ; since Jäger's [8] embedding of  $\text{ATR}_0$  into  $\text{KPi}^0$  does not make use of global  $\Delta_0$  collection, we have that  $\text{KPi}^0$  and  $\text{KPi}^0$  are of the same strength. This picture changes drastically in the presence of formula induction ( $\text{F-I}_\omega$ ) on the natural numbers or induction on the natural numbers for  $\Sigma_1$  formulas, ( $\Sigma_1\text{-I}_\omega$ ). Here we have the following relationship to transfinitely iterated fixed point theories:

$$\text{KPi}^0 + (\text{F-I}_\omega) \equiv \widehat{\text{ID}}_\omega, \quad \text{KPi}^0 + (\Sigma_1\text{-I}_\omega) \equiv \widehat{\text{ID}}_{<\omega^\omega}, \quad \text{KPi}^0 + (\text{F-I}_\omega) \equiv \widehat{\text{ID}}_{<\varepsilon_0}.$$

Lower bounds for these three equivalences are obtained as follows: since  $\text{ATR}$  is contained in  $\text{KPi}^0 + (\text{F-I}_\omega)$  we have that  $\Gamma_{\varepsilon_0}$  is a lower bound of this system; further, since transfinite induction for  $\Sigma_1$  statements is available in  $\text{KPi}^0 + (\Sigma_1\text{-I}_\omega)$  and  $\text{KPi}^0 + (\text{F-I}_\omega)$  below  $\omega^\omega$  and  $\varepsilon_0$ , respectively, the proof of Theorem 7 reveals that  $\widehat{\text{ID}}_{<\omega^\omega}$  and  $\widehat{\text{ID}}_{<\varepsilon_0}$  is contained in  $\text{KPi}^0 + (\Sigma_1\text{-I}_\omega)$  and  $\text{KPi}^0 + (\text{F-I}_\omega)$ , respectively. Moreover, the methods of [10] or [12] can be used in order to show that these bounds are indeed sharp. The system  $\text{KPi}^0 + (\Sigma_1\text{-I}_\omega)$  is not directly comparable to transfinitely iterated fixed point theories. It can be shown that its proof-theoretic ordinal is  $\Gamma_{\omega^\omega}$ .

Let us include a short discussion on systems of *explicit mathematics* and first steps into metapredicativity. Explicit mathematics goes back to Feferman [3, 4]. Its primary aim was to lay a logical basis for constructive mathematics, but it soon turned out to be important in connection with various activities in proof theory, e.g. the reduction of strong classical systems to constructive ones. *Universes* are a frequently studied concept in constructive mathematics at least since the work of Martin-Löf, cf. e.g. Martin-Löf [15] or Palmgren [19] for a survey. They can be considered as types of types (or names) which are closed under previously recognized type formation operations, i.e. a universe *reflects* these operations. Hence, universes are closely related to reflection principles in classical and admissible set theory. Universes were first discussed in the framework of explicit mathematics in Feferman [5] in connection with his proof of Hancock's conjecture. In Marzetta [17, 16] they are introduced via a so-called (non-uniform) limit axiom, thus providing a natural framework of explicit mathematics which has exactly the strength of predicative analysis, cf. also Marzetta and Strahm [18] and Kahle [14].

In Strahm [27] a system of explicit mathematics termed EMU is introduced which incorporates a *uniform* universe construction principle and includes the schema of formula induction on the natural numbers. Universes are closed under elementary comprehension and join (disjoint union). It is shown in [27] that EMU is proof-theoretically equivalent to  $\widehat{\text{ID}}_{<\varepsilon_0}$ . Further, a natural subsystem of EMU is singled out which has the same strength as  $\widehat{\text{ID}}_{<\omega^\omega}$ . Independently and very recently, similar results have been obtained in the context of Frege structures by Kahle [13] and in the framework of Martin-Löf type theory by Rathjen [21].

This concludes our short discussion on systems whose proof-theoretic ordinal can be denoted by means of a Veblen hierarchy above the  $\Gamma$  function. Next steps into metapredicativity are provided by the theories which we have discussed

in this article, namely systems which allow for various forms of autonomous generation of fixed point hierarchies. We have seen that autonomous fixed point theories are related to hyperinaccessibility in Kripke Platek set theory without foundation. More generally, it is shown in Jäger and Strahm [12] and Strahm [28] that the standard theory which formalizes an  $n$ -hyperinaccessible universe of sets without foundation has proof-theoretic ordinal  $\varphi(n+1)00$ . Since the theory  $\text{KPM}^0$  for a recursively Mahlo universe of sets without foundation is proof-theoretically equivalent to the union of these theories for  $n$ -hyperinaccessibility for each finite  $n$  (see [12]), we have that  $\varphi\omega 00$  is the proof-theoretic ordinal of  $\text{KPM}^0$ . Finally, let us mention that there are natural systems of explicit mathematics which correspond to  $\text{KPM}^0$ , see [12] for details.

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