A note on the theory $\mathsf{SID}_{<\omega}$ of stratified induction

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Abstract

We introduce and analyse a theory of finitely stratified general inductive definitions over the natural numbers, $\mathsf{SID}_{<\omega}$, and establish its proof-theoretic ordinal, $\varphi_{\varepsilon_0}(0)$. The definition of $\mathsf{SID}_{<\omega}$ bears some similarities with D. Leivant's ramified theories for finitary inductive definitions.

Keywords: Proof theory, inductive definitions, stratification.

1 Introduction

First-order theories that result from number theory by adding new predicate symbols P and axioms for P are used as a tool to investigate the proof-theoretic strength of various theories (consider [BFPS81] for example). In particular and with focus on the topic of this note, predicates $P^{\mathfrak{A}}$ may formalise for each positive arithmetical operator form $\mathfrak{A}$ a fixed-point F_Φ of the operator $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$, where Φ is the intended interpretation of $\mathfrak{A}$ and $\mathcal{P}(\mathbb{N})$ is the power set of the natural numbers $\mathbb{N}$ (compare [Acz77a] for background information). A famous example of such a formalisation is the impredicative theory ID_1 that allows to axiomatise the least fixed-point I_Φ of such Φ by means of axioms for the closure property and the induction principle assigned to $P^{\mathfrak{A}}$; in the context of Φ this can be expressed by

$$\Phi(I_\Phi) \subseteq I_\Phi \qquad (\Phi\text{-Closure})$$

$$(\forall X \subseteq \mathbb{N})\,(\Phi(X) \subseteq X \to I_\Phi \subseteq X) \qquad (\Phi\text{-Induction})$$

and if considered as a definition of I_Φ, its impredicative characterisation becomes apparent by the unrestricted quantification over subsets of $\mathbb{N}$. Furthermore, one can consider just any fixed-point F_Φ, thus described by the single equation

$$\Phi(F_\Phi) = F_\Phi \qquad (\Phi\text{-Fixed-Point})$$

that is a consequence of (Φ-Closure) and (Φ-Induction).

A theory that formalises fixed-points over positive arithmetical operator forms $\mathfrak{A}$ is the theory $\widehat{\mathsf{ID}}_1$, it was introduced in [Acz77b] and further analysed for the iterated case in [Fef82] and [JKSS99], using predicative methods. While $\widehat{\mathsf{ID}}_1$ has no formalisation for (Φ-Induction) at all, a theory that is predicatively reducible and axiomatises certain (so-called positive) instances of $\Phi(X) \subseteq X \to I_\Phi \subseteq X$ is the theory ID_1^*. It has been analysed in [Pro06] and [AR10], in particular $|\mathsf{ID}_1^*| = |\widehat{\mathsf{ID}}_1| = \varphi_{\varepsilon_0}(0)$ has been shown there for the proof-theoretic ordinal of ID_1^*.

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The aim of this note is to investigate the proof-theoretic strength of a theory of stratified induction $\mathsf{SID}_{<\omega}$. It has a similar approach as in ID_1^*, namely in the sense of formalising certain instances of (Φ-Induction). In order to illustrate the differences, we need to consider the axioms of those two theories (precise formulations for $\mathsf{SID}_{<\omega}$ are given in section 2). Therefore let $\mathcal{L}_{\mathsf{PA}}$ be some usual formulation of the first-order language of Peano arithmetic PA, augmented with countably many set-parameters (i.e., fresh unary relation symbols X) that are only used as place-holders in the sense described further below. A formula is called *arithmetical* if it is formulated in the language $\mathcal{L}_{\mathsf{PA}}$. For any arithmetical formula $\mathfrak{A}(X,x)$ with a designated number variable x that may occur free in it (and that contains no other free number variable) and a designated set-parameter X occurring at most positively—i.e., $\mathfrak{A}(X,x)$ is a positive operator form—let $P^{\mathfrak{A}}$ be a distinguished new unary relation symbol not in $\mathcal{L}_{\mathsf{PA}}$. Then the language obtained by extending $\mathcal{L}_{\mathsf{PA}}$ with such new symbols $P^{\mathfrak{A}}$ is used as the language $\mathcal{L}_{\mathsf{ID}}$ for the theories ID_1, $\widehat{\mathsf{ID}}_1$, and ID_1^*. The first of these theories formalises (Φ-Induction) by means of the axiom scheme

$$(\forall x)(\mathfrak{A}\big(\{z\colon B\},x\big) \to B_z(x)) \to (\forall x)(P^{\mathfrak{A}}(x) \to B_z(x)) \tag{ID}$$

where B can be any $\mathcal{L}_{\mathsf{ID}}$ formula and $B_z(t)$ denotes for any $\mathcal{L}_{\mathsf{PA}}$ term t the substitution of z in B by t, furthermore $\mathfrak{A}\big(\{z\colon B\},x\big)$ expresses the straight-forward substitution of atomic formulas $t \in X$ in $\mathfrak{A}(X,x)$ by $B_z(t)$.

While $\widehat{\mathsf{ID}}_1$ has no instances of (ID), the theory ID_1^* allows for positive induction (ID^*), i.e., it contains instances of (ID) where B may contain $P^{\mathfrak{A}}$ at most positively. The new theory $\mathsf{SID}_{<\omega}$ that we propose and investigate here is used to express a kind of *stratified induction* (over fixed-points) by admitting indexed copies of the above mentioned symbols $P^{\mathfrak{A}}$, namely by replacing $P^{\mathfrak{A}}$ with infinitely many distinguished new unary relation symbols $P^{\mathfrak{A}}_n$ for $1 \le n < \omega$ (i.e., $P^{\mathfrak{A}}_1, P^{\mathfrak{A}}_2, \dots$). Hence, $\mathsf{SID}_{<\omega}$ has a different language than ID_1 and further has stratified induction (over fixed points) via the axiom scheme

$$(\forall x)(\mathfrak{A}\big(\{z\colon B\},x\big) \to B_z(x)) \to (\forall x)(P^{\mathfrak{A}}_n(x) \to B_z(x)) \tag{SID}$$

for $1 \le n < \omega$ and with the restriction that B has to be a formula in this new language which may contain relation symbols $P^{\mathfrak{B}}_l$ only if $l < n$ (where $\mathfrak{B}$ is some operator form).

Let SID_n denote the restriction of the theory $\mathsf{SID}_{<\omega}$ to formulas that contain at most the symbols $P^{\mathfrak{A}}_l$ with $l \le n$. The theory SID_0 is just PA and the theory SID_1 is essentially a weakening of ID_1^* where (ID^*) is further restricted in B to allow only arithmetical formulas. We will investigate the theories SID_n of *finitely stratified induction* and refer for the next question on the treatment of *transfinitely stratified induction* to [JP15]. In this note, we show how we can apply the proof-theoretic technique of asymmetric interpretation very neatly in order to gain proof-theoretic insight into this concept of stratified induction.

Aiming towards a characterisation of $\mathsf{SID}_{<\omega}$, notice that it is the same as $\bigcup_{n<\omega} \mathsf{SID}_n$ and that obviously SID_n embeds into ID_n^* for any $n < \omega$. So we have for the proof-theoretic ordinal $|\mathsf{SID}_{<\omega}| \le |\bigcup_{n<\omega} \mathsf{ID}_n^*| = \Gamma_0$, see [Can85]. We show in this note that actually $|\mathsf{SID}_{<\omega}| = |\widehat{\mathsf{ID}}_1|$ holds. Since $\widehat{\mathsf{ID}}_1$ trivially embeds into $\mathsf{SID}_{<\omega}$, it suffices to show that $\varphi_{\varepsilon_0}(0) = |\widehat{\mathsf{ID}}_1|$ is an upper bound for $|\mathsf{SID}_{<\omega}|$, and this is done via an asymmetric interpretation combined with partial cut-elimination.

The approach of this article bears some similarities to D. Leivant's proof-theoretic approach to computational complexity (cf. e.g. [Lei94]) which makes use of ramified theories over (finitary) inductively generated free algebras. Here we treat ramified general inductive definitions over the natural numbers. W. Buchholz's notes [Buc05] contributed to the presentation of the following material.

2 The theory $\mathsf{SID}_{<\omega}$ of stratified induction

We are now going to render more precisely the notions given in the introduction.

Definition 2.1 (Preliminaries)**.**

- As *basic logical symbols* of first-order predicate logic with equality, take the usual symbols $\neg, \wedge, \vee, \forall, \exists, =$, together with countably many *(number) variables.* We use x, y, z as syntactic variables for those number variables, and for any kind of syntactic variables introduced here, we allow subscripts and vector notation.

- $\mathcal{L}_{\mathsf{PA}}$ denotes the first-order language of *Peano arithmetic* PA (with the usual function and relation symbols for primitive recursive functions and relations) plus *set-parameters*, i.e., countably many unary relation symbols. We use X, Y, Z as syntactic variables for set-parameters.

- $\mathcal{L}_{\mathsf{PA}}$ terms will be just called *terms.* We use s, t as syntactic variables for terms and denote by TER_0 the set of closed terms. In case that t is *closed* (i.e., does not contain a variable) we mean by $t^{\mathbb{N}}$ the *numerical value* of t, i.e., the valuation of t in the standard model $\mathbb{N}$.

  Consider a fixed language $\mathcal{L} \supseteq \mathcal{L}_{\mathsf{PA}}$. We define $\mathcal{L}$ *formulas* as usual inductively from $\mathcal{L}$ and the basic symbols but with the restriction that the negation symbol $\neg$ is only allowed to occur in front of an atomic formula. We use A, B, C, D as syntactic variables for $\mathcal{L}$ formulas. A *literal* is either an atomic formula or its negated version. In case of a compound formula A, its *negation* $\neg A$ stands for the translation of A according to De Morgan's laws and the law of double negation.

  Moreover, we use capital Greek letters Γ, Δ, Σ as syntactic variables for $\mathcal{L}$ *sequents*, i.e., finite (possibly empty) lists of $\mathcal{L}$ formulas (e.g., $A_0, \ldots, A_k$) that are identified with finite sets (i.e., $\{A_0, \ldots, A_k\}$). Therefore, Γ, A is understood as $\Gamma \cup \{A\}$ and accordingly Γ, Δ is identified with $\Gamma \cup \Delta$.

  In order to stress that A is an $\mathcal{L}$ formula, we write ambiguously $A \in \mathcal{L}$. Moreover, $A \to B$ abbreviates just $\neg A \vee B$. If an $\mathcal{L}$ formula is introduced as $A(z)$, this means that A denotes this formula and that the variable z may occur freely in A. We write $\mathrm{FV}(A)$ for the set of free number variables of A. A formula is called *arithmetical* in case of $A \in \mathcal{L}_{\mathsf{PA}}$.

  Substitution of a variable z in A by a term t is denoted by $A_z(t)$, or just by $A(t)$ in case A has been introduced in the form $A(z)$.

  Let P be a unary relation symbol of $\mathcal{L}$ and A an $\mathcal{L}$ formula. Then we say that P occurs *positively* in A if A does not contain the negated formula $\neg P(t)$ for any term t. Moreover, we write often $t \in P$ instead of $P(t)$ and $t \notin P$ instead of $\neg P(t)$.

- By *ordinals* we actually mean ordinals smaller than the first strongly-critical ordinal Γ_0, in particular we will work with the binary Veblen-function φ. In fact, we will need only ordinals below $\varphi_{\varepsilon_0}(0)$ and for most of the results even ordinals below ε_0 will suffice. We use small Greek letters $\alpha, \beta, \gamma, \delta, \xi, \pi, \nu, \tau$ as syntactic variables for ordinals.

- From now on, let X denote a fixed set-parameter. We call an arithmetical formula a *positive (arithmetical) operator form* if it contains X at most *positively* and if it contains *at most one number variable*. In order to stress the special role of such formulas, we use $\mathfrak{A}, \mathfrak{B}$ as syntactic variables for those and consider them implicitly given as $\mathfrak{A}(X, x)$ for some variable x. In particular, we assume $\mathrm{FV}(\mathfrak{A}(X, x)) \subseteq \{x\}$ here, while set-parameters distinct from X may still occur in $\mathfrak{A}$.

  Given a language $\mathcal{L} \supseteq \mathcal{L}_{\mathsf{PA}}$ together with an $\mathcal{L}$ formula B and a term t, we write $\mathfrak{A}(\{z\colon B\}, t)$ for the $\mathcal{L}$ formula obtained from $\mathfrak{A}$ by substituting any atomic formula $X(s)$ with $B_z(s)$ and any occurrence of x with t, while a renaming of bound variables may be necessary as usual. In case B has been introduced as $B(z)$, we write also $\mathfrak{A}(B(z), t)$ for $\mathfrak{A}(\{z\colon B\}, t)$. Moreover, for unary relation symbols $P \in \mathcal{L}$, we abbreviate $\mathfrak{A}(\{z\colon P(z)\}, t)$ by $\mathfrak{A}(P, t)$.

- The proof-theoretic ordinal of a first-order theory T over a language $\mathcal{L} \supseteq \mathcal{L}_{\mathsf{PA}}$ is denoted by $|\mathsf{T}|$. We refer to [Poh09] for background information.

Definition 2.2. For each $\mathfrak{A}$ and $1 \leq n < \omega$ let $P_n^{\mathfrak{A}}$ denote a new and distinguished unary relation symbol. Furthermore, define for each $n < \omega$:

$$\mathcal{L}_0 := \mathcal{L}_{\mathsf{PA}} \qquad\qquad \mathcal{L}_{n+1} := \mathcal{L}_n \cup \{\, P_{n+1}^{\mathfrak{A}} \colon \mathfrak{A} \text{ is a positive operator form} \,\}$$

From now on, let A, B, C, D range over formulas of the language $\mathcal{L}_{<\omega} := \bigcup_{n<\omega} \mathcal{L}_n$.

Definition 2.3. For each $n < \omega$, the theory SID_n with language $\mathcal{L}_n$ consists of the following axioms.

I. Number-theoretic and logical axioms:

The axioms of PA with the scheme of complete induction for all $\mathcal{L}_n$ formulas.

II. Stratified induction axioms for $1 \leq m \leq n$ and $B(z) \in \mathcal{L}_{m-1}$:

$$\forall x(\mathfrak{A}(B(z), x) \to B(x)) \to \forall x(x \in P_m^{\mathfrak{A}} \to B(x))$$

III. Fixed-point axioms for $1 \leq m \leq n$:

$$\forall x(\mathfrak{A}(P_m^{\mathfrak{A}}, x) \leftrightarrow x \in P_m^{\mathfrak{A}})$$

The theory $\mathsf{SID}_{<\omega}$ with language $\mathcal{L}_{<\omega}$ is the collection $\bigcup_{n<\omega} \mathsf{SID}_n$. Furthermore, we also presume that a derivability notion $\mathsf{SID}_n \vdash A$ is given for each $n < \omega$ and $A \in \mathcal{L}_n$ via a standard first-order Hilbert-style predicate calculus. Accordingly, $\mathsf{SID}_{<\omega} \vdash A$ for $A \in \mathcal{L}_{<\omega}$ just means that $A \in \mathcal{L}_n$ and $\mathsf{SID}_n \vdash A$ hold for some $n < \omega$.

Theorem 2.4 (Lower bound of $|\mathsf{SID}_{<\omega}|$).

$$\widehat{\mathsf{ID}}_1 \vdash A \implies \mathsf{SID}_1 \vdash A$$

holds for each $A \in \mathcal{L}_{\mathsf{PA}}$. Therefore $\varphi_{\varepsilon_0}(0) \leq |\mathsf{SID}_{<\omega}|$.

Proof. Recall that $|\widehat{\mathsf{ID}}_1| = \varphi_{\varepsilon_0}(0)$ and notice that $\widehat{\mathsf{ID}}_1$ is essentially SID_1 without **II** from its definition. □

Strategy for the upper bound of $\mathsf{SID}_{<\omega}$. We will work with infinitary proof systems SID_n^∞ with $n < \omega$ that are suitable for partial cut elimination, asymmetric interpretation, and in case of $n = 0$ full predicative cut-elimination. The steps to reach the main result of section 4 will be the following:

1. Unary relation symbols $Q_{\mathfrak{A}}^{<\xi}$ for each $\mathfrak{A}$ and ξ are added to the language.
2. For each $n < \omega$, set up an infinitary proof-system SID_n^∞. For $n > 0$, we obtain a useful result on *partial cut elimination (p.c.e.)*, while for the case $n = 0$, we can even achieve *full predicative cut-elimination (f.c.e.)*.
3. *Asymmetric interpretation (a.i.)* is used to establish the connection between the systems $\mathsf{SID}_{n+1}^\infty$ and SID_n^∞ for any $n < \omega$, given that we deal with derivations where we partially removed cuts first. In particular, the symbols $P_{n+1}^{\mathfrak{A}}$ are interpreted by $Q_{\mathfrak{A}}^{<\xi}$ for suitable ξ.
4. The theme is to start with a formal derivation in SID_{n+1} of an *arithmetical* formula A, embed it into $\mathsf{SID}_{n+1}^\infty$ such that the proof complexity stays below ε_0, combine a p.c.e. followed by an a.i. iteratively, and end up with a derivation in SID_0^∞ with proof complexity still below ε_0. Then f.c.e. yields the desired sharp bound $\varphi_{\varepsilon_0}(0)$ for $|\mathsf{SID}_{<\omega}|$ via a standard boundedness argument:

$$\mathsf{SID}_{n+1} \overset{\text{embed}}{\rightsquigarrow} \mathsf{SID}_{n+1}^\infty \overset{\text{p.c.e.}}{\rightsquigarrow} \mathsf{SID}_{n+1}^\infty \overset{\text{a.i.}}{\rightsquigarrow} \mathsf{SID}_n^\infty \rightsquigarrow \cdots \rightsquigarrow \mathsf{SID}_0^\infty \overset{\text{f.c.e.}}{\rightsquigarrow} \mathsf{SID}_0^\infty$$

   Besides the care needed to maintain a proof-complexity below ε_0, we also have to cope with the fact that in general an infinitary proof system may yield derivations whose cuts cannot be globally bounded. In particular for our iterative use of p.c.e. that started with embedding a formal derivation (e.g., from SID_{n+1} into $\mathsf{SID}_{n+1}^\infty$), we depend on the method of a.i. to provide always a derivation whose cut-formulas are bounded by a finite ordinal. To guarantee this, we shall fix a finite ordinal ℓ and restrict the derivability relation for SID_n^∞ with $n > 0$ such that the cut-formulas have to be globally bounded by ℓ.

3 The infinitary proof system SID_n^∞ for $n < \omega$

Convention. For the rest of this section, we fix some finite ordinal ℓ. In particular, we will define the derivability relation for the proof systems SID_n^∞ such that ℓ globally bounds the length of the cut-formulas that are allowed in an application of a cut-rule if $n > 0$. Compare the proof of lemma 3.9 to see why this bound should not hold for the case $n = 0$.

Definition 3.1. Let $Q_{\mathfrak{A}}^{<\xi}$ be a fresh unary relation symbol for each $\mathfrak{A}$ and ξ. For each $n < \omega$, let $\mathcal{L}_n^\infty := \mathcal{L}_n \cup \{\, Q_{\mathfrak{A}}^{<\xi} \colon \xi < \Gamma_0 \;\; \& \;\; \mathfrak{A} \text{ is a positive operator form}\,\}$. In the following, let A, B, C, D range over formulas of the language $\mathcal{L}_{<\omega}^\infty := \bigcup_{n<\omega} \mathcal{L}_n^\infty$.

Definition 3.2. The *length* $\mathrm{lh}(A)$ of a formula A is defined as the number of basic logical symbols that occur in A. In particular, $\mathrm{lh}(A) = \mathrm{lh}(A_x(t))$ holds for all terms t.

Definition 3.3. Let $\mathrm{rk}_0(A) := 0$ for each $A \in \mathcal{L}_0^\infty$. For $1 \le n < \omega$, we say that $A \in \mathcal{L}_n^\infty$ is *n-atomic* if $A \in \mathcal{L}_{n-1}^\infty$ or if it is a literal of the form $t \in P_n^{\mathfrak{A}}$ or $t \notin P_n^{\mathfrak{A}}$.

The *n-rank* $\mathrm{rk}_n(A) < \omega$ is defined for $1 \le n < \omega$ and formulas $A \in \mathcal{L}_n^\infty$ by

$$\mathrm{rk}_n(A) := \begin{cases} 0 & \text{if } A \text{ is } n\text{-atomic, or otherwise} \\ \max(\mathrm{rk}_n(B), \mathrm{rk}_n(C)) + 1 & \text{if } A = B \wedge C \text{ or } A = B \vee C \\ \mathrm{rk}_n(B) + 1 & \text{if } A = \forall x B \text{ or } A = \exists x B \end{cases}$$

The *ordinal-rank* $\mathrm{rk}(A) < \Gamma_0$ is defined for formulas $A \in \mathcal{L}_{<\omega}^\infty$ by

$$\mathrm{rk}(A) := \begin{cases} 0 & \text{if } A \text{ is a literal and } A \in \mathcal{L}_{<\omega} \\ \omega \cdot \xi & \text{if } A = t \in Q_{\mathfrak{A}}^{<\xi} \text{ or } A = t \notin Q_{\mathfrak{A}}^{<\xi} \\ \max(\mathrm{rk}(B), \mathrm{rk}(C)) + 1 & \text{if } A = B \wedge C \text{ or } A = B \vee C \\ \mathrm{rk}(B) + 1 & \text{if } A = \forall x B \text{ or } A = \exists x B \end{cases}$$

Furthermore for $1 \le n < \omega$ and $A \in \mathcal{L}_n^\infty$, we write $A \in \mathrm{Pos}_n$ to denote that $P_n^{\mathfrak{A}}$ occurs at most positively in A for every $\mathfrak{A}$, and we write $A \in \mathrm{Neg}_n$ to denote $\neg A \in \mathrm{Pos}_n$.

Remark. For $A \in \mathcal{L}_n^\infty$ and $1 \le n < \omega$, we have that $\mathrm{lh}(A) < \ell$ implies $\mathrm{rk}_n(A) < \ell$, and that $\mathrm{rk}_n(A) \ne 0$ implies that A is not a literal.

Definition 3.4. For each $n < \omega$, the infinitary Tait-style proof system SID_n^∞ with language $\mathcal{L}_n^\infty$ is defined by means of the following inferences (i.e., axioms and inference rules). SID_n^∞ shall derive $\mathcal{L}_n^\infty$ sequents that consist of *closed* formulas only, therefore we assume in this definition that the sequents of the axioms and the sequents that occur in the premiss of a rule consist of closed $\mathcal{L}_n^\infty$ formulas only. Note that the inference rules $(\bigwedge_{\forall x A})$ and $(\bigwedge_{t \notin Q_{\mathfrak{A}}^{<\tau}})$ have infinitely many premisses.

I. Number-theoretic and logical axioms:

$$\Gamma, A \qquad \text{if } A \text{ is a true } \mathcal{L}_{\mathsf{PA}} \text{ literal without set-parameters}$$

$$\Gamma, A(s), \neg A(t) \qquad \text{if } s^{\mathbb{N}} = t^{\mathbb{N}} \text{ and } A(z) \in \mathcal{L}_n \text{ is atomic}$$

II. Stratified induction axioms for each $1 \le m \le n$ and $B(z) \in \mathcal{L}_{m-1}$:

$$\Gamma, \exists x(\mathfrak{A}(B(z), x) \wedge \neg B(x)), t \notin P_m^{\mathfrak{A}}, B(t)$$

III. Fixed-point rules for $1 \le m \le n$:

$$\frac{\Gamma, \mathfrak{A}(P_m^{\mathfrak{A}}, t)}{\Gamma, t \in P_m^{\mathfrak{A}}} \ (\mathsf{Fix}_{t \in P_m^{\mathfrak{A}}}) \qquad \frac{\Gamma, \neg\mathfrak{A}(P_m^{\mathfrak{A}}, t)}{\Gamma, t \notin P_m^{\mathfrak{A}}} \ (\mathsf{Fix}_{t \notin P_m^{\mathfrak{A}}})$$

IV. Predicative rules:

$$\frac{\Gamma, A}{\Gamma, A \vee B} \ (\textstyle\bigvee_{A \vee B}^{A}) \qquad \frac{\Gamma, B}{\Gamma, A \vee B} \ (\textstyle\bigvee_{A \vee B}^{B}) \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B} \ (\textstyle\bigwedge_{A \wedge B})$$

$$\frac{\Gamma, A_x(t)}{\Gamma, \exists x A} \ (\textstyle\bigvee_{\exists x A}^{t}) \text{ for } t \in \mathrm{TER}_0 \qquad \frac{\ldots \ \Gamma, A_x(t) \ \ldots \ (t \in \mathrm{TER}_0)}{\Gamma, \forall x A} \ (\textstyle\bigwedge_{\forall x A})$$

$$\frac{\Gamma, \mathfrak{A}(Q_{\mathfrak{A}}^{<\xi}, t)}{\Gamma, t \in Q_{\mathfrak{A}}^{<\tau}} \ (\textstyle\bigvee_{t \in Q_{\mathfrak{A}}^{<\tau}}^{\xi}) \text{ for } \xi < \tau \qquad \frac{\ldots \ \Gamma, \neg\mathfrak{A}(Q_{\mathfrak{A}}^{<\xi}, t) \ \ldots \ (\xi < \tau)}{\Gamma, t \notin Q_{\mathfrak{A}}^{<\tau}} \ (\textstyle\bigwedge_{t \notin Q_{\mathfrak{A}}^{<\tau}})$$

V. Cut rule:

$$\frac{\Gamma, C \qquad \Gamma, \neg C}{\Gamma}\ (\mathsf{Cut}_C)$$

For each of the above mentioned inferences, we define the notions *side formula*, *minor formula*, and *main formula* as usual. In particular, (Cut_C) has no main formulas, the axioms in **I** and **II** do not have minor formulas, and for every inference the formulas in the sequent Γ are the side formulas.

Definition 3.5. The derivability notion $\mathsf{SID}_n^\infty \vdash_{\rho,r}^\alpha \Gamma$ for $n, r < \omega$ is defined inductively on α:

- $\mathsf{SID}_n^\infty \vdash_{\rho,r}^\alpha \Gamma$ holds for all α, ρ, and $r < \omega$ if Γ is an axiom of SID_n^∞.
- $\mathsf{SID}_n^\infty \vdash_{\rho,r}^\alpha \Gamma$ holds if there is a rule of SID_n^∞ in **III** or **IV** such that Γ is its conclusion and $\mathsf{SID}_n^\infty \vdash_{\rho,r}^{\alpha_\iota} \Gamma_\iota$ holds for each of its premisses Γ_ι with some $\alpha_\iota < \alpha$.
- $\mathsf{SID}_n^\infty \vdash_{\rho,r}^\alpha \Gamma$ holds if $\mathsf{SID}_n^\infty \vdash_{\rho,r}^{\alpha_0} \Gamma, C$ and $\mathsf{SID}_n^\infty \vdash_{\rho,r}^{\alpha_1} \Gamma, \neg C$ hold for some $\alpha_0, \alpha_1 < \alpha$ and we have $\mathrm{rk}(C) < \rho$, $\mathrm{rk}_n(C) < r$, and in case of $n > 0$ also $\mathrm{lh}(C) < \ell$.

Moreover, $\mathsf{SID}_n^\infty \vdash_{\rho,r}^{<\alpha} \Gamma$ means that $\mathsf{SID}_n^\infty \vdash_{\rho,r}^{\alpha_0} \Gamma$ holds for some $\alpha_0 < \alpha$.

Remark. Recalling the end of section 2 where we explained the strategy of this article, we notice here that for $n > 0$, the condition $\mathrm{lh}(C) < \ell$ in the third case of the above definition is needed in order to globally bound the occurring (cut-)formulas' syntactical complexity by a finite ordinal, namely ℓ. Having in mind the property of most derivability notions for infinitary proof systems that the underlying derivations may contain cut-formulas whose complexity cannot be globally bounded by a finite ordinal, we decided to add the condition $\mathrm{lh}(C) < \ell$ since otherwise it would have been more cumbersome to check and guarantee the well-behaviour of our iterative use of partial cut elimination and asymmetric interpretation that we are going to apply below. Furthermore, we put no extra effort in encoding such a property into rk_n because we wanted to keep rk_n as perspicuous as possible.

Lemma 3.6 (Weakening)**.**

$$\mathsf{SID}_n^\infty \vdash_{\rho,r}^\alpha \Gamma \ \ \& \ \ \alpha \le \beta \ \ \& \ \ \rho \le \eta \ \ \& \ \ r \le k \ \ \& \ \ \Gamma \subseteq \Delta \implies \mathsf{SID}_n^\infty \vdash_{\eta,k}^\beta \Delta$$

Proof. By a straight-forward induction on α. Notice that the condition concerning ℓ can be preserved here. □

Remark 3.7. $\mathsf{SID}_n^\infty \vdash_{\rho,r}^\alpha \Gamma$ with $\rho = 0$ or $r = 0$ implies $\mathsf{SID}_n^\infty \vdash_{0,0}^\alpha \Gamma$. Notice also that $\mathsf{SID}_0^\infty \vdash_{\rho,r}^\alpha \Gamma$ implies $\mathsf{SID}_0^\infty \vdash_{\rho,1}^\alpha \Gamma$ since $\mathrm{rk}_0(A) = 0$ for each $A \in \mathcal{L}_0^\infty$. Furthermore, we notice that in the following we will not mention every use of lemma 3.6 explicitly.

3.1 Partial and full cut-elimination

Lemma 3.8. *For each* $1 \le n < \omega$ *and* $C \in \mathcal{L}_n^\infty$ *with* $\mathrm{lh}(C) < \ell$, *we have*

$$\mathrm{rk}_n(C) = 1 + r \ \ \& \ \ \mathsf{SID}_n^\infty \vdash_{\rho,1+r}^\alpha \Gamma, C \ \ \& \ \ \mathsf{SID}_n^\infty \vdash_{\rho,1+r}^\beta \Gamma, \neg C \implies \mathsf{SID}_n^\infty \vdash_{\rho,1+r}^{\alpha\#\beta} \Gamma$$

Proof. By induction on $\alpha \# \beta$ and the following case distinction.
1. C or $\neg C$ is not among the main formulas of the last inference of $\mathsf{SID}_n^\infty \vdash^\alpha_{\rho,1+r} \Gamma, C$ or $\mathsf{SID}_n^\infty \vdash^\beta_{\rho,1+r} \Gamma, \neg C$, respectively: The claim follows immediately from the I.H. or, in case of an axiom, by reapplying the inference with suitable side formulas.
2. Otherwise, we notice first that $\mathrm{rk}_n(C) \neq 0$, hence C is not n-atomic and only the following cases are possible:
2.1. $C = C_0 \vee C_1$ and $\mathsf{SID}_n^\infty \vdash^{\alpha_0}_{\rho,1+r} \Gamma, C, C_0$ for some $\alpha_0 < \alpha$: Then we also get $\mathsf{SID}_n^\infty \vdash^{\beta_0}_{\rho,1+r} \Gamma, \neg C, \neg C_0$ for some $\beta_0 < \beta$, so by I.H. we get $\mathsf{SID}_n^\infty \vdash^{\alpha_0 \# \beta}_{\rho,1+r} \Gamma, C_0$ and $\mathsf{SID}_n^\infty \vdash^{\alpha \# \beta_0}_{\rho,1+r} \Gamma, \neg C_0$. Since $\alpha_0 \# \beta, \alpha \# \beta_0 < \alpha \# \beta$, $\mathrm{rk}_n(C_0) < \mathrm{rk}_n(C) = 1+r$, and also $\mathrm{lh}(C_0) < \mathrm{lh}(C)$ hold, we can apply (Cut_{C_0}) in order to obtain $\mathsf{SID}_n^\infty \vdash^{\alpha \# \beta}_{\rho,1+r} \Gamma$. The other cases where $\mathsf{SID}_n^\infty \vdash^{\alpha_0}_{\rho,1+r} \Gamma, C, C_1$ or $C = C_0 \wedge C_1$ holds are treated similarly.
2.2. $C = \exists x D$ or $C = \forall x D$: The claim follows similar to the previous case, notice that $\mathrm{lh}(D_x(t)) = \mathrm{lh}(D) < \mathrm{lh}(C)$ holds for any term t. □

Lemma 3.9. *For each $C \in \mathcal{L}_0^\infty$, we have*

$$\mathrm{rk}(C) = \rho \ \& \ \mathsf{SID}_0^\infty \vdash^\alpha_{\rho,r} \Gamma, C \ \& \ \mathsf{SID}_0^\infty \vdash^\beta_{\rho,r} \Gamma, \neg C \implies \mathsf{SID}_0^\infty \vdash^{\alpha \# \beta}_{\rho,r} \Gamma$$

Proof. By induction on $\alpha \# \beta$ and almost literally as lemma 3.8 because of a similar behaviour of the n-rank rk_n and the ordinal-rank rk in combination with the built-up of formulas. The following two special situations illustrate the advantage of the ordinal-rank rk and why this does not work for SID_n^∞ with $n > 0$. Assume that both C and $\neg C$ are among the main formulas of the last inference.
1. C is the main formula of an axiom: Then it can only be due to an instance of **I**, so C and $\neg C$ are $\mathcal{L}_{\mathsf{PA}}$ literals. If $C = Y(s)$ for some set-parameter Y and term s, then we have $\neg Y(t), Y(t') \in \Gamma$ for some t, t' with $t^{\mathbb{N}} = s^{\mathbb{N}} = t'^{\mathbb{N}}$, and hence Γ is already an instance of **I**. Otherwise, if C does not contain a set-parameter, the claim again follows easily from **I**.
2. $C = t \in Q_{\mathfrak{A}}^{<\tau}$ with $\mathsf{SID}_0^\infty \vdash^{\alpha_\xi}_{\rho,r} \Gamma, C, \mathfrak{A}(Q_{\mathfrak{A}}^{<\xi}, t)$ for some $\xi < \tau$ and $\alpha_\xi < \alpha$: Now $\rho = \omega \cdot \tau$ and $\neg C = t \notin Q_{\mathfrak{A}}^{<\tau}$. Because of the definition of SID_0^∞, we do not have $\mathsf{SID}_0^\infty \vdash^\beta_{\rho,r} \Gamma, \neg C$ due to a logical axiom and hence $\neg C$ must be the main formula of $(\bigwedge_{t \notin Q_{\mathfrak{A}}^{<\tau}})$. Then we have $\mathsf{SID}_0^\infty \vdash^{\beta_\xi}_{\rho,r} \Gamma, \neg C, \neg \mathfrak{A}(Q_{\mathfrak{A}}^{<\xi}, t)$ available with $\beta_\xi < \beta$ for every $\xi < \tau$, so the claim follows very similar as in the proof of lemma 3.8. Notice that in the setting of SID_0^∞, we do not have to guarantee $\mathrm{lh}(\mathfrak{A}(Q_{\mathfrak{A}}^{<\xi}, t)) < \ell$, and that we have $\mathrm{rk}(\mathfrak{A}(Q_{\mathfrak{A}}^{<\xi}, t)) < \omega \cdot (\xi + 1) \leq \rho$ because of $\xi < \tau$. □

Theorem 3.10 (Cut-elimination)**.**

(a) *Partial cut-elimination:* $\mathsf{SID}_n^\infty \vdash^\alpha_{\rho,1+r} \Gamma$ *implies* $\mathsf{SID}_n^\infty \vdash^{\omega_r(\alpha)}_{\rho,1} \Gamma$ *for each* $1 \leq n < \omega$, *where* $\omega_0(\alpha) := \alpha$ *and* $\omega_{k+1}(\alpha) := \omega_k(\omega^\alpha)$.

(b) *Full predicative cut-elimination:* $\mathsf{SID}_0^\infty \vdash^\alpha_{\gamma+\omega^\delta,1} \Gamma$ *implies* $\mathsf{SID}_0^\infty \vdash^{\varphi\delta(\alpha)}_{\gamma,1} \Gamma$.

Proof. The theorem follows from the previous lemmas by a standard argument, and we refer to [Poh09] for details. □

3.2 Asymmetric interpretation

Convention. We fix $n < \omega$ for this subsection and will only deal with the proof systems SID_n^∞ and $\mathsf{SID}_{n+1}^\infty$.

Definition 3.11. For $\mathcal{L}_{n+1}^\infty$ formulas A, $\mathcal{L}_{n+1}^\infty$ sequents Γ, and ordinals $\xi, \xi_1, \dots, \xi_k$, we write

A^ξ	for the $\mathcal{L}_n^\infty$ formula that is obtained from A by substituting any $P_{n+1}^{\mathfrak{A}}$ that occurs in A with the corresponding symbol $Q_{\mathfrak{A}}^{<\xi}$,
$[\Gamma]^\xi$	for the $\mathcal{L}_n^\infty$ sequent obtained from Γ by substituting every occurring formula A with A^ξ,

and if Γ is explicitly given as a list $A_1, \dots, A_k$, we write

$[\Gamma]^{\xi_1,\dots,\xi_k}$	for the $\mathcal{L}_n^\infty$ sequent $A_1^{\xi_1}, \dots, A_k^{\xi_k}$

Lemma 3.12.

(a) $\mathsf{SID}_n^\infty \vdash_{\rho,r}^{\alpha} \Gamma, B(s_1), \neg B'(s_2)$ *for each* s_1, s_2 *with* $s_1^{\mathbb{N}} = s_2^{\mathbb{N}}$ *implies that for each* t_1, t_2 *with* $t_1^{\mathbb{N}} = t_2^{\mathbb{N}}$ *also* $\mathsf{SID}_n^\infty \vdash_{\rho,r}^{\alpha+2\cdot\mathrm{rk}(\mathfrak{A})} \Gamma, \mathfrak{A}(B(z), t_1), \neg\mathfrak{A}(B'(z), t_2)$ *holds.*

(b) $s^{\mathbb{N}} = t^{\mathbb{N}}$ *and* $\nu \le \pi$ *imply* $\mathsf{SID}_n^\infty \vdash_{0,0}^{\omega\cdot\nu} s \in Q_{\mathfrak{A}}^{<\pi}, t \notin Q_{\mathfrak{A}}^{<\nu}$.

(c) $s^{\mathbb{N}} = t^{\mathbb{N}}$ *and* $A(z) \in \mathcal{L}_n^\infty$ *imply* $\mathsf{SID}_n^\infty \vdash_{0,0}^{2\cdot\mathrm{rk}(A)} A(s), \neg A(t)$

(d) $B(z) \in \mathcal{L}_n$ *implies* $\mathsf{SID}_n^\infty \vdash_{0,0}^{\omega\cdot\tau}, \exists x(\mathfrak{A}(B(z), x) \wedge \neg B(x)), t \notin Q_{\mathfrak{A}}^{<\tau}, B(t)$.

Proof. Statement (a) is proven by a straight-forward induction on $\mathrm{rk}(\mathfrak{A}) < \omega$ and we leave the proof to the reader. Statement (b) is proven by induction on ν: The case $\nu = 0$ follows from $(\bigwedge_{t \notin Q_{\mathfrak{A}}^{<0}})$. If $\nu > 0$, the I.H. and (a) yield $\mathsf{SID}_n^\infty \vdash_0^{\omega\cdot\xi+2\cdot\mathrm{rk}(\mathfrak{A})} \mathfrak{A}(Q_{\mathfrak{A}}^{<\xi}, s), \neg\mathfrak{A}(Q_{\mathfrak{A}}^{<\xi}, t)$ for all $\xi < \nu$. Since $\nu \le \pi$, the claim follows from $(\bigvee_{t \in Q_{\mathfrak{A}}^{<\pi}}^{\xi})$ and $(\bigwedge_{s \notin Q_{\mathfrak{A}}^{<\nu}})$, and notice that $\mathfrak{A} \in \mathcal{L}_{\mathsf{PA}}$ implies $\mathrm{rk}(\mathfrak{A}) < \omega$ and hence $\omega\cdot\xi + 2\cdot\mathrm{rk}(\mathfrak{A}) + 1 < \omega\cdot(\xi+1) \le \omega\cdot\nu$ holds for all $\xi < \nu$. Statement (c) is proven by a straight-forward induction on $\mathrm{rk}(A)$, and we leave the proof to the reader, noticing that (b) is used for the case that A is of the form $r \in Q_{\mathfrak{A}}^{<\xi}$. Finally, statement (d) is proven by induction on τ and we let $D := \exists x(\mathfrak{A}(B(z), x) \wedge \neg B(x))$. If $\tau = 0$, we immediately get $\mathsf{SID}_n^\infty \vdash_{0,0}^{0} D, B(t), t \notin Q_{\mathfrak{A}}^{<0}$ from $(\bigwedge_{t \notin Q_{\mathfrak{A}}^{<0}})$. If $\tau > 0$, we first get

$$\mathsf{SID}_n^\infty \vdash_{0,0}^{\omega\cdot\xi} D, B(t), t \notin Q_{\mathfrak{A}}^{<\xi} \tag{$*$}$$

by I.H. for all $\xi < \tau$ and all t. Using (a) with $(*)$ and (c) with $B(t)$ yields

$$\mathsf{SID}_n^\infty \vdash_{0,0}^{\omega\cdot\xi+2\cdot\mathrm{rk}(\mathfrak{A})} D, \mathfrak{A}(B(z), t), \neg\mathfrak{A}(Q_{\mathfrak{A}}^{<\xi}, t)$$
$$\mathsf{SID}_n^\infty \vdash_{0,0}^{2\cdot\mathrm{rk}(B(t))} D, \neg\mathfrak{A}(Q_{\mathfrak{A}}^{<\xi}, t), B(t), \neg B(t)$$

Since $B(t) \in \mathcal{L}_n$, we have $\mathrm{rk}(B(t)) < \omega$ and hence we get for some $m < \omega$

$$\mathsf{SID}_n^\infty \vdash_{0,0}^{\omega\cdot\xi+m} D, \mathfrak{A}(B(z), t) \wedge \neg B(t), \neg\mathfrak{A}(Q_{\mathfrak{A}}^{<\xi}, t), B(t)$$

Using $(\bigvee_D^t)$ and that $\omega\cdot\xi + m + 1 < \omega\cdot(\xi+1) \le \omega\cdot\tau$ holds for each $\xi < \tau$, the claim follows with an $(\bigwedge_{t \notin Q_{\mathfrak{A}}^{<\tau}})$ inference. $\square$

Lemma 3.13 (Persistence)**.** *Let $\mathcal{L}_{n+1}^{\infty}$ sequents $\Delta^- := A_0, \dots, A_q$ and $\Delta^+ := B_0, \dots, B_r$ be given with $\Delta^- \subseteq \mathrm{Neg}_{n+1}$ and $\Delta^+ \subseteq \mathrm{Pos}_{n+1}$, then for all ordinals $\nu_0, \nu_0', \dots, \nu_q, \nu_q'$ with $(\forall i \leq q)(\nu_i' \leq \nu_i)$, all ordinals $\pi_0, \pi_0', \dots, \pi_p, \pi_p'$ with $(\forall i \leq p)(\pi_i \leq \pi_i')$, and each $\mathcal{L}_n^{\infty}$ sequent Γ, we have*

$$\mathsf{SID}_n^{\infty} \vdash_{\rho,r}^{\alpha} \Gamma, [\Delta^-]^{\nu_0,\dots,\nu_q}, [\Delta^+]^{\pi_0,\dots,\pi_p} \implies \mathsf{SID}_n^{\infty} \vdash_{\rho,r}^{\alpha} \Gamma, [\Delta^-]^{\nu_0',\dots,\nu_q'}, [\Delta^+]^{\pi_0',\dots,\pi_p'}$$

Proof. By induction on α. In case that all main formulas of the last inference are among Γ or if the last inference is an instance **I** or **II**, a fixed-point rule in **III**, or a cut-rule in **V**, then the proof is straight-forward. Otherwise the last inference is a rule in **IV** and we consider the following cases:
1. $(\bigvee_C^{\xi})$ with $\xi < \pi_i$ and $C = t \in Q_{\mathfrak{A}}^{<\pi_i}$ for some $1 \leq i \leq p$: Then we have

$$\mathsf{SID}_n^{\infty} \vdash_{r,\rho}^{\alpha_0} \Gamma, [\Delta^-]^{\nu_0,\dots,\nu_q}, [\Delta^+]^{\pi_0,\dots,\pi_p}, \mathfrak{A}(Q_{\mathfrak{A}}^{<\xi}, t)$$

and $\alpha_0 < \alpha$. The I.H. (keeping $\mathfrak{A}(Q_{\mathfrak{A}}^{<\xi}, t)$ unchanged) and $(\bigvee_{C'}^{\xi})$ with $C' := t \in Q_{\mathfrak{A}}^{<\pi_i'}$ yield the claim since $\xi < \pi_i'$ holds due to $\pi_i \leq \pi_i'$.
2. $(\bigwedge_C)$ with $C = t \notin Q_{\mathfrak{A}}^{<\nu_i}$ for some $1 \leq i \leq q$: As in the previous case, using $\nu_i' \leq \nu_i$.
3. $(\bigwedge_C)$ with $C = C_0 \wedge C_1$ and w.l.o.g., let $C = A_0^{\nu_0}$: Then $C_0 = D_0^{\nu_0}$ and $C_1 = D_1^{\nu_0}$ for some $D_0, D_1 \in \mathrm{Neg}_{n+1}$: We can apply the I.H. here as well but change C_0, C_1 now to $D_0^{\nu_0'}$ and $D_1^{\nu_0'}$, respectively. $(\bigwedge_{C'})$ with $C' := D_0^{\nu_0'} \wedge D_1^{\nu_0'}$ yields the claim.
4. Another rule of inference from **IV**: Similar as in the previous case. □

Theorem 3.14 (Asymmetric interpretation)**.** *Assume that we have*

$$\mathsf{SID}_{n+1}^{\infty} \vdash_{\rho,1}^{\alpha} \Delta^-, \Delta^+$$

for some $\Delta^- \subseteq \mathrm{Neg}_{n+1}$ and $\Delta^+ \subseteq \mathrm{Pos}_{n+1}$. Let ν and π be given such that $\pi = \nu + 2^{\alpha}$ and $\rho \leq \omega \cdot \pi$ hold, then we have

$$\mathsf{SID}_n^{\infty} \vdash_{\omega\cdot\pi,\ell}^{\omega\cdot\pi+\alpha} [\Delta^-]^{\nu}, [\Delta^+]^{\pi}$$

Proof. By induction on α and a case distinction for the last inference.
1. Axioms in **I**: In case of $t \in P_{n+1}^{\mathfrak{A}} \in \Delta^+$ and $s \notin P_{n+1}^{\mathfrak{A}} \in \Delta^-$ with $s^{\mathbb{N}} = t^{\mathbb{N}}$, we can use (b) in lemma 3.12 for $t \in Q_{\mathfrak{A}}^{<\pi}$ and $s \notin Q_{\mathfrak{A}}^{<\nu}$. The other cases are trivial by taking appropriate instances of the corresponding axiom schemes.
2. Axioms in **II**: If we have an instance for some $P_m^{\mathfrak{A}}$ with $1 \leq m \leq n$, the axiom can be reused immediately. Otherwise it is an instance for some $P_{n+1}^{\mathfrak{A}}$, and then the claim follows by using (d) in lemma 3.12 for $Q_{\mathfrak{A}}^{<\nu}$.
3. (Cut_C) with $\mathrm{rk}(C) < \rho \leq \omega \cdot \pi$ and $\mathrm{rk}_{n+1}(C) = 0$ (and also $\mathrm{lh}(C) < \ell$):
3.1. If C is of the form $t \in P_{n+1}^{\mathfrak{A}}$ (or $t \notin P_{n+1}^{\mathfrak{A}}$): We have $\mathsf{SID}_{n+1}^{\infty} \vdash_{\rho,1}^{\alpha_0} \Delta^-, \Delta^+, t \in P_{n+1}^{\mathfrak{A}}$ and $\mathsf{SID}_{n+1}^{\infty} \vdash_{\rho,1}^{\alpha_1} \Delta^-, \Delta^+, t \notin P_{n+1}^{\mathfrak{A}}$ for some $\alpha_0, \alpha_1 < \alpha$. The I.H. yields with ν and $\pi_0 := \nu + 2^{\alpha_0}$

$$\mathsf{SID}_n^{\infty} \vdash_{\omega\cdot\pi_0,\ell}^{\omega\cdot\pi_0+\alpha_0} [\Delta^-]^{\nu}, [\Delta^+]^{\pi_0}, t \in Q_{\mathfrak{A}}^{<\pi_0}$$

and it also yields with π_0 and $\pi_1 := \pi_0 + 2^{\alpha_1}$

$$\mathsf{SID}_n^{\infty} \vdash_{\omega\cdot\pi_1,\ell}^{\omega\cdot\pi_1+\alpha_1} [\Delta^-]^{\pi_0}, [\Delta^+]^{\pi_1}, t \notin Q_{\mathfrak{A}}^{<\pi_0}$$

After some weakening and applying lemma 3.13 (using in particular $\nu < \pi_0$ and $\pi_1 = \pi_0 + 2^{\alpha_1} \leq \nu + 2^{\alpha} = \pi$), the claim follows by $(\mathsf{Cut}_{t \in Q_{\mathfrak{A}}^{<\pi_0}})$ since we have $\mathrm{rk}(t \in Q_{\mathfrak{A}}^{<\pi_0}) = \omega \cdot \pi_0 < \omega \cdot \pi$, $\mathrm{rk}_n(t \in Q_{\mathfrak{A}}^{<\pi_0}) = 0$, and in case of $n > 0$, we also have $\mathrm{lh}(t \in Q_{\mathfrak{A}}^{<\pi_0}) = \mathrm{lh}(C) < \ell$.

3.2. Otherwise $C \in \mathcal{L}_n^\infty$: First notice that we have $\mathrm{rk}_n(C) \leq \mathrm{lh}(C) < \ell$, so we can use the I.H. and then reuse (Cut_C) in SID_n^∞ to obtain the claim.
4. Fixed-point rules in **III**:
4.1. $(\mathsf{Fix}_{t \in P_{n+1}^{\mathfrak{A}}})$: We have $\mathsf{SID}_{n+1}^\infty \vdash_{\rho,1}^{\alpha_0} \Delta^-, \Delta^+, \mathfrak{A}(P_{n+1}^{\mathfrak{A}}, t)$ and $\mathfrak{A}(P_{n+1}^{\mathfrak{A}}, t) \in \mathrm{Pos}_{n+1}$ for some $\alpha_0 < \alpha$, and hence the I.H. with ν and $\pi_0 := \nu + 2^{\alpha_0} < \pi$ yields $\mathsf{SID}_n^\infty \vdash_{\omega\cdot\pi_0,\ell}^{\omega\cdot\pi_0+\alpha_0} [\Delta^-]^\nu, [\Delta^+]^{\pi_0}, \mathfrak{A}(Q_{\mathfrak{A}}^{<\pi_0}, t)$. Then the claim follows from $(\bigvee_{t \in Q_{\mathfrak{A}}^{<\pi}}^{\pi_0})$, lemma 3.13, and some weakening.
4.2. $(\mathsf{Fix}_{t \notin P_{n+1}^{\mathfrak{A}}})$: We have $\mathsf{SID}_{n+1}^\infty \vdash_{\rho,1}^{\alpha_0} \Delta^-, \Delta^+, \neg\mathfrak{A}(P_{n+1}^{\mathfrak{A}}, t)$ and $\neg\mathfrak{A}(P_{n+1}^{\mathfrak{A}}, t) \in \mathrm{Neg}_{n+1}$ for some $\alpha_0 < \alpha$, so we get with $\pi_0 := \nu + 2^{\alpha_0}$ by the I.H.

$$\mathsf{SID}_n^\infty \vdash_{\omega\cdot\pi_0,\ell}^{\omega\cdot\pi_0+\alpha_0} [\Delta^-]^\nu, [\Delta^+]^{\pi_0}, \neg\mathfrak{A}(Q_{\mathfrak{A}}^{<\nu}, t) \tag{$*$}$$

and hence by lemma 3.13 and some weakening, we get for each $\xi < \nu$

$$\mathsf{SID}_n^\infty \vdash_{\omega\cdot\pi,\ell}^{\omega\cdot\pi+\alpha_0} [\Delta^-]^\nu, [\Delta^+]^{\pi}, \neg\mathfrak{A}(Q_{\mathfrak{A}}^{<\xi}, t)$$

By using $(\bigwedge_{t \notin Q_{\mathfrak{A}}^{<\nu}})$, the claim follows.
4.3. $(\mathsf{Fix}_{t \in P_m^{\mathfrak{A}}})$ or $(\mathsf{Fix}_{t \notin P_m^{\mathfrak{A}}})$ for some $1 \leq m \leq n$: We can apply the I.H. for the premise and reuse the rule because it is available in SID_n^∞ and its minor formulas do not contain $P_{n+1}^{\mathfrak{A}}$.
5. Predicative rules in **IV**: Use the I.H. and repeat the rule with an appropriate instance. □

Remark. An inspection of the proof of theorem 3.14 yields that in case of $\rho = 0$, we even obtain $\mathsf{SID}_n^\infty \vdash_{0,0}^{\omega\cdot\pi+\alpha} [\Delta^-]^\nu, [\Delta^+]^\pi$ in the conclusion of theorem 3.14. We do not need this stronger result, though.

3.3 Arithmetical derivability

Theorem 3.15 (Arithmetical derivability)**.** *Let $\Gamma \subseteq \mathcal{L}_{\mathsf{PA}}$ and $r, n < \omega$. If $\mathsf{SID}_n^\infty \vdash_{\rho,r}^{<\varepsilon_0} \Gamma$ holds for some $\rho < \varepsilon_0$, then $\mathsf{SID}_0^\infty \vdash_{\eta,1}^{<\varepsilon_0} \Gamma$ holds for some $\eta < \varepsilon_0$.*

Proof. By induction on n. The case $n = 0$ is clear (see remark 3.7). We can also assume $r > 0$ w.l.o.g. and get $\mathsf{SID}_n^\infty \vdash_{\rho,1}^{<\varepsilon_0} \Gamma$ by theorem 3.10.(a). Now theorem 3.14 yields $\mathsf{SID}_{n-1}^\infty \vdash_{\eta,\ell}^{<\varepsilon_0} \Gamma$ for some $\eta < \varepsilon_0$ and hence the claim by the I.H. □

4 The upper bound of $\mathsf{SID}_{<\omega}$

Theorem 4.1. *If $\mathsf{SID}_n \vdash A$ for a closed $\mathcal{L}_n$ formula A, then there is an $\ell < \omega$ such that the derivability relation for SID_n^∞ and this ℓ yields $\mathsf{SID}_n^\infty \vdash_{\ell,\ell}^{<\omega+\omega} A$.*

Proof. As usual and inductively with respect to the underlying derivability notion $\mathsf{SID}_n \vdash A$. Notice that complete induction can be proven by use of the infinitary inference rule $(\bigwedge_{\forall xB})$ and that no inferences are needed that involve symbols of the form $Q_{\mathfrak{A}}^{<\xi}$ when inductively translating from $\mathsf{SID}_n \vdash A$ to the proof-system SID_n^∞ (hence cuts of finite rank ℓ are sufficient). □

Corollary 4.2. $|\mathsf{SID}_{<\omega}| \leq \varphi_{\varepsilon_0}(0)$.

Proof. For any closed arithmetical formula A with $\mathsf{SID}_n \vdash A$, we know from theorem 4.1 that $\mathsf{SID}_n^\infty \vdash_{\ell,\ell}^{<\varepsilon_0} A$ holds for some $\ell < \omega$. According to theorem 3.15, this means $\mathsf{SID}_0^\infty \vdash_{\rho,1}^{<\varepsilon_0} A$ for some $\rho < \varepsilon_0$. By weakening we have $\mathsf{SID}_0^\infty \vdash_{\omega^\rho,1}^{<\varepsilon_0} A$ since $\rho \leq \omega^\rho (< \varepsilon_0)$, so theorem 3.10.(b) yields $\mathsf{SID}_0^\infty \vdash_{0,0}^{<\varphi_{\varepsilon_0}(0)} A$ because $\alpha, \rho < \varepsilon_0$ implies $\varphi_\rho(\alpha) < \varphi_\rho(\varphi_{\varepsilon_0}(0)) = \varphi_{\varepsilon_0}(0)$, using $\varepsilon_0 < \varphi_{\varepsilon_0}(0)$. Finally, we get $|\mathsf{SID}_{<\omega}| \leq \varphi_{\varepsilon_0}(0)$ by a standard boundedness argument. □

5 Concluding remarks

We finish this note on the theory $\mathsf{SID}_{<\omega}$ of finitely stratified induction over fixed-points with some remarks on the proof-theoretic methods that we applied here and the generalisation to transfinitely stratified induction. In this context, an immediate question is the relation of transfinite stratification to the *iteration of fixed-point definitions*. We established the connection of $\mathsf{SID}_{<\omega}$ to the non-iterated theory $\widehat{\mathsf{ID}}_1$ and will now briefly explain the concept of (finite) iteration of fixed-point definitions: Since $\widehat{\mathsf{ID}}_1$ is based on positive operator forms $\mathfrak{A}_1$ that are formulated in the language $\mathcal{L}_{\mathsf{PA}}$, the theory $\widehat{\mathsf{ID}}_2$ is based on positive operator forms $\mathfrak{A}_2$ that are formulated in the language $\widehat{\mathcal{L}}_1$ (i.e., $\widehat{\mathsf{ID}}_2$ axiomatises fixed-points of $\mathfrak{A}_2$ by means of new unary relation symbols $P^{\mathfrak{A}_2}$ for each such $\mathfrak{A}_2$, resulting in the language $\widehat{\mathcal{L}}_2$ of $\widehat{\mathsf{ID}}_2$). This is similarly defined for $\widehat{\mathsf{ID}}_n$ with arbitrary $2 < n < \omega$, and it further extends to transfinite iterations of fixed-point definitions $\widehat{\mathsf{ID}}_\alpha$. As remarked in the introduction, we know for instance that $|\mathsf{ID}^*_\beta| = |\widehat{\mathsf{ID}}_\beta|$ holds for any ordinal β, and we refer to [JKSS99] and [Pro06] for details on results and definitions.

Comparison with the proof-theoretic methods for $\widehat{\mathsf{ID}}_n$ Considering only the case $n = 2$ and the reduction of $\widehat{\mathsf{ID}}_2$ to $\widehat{\mathsf{ID}}_1$, we first notice that similar methods (e.g., asymmetric interpretation) are used as in the reduction of SID_2 to SID_1 but with the difference that $|\widehat{\mathsf{ID}}_1| < |\widehat{\mathsf{ID}}_2|$ holds and that we actually established $|\mathsf{SID}_1| = |\mathsf{SID}_2|$ here. This is due to the following observation: Without going into too many details, let $\widehat{\mathsf{ID}}_2^\infty$ and $\widehat{\mathsf{ID}}_1^\infty$ be the infinitary proof-systems assigned to $\widehat{\mathsf{ID}}_2$ and $\widehat{\mathsf{ID}}_1$, respectively, which are defined in a similar way as the infinitary proof-systems in section 3. The difference is that stratified induction axioms are missing and that for $\widehat{\mathsf{ID}}_2^\infty$, we have fixed-point rules

$$\frac{\Gamma, \mathfrak{A}_2(P^{\mathfrak{A}_2}, t)}{\Gamma, t \in P^{\mathfrak{A}_2}}\ (\mathsf{Fix}_{t \in P^{\mathfrak{A}_2}}) \qquad \frac{\Gamma, \neg\mathfrak{A}_2(P^{\mathfrak{A}_2}, t)}{\Gamma, t \notin P^{\mathfrak{A}_2}}\ (\mathsf{Fix}_{t \notin P^{\mathfrak{A}_2}})$$

for positive operator forms $\mathfrak{A}_2(X, x) \in \widehat{\mathcal{L}}_1$ that may contain symbols $P^{\mathfrak{A}_1}$ for positive operator forms $\mathfrak{A}_1 \in \widehat{\mathcal{L}}_0 (= \mathcal{L}_{\mathsf{PA}})$ in arbitrary position. This is not the case for SID_2 where the operator form is arithmetical. As remarked above, the reduction from $\widehat{\mathsf{ID}}_2$ to $\widehat{\mathsf{ID}}_1$ uses asymmetric interpretation of $\widehat{\mathsf{ID}}_2^\infty$ in $\widehat{\mathsf{ID}}_1^\infty$, therefore $\widehat{\mathsf{ID}}_1^\infty$ has for example predicative rules of the form

$$\frac{\Gamma, \mathfrak{A}_2(Q^{<\xi}_{\mathfrak{A}_2}, t)}{\Gamma, t \in Q^{<\tau}_{\mathfrak{A}_2}}\ (\textstyle\bigvee^\xi_{t \in Q^{<\tau}_{\mathfrak{A}_2}}) \text{ for } \xi < \tau \tag{\#}$$

with $\mathfrak{A}_2$ being a positive operator form over the language $\widehat{\mathcal{L}}_2$ rather than $\widehat{\mathcal{L}}_1$. This is needed in order to be able to interpret a $(\mathsf{Fix}_{t \in P^{\mathfrak{A}_2}})$ inference, but it also makes it more difficult to remove cuts partially. Recall that in order to be able to use theorem 3.14, we first had to partially remove cuts in SID_2^∞ before doing an asymmetric interpretation (this was needed to make the proof by induction of theorem 3.14 work). Similarly, $\widehat{\mathsf{ID}}_2^\infty$ needs first to partially remove cuts, and because of the existence of rule of inferences such as (#) this is only possible by doing a partial cut-elimination that involves a cut-reduction for formulas of transfinite rank (compare lemma 3.9). In contrast to this, we were able to avoid such cut-reductions for SID_2^∞ so that it was needed only once in the very end for SID_0^∞. We refer to the references for more details on the proof-theoretic analysis of $\widehat{\mathsf{ID}}_n$ for $n < \omega$ (yielding $|\widehat{\mathsf{ID}}_{<\omega}| = \Gamma_0$) and the generalisation to the transfinite.

Transfinite stratification The equality $\varphi_{\varepsilon_0}(0) = |\widehat{\mathsf{ID}}_1| = |\mathsf{SID}_n| = |\mathsf{SID}_{<\omega}|$ (with $n < \omega$) established here still leaves the question open concerning the relationship of stratification to iteration. For this, we refer to [JP15] where a generalisation of stratification to the transfinite gives an answer to it. The following picture captures line by line some aspects of this relationship and we refer again to [JP15] for the meaning and characterisation of the last three rows.

ordinal	stratification	iteration
ε_0	SID_0	$\widehat{\mathsf{ID}}_0$
$\varphi_{\varepsilon_0}(0)$	$\mathsf{SID}_{<\omega}$	$\widehat{\mathsf{ID}}_1$
$\varphi_{\varepsilon_{\varepsilon_0}}(0)$	$\mathsf{SID}_{<\omega+\omega}$	—
$\varphi_{\varphi_\omega(0)}(0)$	$\mathsf{SID}_{<\omega^\omega}$	—
$\varphi_{\varphi_{\varepsilon_0}(0)}(0)$	$\mathsf{SID}_{<\varepsilon_0}$	$\widehat{\mathsf{ID}}_2$

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