The non-constructive μ operator, fixed point theories with ordinals, and the bar rule

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Abstract

This paper deals with the proof theory of first order applicative theories with non-constructive μ operator and a form of the bar rule, yielding systems of ordinal strength Γ_0 and $\varphi 20$, respectively. Relevant use is made of fixed point theories with ordinals plus bar rule.

1 Introduction

In the past few years there have been rather extensive proof-theoretic investigations on Feferman's explicit mathematics (cf. [6, 7]) with a predicatively justified quantification operator μ, cf. the papers Feferman and Jäger [9, 10], Glaß and Strahm [13], Jäger and Strahm [17, 18] and Marzetta and Strahm [19]. The systems studied in the context of μ range from pure first order applicative frameworks to theories of types and names with universes.

It is the aim of the present article to continue these investigations; more precisely, we want to study the role of the bar rule in pure applicative theories with non-constructive μ operator. We will show that the corresponding theory $\mathsf{AutBON}(\mu)$ has the same proof-theoretic strength as predicative analysis and, hence, its proof-theoretic ordinal is exactly the Feferman-Schütte ordinal Γ_0. Further, we will shortly discuss the effect of replacing the applicative basis of $\mathsf{AutBON}(\mu)$ by Schlüter's [21] applicative axioms for primitive recursive application; we will see that the so-obtained theory with μ operator and bar rule has proof-theoretic ordinal $\varphi 20$.

The upper bound computations for explicit mathematics with μ carried through in [9, 10, 13, 17, 18, 19] have made substantial use of so-called fixed point theories with ordinals which go back to Jäger [15]. The adequate system for the treatment of $\mathsf{AutBON}(\mu)$ is the system $\mathsf{PA}^{\mathsf{w}}_{\Omega}$ of [15] plus a suitable substitution rule (Subst). In this paper we establish the upper bound Γ_0 for an extension of $\mathsf{PA}^{\mathsf{w}}_{\Omega} + (\mathsf{Subst})$, namely $\mathsf{PA}^{+}_{\Omega} + (\mathsf{Subst})$; the latter system includes induction on the ordinals for statements which are Σ in the ordinals. $\mathsf{PA}^{+}_{\Omega} + (\mathsf{Subst})$ is also used in a crucial way for establishing proof-theoretic bounds of theories for least fixed point recursion in the paper Feferman and Strahm [11].

The exact procedure of this paper is as follows. In Section 2 we introduce the formal framework of the theory $\mathsf{AutBON}(\mu)$. Section 3 is devoted to the lower bound computation for $\mathsf{AutBON}(\mu)$: we show that the theory $\mathsf{Aut}(\Pi^1_0)$ for autonomously iterated Π^1_0 jumps is contained in $\mathsf{AutBON}(\mu)$. In Section 4 we discuss fixed point theories with ordinals and a substitution rule. In particular, we give a complete ordinal analysis of the system $\mathsf{PA}^+_\Omega + (\mathsf{Subst})$. In Section 5 we conclude the upper bound computation for $\mathsf{AutBON}(\mu)$, and in Section 6 theories with primitive recursive operations plus μ and bar rule are considered.

2 The theory $\mathsf{AutBON}(\mu)$

In this section we introduce the applicative framework $\mathsf{AutBON}(\mu)$, which is obtained from the basic theory of operations and numbers BON (cf. [9]) by adding a suitable axiomatization of the non-constructive μ operator and a form of the bar rule.

2.1 The basic theory of operations and numbers BON

Our applicative language $\mathcal{L}$ is a first order language of partial terms with individual variables $a, b, c, x, y, z, u, v, w, f, g, h, \ldots$ (possibly with subscripts). $\mathcal{L}$ includes individual constants k, s (combinators), $\mathsf{p}, \mathsf{p}_0, \mathsf{p}_1$ (pairing and unpairing), $\mathsf{0}$ (zero), $\mathsf{s_N}$ (numerical successor), $\mathsf{p_N}$ (numerical predecessor), $\mathsf{d_N}$ (definition by numerical cases), μ (unbounded minimum operator), and $\mathsf{c_U}$ (characteristic function of U). Further, $\mathcal{L}$ has a binary function symbol $\cdot$ for (partial) term application, unary relation symbols $\downarrow$ (defined), N (natural numbers) and U (free relation symbol) as well as a binary relation symbol $=$ (equality). The free relation symbol U will be used in order to formulate the substitution rule below.

The *individual terms* $(r, s, t, r_1, s_1, t_1, \ldots)$ of $\mathcal{L}$ are inductively defined as follows:

1. The individual variables and individual constants are individual terms.
2. If s and t are individual terms, then so also is $(s \cdot t)$.

In the following we write (st) or just st instead of $(s \cdot t)$, and we adopt the convention of association to the left, i.e. $s_1 s_2 \ldots s_n$ stands for $(\ldots(s_1 s_2) \ldots s_n)$. We also write (t_1, t_2) for $\mathsf{p} t_1 t_2$ and $(t_1, t_2, \ldots, t_n)$ for $(t_1, (t_2, \ldots, t_n))$. Further we put $t' := \mathsf{s_N} t$ and $\mathsf{1} := \mathsf{0}'$.

The *formulas* $(A, B, C, A_1, B_1, C_1, \ldots)$ of $\mathcal{L}$ are inductively defined as follows:

1. Each atomic formula $\mathsf{N}(t)$, $\mathsf{U}(t)$, $t{\downarrow}$ and $(s = t)$ is a formula.
2. If A and B are formulas, then so also are $\neg A$, $(A \vee B)$, $(A \wedge B)$ and $(A \rightarrow B)$.
3. If A is a formula, then so also are $(\exists x)A$ and $(\forall x)A$.

Our applicative theories are based on *partial* term application. Hence, it is not guaranteed that terms have a value, and $t\downarrow$ is read as "t is defined" or "t has a value". The *partial equality relation* $\simeq$ is introduced by

$$s \simeq t \ := \ (s\downarrow \vee t\downarrow) \rightarrow (s = t).$$

In addition, we write $(s \neq t)$ for $(s\downarrow \wedge t\downarrow \wedge \neg(s = t))$. Finally, we use the following abbreviations concerning the predicate N:

$$\begin{array}{rcl} t \in \mathsf{N} & := & \mathsf{N}(t), \\ (\exists x \in \mathsf{N})A & := & (\exists x)(x \in \mathsf{N} \wedge A), \\ (\forall x \in \mathsf{N})A & := & (\forall x)(x \in \mathsf{N} \rightarrow A), \\ (t : \mathsf{N} \rightarrow \mathsf{N}) & := & (\forall x \in \mathsf{N})(tx \in \mathsf{N}), \\ (t : \mathsf{N}^{m+1} \rightarrow \mathsf{N}) & := & (\forall x \in \mathsf{N})(tx : \mathsf{N}^m \rightarrow \mathsf{N}). \end{array}$$

The underlying logic of BON is the *classical logic of partial terms* due to Beeson [1]; it corresponds to E^+ logic with strictness and equality of Troelstra and Van Dalen [25]. The non-logical axioms of BON are divided into the following five groups.

I. Partial combinatory algebra.

(1) $\mathsf{k}xy = x$,

(2) $\mathsf{s}xy\downarrow \ \wedge \ \mathsf{s}xyz \simeq xz(yz)$.

II. Pairing and projection.

(3) $\mathsf{p}_0(x, y) = x \ \wedge \ \mathsf{p}_1(x, y) = y$.

III. Natural numbers.

(4) $0 \in \mathsf{N} \ \wedge \ (\forall x \in \mathsf{N})(x' \in \mathsf{N})$,

(5) $(\forall x \in \mathsf{N})(x' \neq 0 \wedge \mathsf{p}_\mathsf{N}(x') = x)$,

(6) $(\forall x \in \mathsf{N})(x \neq 0 \ \rightarrow \ \mathsf{p}_\mathsf{N}x \in \mathsf{N} \wedge (\mathsf{p}_\mathsf{N}x)' = x)$.

IV. Definition by numerical cases.

(7) $a \in \mathsf{N} \wedge b \in \mathsf{N} \ \wedge \ a = b \rightarrow \mathsf{d}_\mathsf{N}xyab = x$,

(8) $a \in \mathsf{N} \wedge b \in \mathsf{N} \ \wedge \ a \neq b \rightarrow \mathsf{d}_\mathsf{N}xyab = y$.

V. Characteristic function of U.

(9) $(\forall x \in \mathsf{N})(\mathsf{c}_\mathsf{U}x = 0 \ \vee \ \mathsf{c}_\mathsf{U}x = 1)$,

(10) $(\forall x \in \mathsf{N})(\mathsf{U}(x) \ \leftrightarrow \ \mathsf{c}_\mathsf{U}x = 0)$.

As usual the axioms of a partial combinatory algebra allow one to define λ abstraction and to prove a recursion or fixed point theorem. For proofs of these standard results the reader is referred to [1, 6].

In contrast to the axiomatization of BON e.g. in Feferman and Jäger [9], we omit axioms about primitive recursion on N. This is justified by the fact that we will not consider BON in the context of restricted induction principles and, hence, axioms V. of [9] become derivable by means of the recursion theorem and a certain amount of complete induction on N.

2.2 The non-constructive μ operator and the bar rule

On our way to the exact formulation of the theory $\mathsf{AutBON}(\mu)$, let us now consider the non-constructive μ operator. We follow its axiomatization in Jäger and Strahm [17, 18]. For a discussion of different formulations of μ, the reader is referred to [18].

The unbounded minimum operator

(μ.1) $(f : \mathsf{N} \to \mathsf{N}) \leftrightarrow \mu f \in \mathsf{N}$,

(μ.2) $(f : \mathsf{N} \to \mathsf{N}) \land (\exists x \in \mathsf{N})(fx = 0) \to f(\mu f) = 0$.

In a next step we turn to the formulation of the bar rule. For that purpose, let $\prec$ be a binary primitive recursive relation which is naturally represented in our applicative framework as usual. Then we set

$$\begin{aligned} \mathit{Prog}(\prec, A) &:= (\forall x \in \mathsf{N})((\forall y \in \mathsf{N})(y \prec x \to A(y)) \to A(x)), \\ \mathit{TI}(\prec, A) &:= \mathit{Prog}(\prec, A) \to (\forall x \in \mathsf{N})A(x). \end{aligned}$$

An instance of the bar rule (BR) has now the form

$$(\mathsf{BR}) \qquad \frac{\mathit{TI}(\prec, \mathsf{U})}{\mathit{TI}(\prec, A)},$$

for $\prec$ a primitive recursive relation and $A(x)$ and arbitrary formula of $\mathcal{L}$. Further, the schema of complete induction on the natural numbers (IND) is spelled out as

$$(\mathsf{IND}) \qquad A(0) \land (\forall x \in \mathsf{N})(A(x) \to A(x')) \to (\forall x \in \mathsf{N})A(x)$$

for $A(x)$ again an arbitrary formula of $\mathcal{L}$. The theory $\mathsf{AutBON}(\mu)$ is now obtained from BON by adding axioms (μ.1) and (μ.2) for the unbounded minimum operator, all instances of the bar rule (BR) and complete induction on the natural numbers.

We call an ordinal α provable in the theory T if there exists a primitive recursive wellordering $\prec$ of ordertype α so that $\mathsf{T} \vdash \mathit{TI}(\prec, \mathsf{U})$. The least ordinal α that is not provable in T is called the *proof-theoretic ordinal of* T, in symbols, $|\mathsf{T}|$.

In the sequel we show that $\mathsf{AutBON}(\mu)$ is proof-theoretically equivalent to predicative analysis and, hence, its proof-theoretic ordinal is exactly the Feferman-Schütte ordinal Γ_0.

3 A lower bound of $\mathsf{AutBON}(\mu)$

In this section we establish that the subsystem of second order arithmetic $\mathsf{Aut}(\Pi^1_0)$ for autonomously iterated Π^1_0 jumps is contained in $\mathsf{AutBON}(\mu)$. Since $\mathsf{Aut}(\Pi^1_0)$ has the same strength as predicative analysis, $\mathsf{RA}_{<\Gamma_0}$, this yields Γ_0 as a lower bound for the proof-theoretic strength of $\mathsf{AutBON}(\mu)$.

3.1 The theory $\mathsf{Aut}(\Pi^1_0)$

In the following we introduce various theories for iterating arithmetic comprehension, in particular the theory $\mathsf{Aut}(\Pi^1_0)$, cf. e.g. Feferman and Jäger [8].

Let $\mathcal{L}_1$ denote the usual first order language of arithmetic with number variables $a, b, c, u, v, w, x, y, z, \ldots$, the constant 0 as well as function symbols for all primitive recursive functions. We further assume that $\mathcal{L}_1$ contains the free unary relation symbol U. The language $\mathcal{L}_2$ of second order arithmetic extends $\mathcal{L}_1$ by set variables $X, Y, Z, \ldots$ (possibly with subscripts) and the binary relation symbol $\in$ for elementhood between numbers and sets. Terms and formulas of $\mathcal{L}_2$ are defined as usual. We write $s \in (X)_t$ for $\langle s, t\rangle \in X$, where $\langle \cdot, \cdot \rangle$ is a standard primitive recursive pairing function with associated projections $(\cdot)_0$ and $(\cdot)_1$. An $\mathcal{L}_2$ formula is called *arithmetic*, if it does not contain bound set variables; let Π^1_0 denote the class of arithmetic $\mathcal{L}_2$ formulas. All $\mathcal{L}_2$ theories considered in this article contain the system $(\Pi^1_0\text{-}\mathsf{CA})\!\restriction$ of arithmetic comprehension together with the induction axiom.

Let us now turn to theories for iterating Π^1_0 comprehension. We assume that the reader is familiar with the Veblen functions $\varphi\alpha$, the ordinal Γ_0 as well as primitive recursive standard wellorderings of order type up to Γ_0, cf. [20, 22]. For such a wellordering $\prec$ and a Π^1_0 formula $A(X, Y, a, y)$ with at most X, Y, a, y free, we can define the A *jump hierarchy along* $\prec$ *with parameter* X by the following transfinite recursion:

$$(Y)_a \;=\; \{m : A(X, (Y)^a, a, m)\},$$

where $(Y)^a$ denotes the set $\{\langle m, b\rangle : b \prec a \wedge m \in (Y)_b\}$. Let us write $\mathcal{H}^{\prec}_A(X, Y)$ for the arithmetic $\mathcal{L}_2$ formula which formalizes this definition. For $\alpha < \Gamma_0$, the $\mathcal{L}_2$ theory $(\Pi^1_0\text{-}\mathsf{CA})_\alpha$ is defined to be $(\Pi^1_0\text{-}\mathsf{CA})\!\restriction$ plus the axioms

$$(\forall X)(\exists Y)\mathcal{H}^{\prec}_A(X, Y) \quad \text{and} \quad TI(\prec, B),$$

for $\prec$ a primitive recursive standard wellordering of ordertype α, $A(X, Y, a, y)$ a Π^1_0 formula and B an arbitrary $\mathcal{L}_2$ formula. The union of the theories $(\Pi^1_0\text{-}\mathsf{CA})_\beta$ for $\beta < \alpha$ is denoted by $(\Pi^1_0\text{-}\mathsf{CA})_{<\alpha}$.

Instead of iterating arithmetic comprehension along *externally* given wellorderings, the theory $\mathsf{Aut}(\Pi^1_0)$ claims the existence of the A jump hierarchy only along those primitive recursive $\prec$, whose wellfoundedness has previously been established *within*

$\mathsf{Aut}(\Pi^1_0)$. Accordingly, the theory $\mathsf{Aut}(\Pi^1_0)$ of autonomously iterated arithmetic comprehension is defined to be $(\Pi^1_0\text{-}\mathsf{CA})\restriction$ plus the bar rule (BR) and the following rule of inference:

$$\frac{TI(\prec, \mathsf{U})}{(\forall X)(\exists Y)\mathcal{H}^{\prec}_A(X,Y)}.$$

Here $\prec$ denotes a primitive recursive relation and A a Π^1_0 formula. It is well-known that $\mathsf{Aut}(\Pi^1_0)$ is equivalent to $(\Pi^1_0\text{-}\mathsf{CA})_{<\Gamma_0}$, and the proof-theoretic ordinal of both systems is Γ_0, cf. [3, 8]. For more information about theories with iterated comprehension and autonomous processes, the reader is referred to [4, 5, 12, 24].

3.2 Embedding $\mathsf{Aut}(\Pi^1_0)$ into $\mathsf{AutBON}(\mu)$

In this paragraph we sketch the main lines of an embedding of the theory $\mathsf{Aut}(\Pi^1_0)$ into $\mathsf{AutBON}(\mu)$, thus generalizing similar lower bound arguments given in Feferman and Jäger [9, 10] and Glaß and Strahm [13].

Let us first recall that sets of natural numbers are most naturally understood in our applicative framework via their characteristic functions which are total on N. Accordingly, we define

$$f \in \mathcal{P}(\mathsf{N}) \;:=\; (\forall x \in \mathsf{N})(fx = \mathsf{0} \vee fx = \mathsf{1}),$$

with the intention that an object x belongs to the set $f \in \mathcal{P}(\mathsf{N})$ if and only if $(fx = \mathsf{0})$. For example, axiom (9) of BON claims that $\mathsf{c_U} \in \mathcal{P}(\mathsf{N})$.

There is an obvious embedding $(\cdot)^{\mathsf{N}}$ of the language $\mathcal{L}_1$ into $\mathcal{L}$. We now extend this embedding to the language $\mathcal{L}_2$ as follows. The set variables of $\mathcal{L}_2$ are supposed to range over $\mathcal{P}(\mathsf{N})$ and, accordingly, an atomic $\mathcal{L}_2$ formula $(x \in Y)$ is translated into $(yx = \mathsf{0})$, where x and y are the variables of $\mathcal{L}$ which are associated to the variables x and Y of $\mathcal{L}_2$, respectively. Hence, the extended translation $(\cdot)^{\mathsf{N}}$ is such that

$$((\exists X)A(X))^{\mathsf{N}} = (\exists x \in \mathcal{P}(\mathsf{N}))A^{\mathsf{N}}(x),$$

and similarly for universal quantifiers. In order to simplify the notation, we identify terms and formulas of $\mathcal{L}_2$ and their translation into $\mathcal{L}$ when there is no danger of confusion.

This is the right place to mention a crucial application of the unbounded μ operator, namely elimination of number quantifiers (cf. [9]). The following lemma is proved by straightforward induction on the complexity of A.

Lemma 1 *For every arithmetic $\mathcal{L}_2$ formula $A(\vec{X}, \vec{y})$ with at most $\vec{X}, \vec{y}$ free there exists an individual term t_A of $\mathcal{L}$ so that*

1. $\mathsf{AutBON}(\mu) \vdash (\forall \vec{x} \in \mathcal{P}(\mathsf{N}))(\forall \vec{y} \in \mathsf{N})(t_A\vec{x}\vec{y} = \mathsf{0} \vee t_A\vec{x}\vec{y} = \mathsf{1})$,
2. $\mathsf{AutBON}(\mu) \vdash (\forall \vec{x} \in \mathcal{P}(\mathsf{N}))(\forall \vec{y} \in \mathsf{N})(A^{\mathsf{N}}(\vec{x}, \vec{y}) \leftrightarrow t_A\vec{x}\vec{y} = \mathsf{0})$.

The central step in verifying the $(\cdot)^{\mathsf{N}}$ embedding of $\mathsf{Aut}(\Pi^1_0)$ in $\mathsf{AutBON}(\mu)$ consists in proving the existence of the A jump hierarchy along a provably wellfounded primitive recursive $\prec$. The proof of the following lemma is a generalization of similar arguments in Feferman and Jäger [9] and Glaß and Strahm [13].

Lemma 2 *Let $\prec$ be primitive recursive so that $\mathsf{AutBON}(\mu) \vdash TI(\prec, \mathsf{U})$. Further assume that $A(X, Y, a, y)$ is an arithmetic $\mathcal{L}_2$ formula. Then there exists a closed $\mathcal{L}$ term h so that we have:*

$$\mathsf{AutBON}(\mu) \vdash x \in \mathcal{P}(\mathsf{N}) \;\rightarrow\; \mathsf{h}x \in \mathcal{P}(\mathsf{N}) \wedge \mathcal{H}^{\prec}_{A}(x, \mathsf{h}x).$$

Proof. Let t_A and t_B be the $\mathcal{L}$ terms which are associated to the arithmetic $\mathcal{L}_2$ formulas $A(X, a, y)$ and $B(z, a)$ by the previous lemma, where $B(z, a)$ denotes the formula

$$z = \langle (z)_0, (z)_1 \rangle \wedge (z)_1 \prec a. \tag{1}$$

Further, the operation f is given by

$$f \;:=\; \lambda xaz.(\mathsf{d_N}(x(z)_1(z)_0)\mathsf{1}(t_B za)\mathsf{0}). \tag{2}$$

If x is assumed to be an operation which enumerates the sets xb, then fxa is a characteristic function of the disjoint union of the sets $(xb)_{b \prec a}$. In the following we work informally in the theory $\mathsf{AutBON}(\mu)$. By the recursion theorem we can find an operation g which satisfies the following recursion equation:

$$gxay \;\simeq\; t_A x(f(gx)a)ay.$$

We have that gxa represents the ath level of the A jump hierarchy with parameter x, but it remains to show that indeed $gxa \in \mathcal{P}(\mathsf{N})$ provided $x \in \mathcal{P}(\mathsf{N})$. The natural way to prove this is by transfinite induction along $\prec$, of course. By assumption we know $TI(\prec, \mathsf{U})$, which by (BR) yields $TI(\prec, C)$ for arbitrary $\mathcal{L}$ statements C. One readily verifies

$$x \in \mathcal{P}(\mathsf{N}) \rightarrow Prog(\prec, gxa \in \mathcal{P}(\mathsf{N})), \tag{3}$$

and, therefore, we get by transfinite induction

$$x \in \mathcal{P}(\mathsf{N}) \rightarrow (\forall a \in \mathsf{N}) gxa \in \mathcal{P}(\mathsf{N}). \tag{4}$$

Our argument is finished by setting $\mathsf{h} := \lambda xy.gx(y)_1(y)_0$. $\square$

As the remaining axioms and rules of $\mathsf{Aut}(\Pi^1_0)$ are easily dealt with, too, we are now in a position to state the following embedding theorem.

Theorem 3 *We have for every $\mathcal{L}_2$ formula $A(\vec{X}, \vec{y})$ with at most $\vec{X}, \vec{y}$ free:*

$$\mathsf{Aut}(\Pi^1_0) \vdash A(\vec{X}, \vec{y}) \quad\Longrightarrow\quad \mathsf{AutBON}(\mu) \vdash \vec{x} \in \mathcal{P}(\mathsf{N}) \wedge \vec{y} \in \mathsf{N} \;\rightarrow\; A^{\mathsf{N}}(\vec{x}, \vec{y}).$$

Using the definition of proof-theoretic ordinal of Section 3.1, we have thus established the following corollary.

Corollary 4 $\Gamma_0 \leq |\mathsf{AutBON}(\mu)|$.

In the following two sections we will show that Γ_0 is in fact also an upper bound for the proof-theoretic ordinal of $\mathsf{AutBON}(\mu)$.

4 Fixed point theories over Peano arithmetic with ordinals and a substitution rule

Fixed point theories over Peano arithmetic with ordinals have been introduced in Jäger [15], and extended to second order theories with ordinals in Jäger and Strahm [16]. They have been used in the proof-theoretic analysis of systems of explicit mathematics with the non-constructive μ operator in an essential way, cf. Feferman and Jäger [9, 10], Glaß and Strahm [13], Jäger and Strahm [17, 18], and Marzetta and Strahm [19].

The system that is adequate for the treatment of $\mathsf{AutBON}(\mu)$ is the theory $\mathsf{PA}^{\mathsf{w}}_{\Omega}$ of [15] plus a suitable substitution rule (Subst); in Section 5 we will describe an embedding of $\mathsf{AutBON}(\mu)$ into $\mathsf{PA}^{\mathsf{w}}_{\Omega} + (\mathsf{Subst})$. The theory $\mathsf{PA}^{\mathsf{w}}_{\Omega}$ includes a very weak form of induction on the ordinals, so-called Δ^{Ω}_{0} induction. Although Δ^{Ω}_{0} induction on the ordinals is sufficient for the treatment of $\mathsf{AutBON}(\mu)$, it turns out that one can even allow induction on the ordinals with respect to statements which are Σ in the ordinals: the corresponding theory PA^{+}_{Ω} together with the substitution rule (Subst) is still predicative. In the following we give a complete ordinal analysis of $\mathsf{PA}^{+}_{\Omega} + (\mathsf{Subst})$, thereby showing that $|\mathsf{PA}^{+}_{\Omega} + (\mathsf{Subst})| \leq \Gamma_0$.

As we have already mentioned in the introduction, the theory $\mathsf{PA}^{+}_{\Omega} + (\mathsf{Subst})$ is also crucial for establishing some proof-theoretic results due to S. Feferman and the author, which are reported in Feferman and Strahm [11]. For this work, the presence of Σ^{Ω} induction on the ordinals is essential.

4.1 The theories $\mathsf{PA}^{\mathsf{w}}_{\Omega} + (\mathsf{Subst})$ and $\mathsf{PA}^{+}_{\Omega} + (\mathsf{Subst})$

Let us now specify the exact formulation of fixed point theories over Peano arithmetic with ordinals plus substitution rule.

We first introduce the notion of an inductive operator form. Let P be an n-ary relation symbol which does not belong to the language $\mathcal{L}_1$, and let $\mathcal{L}_1(P)$ denote the extension of $\mathcal{L}_1$ by P. An $\mathcal{L}_1(P)$ formula is called P *positive* if each occurrence of P in it is positive. We call P positive formulas which contain at most $\vec{x} = x_1, \ldots, x_n$ free *inductive operator forms*; we let $\mathcal{A}(P, \vec{x})$ range over such forms. Observe that the relation symbol U can have positive *and* negative occurrences in an inductive operator form $\mathcal{A}(P, \vec{x})$.

Now we extend $\mathcal{L}_1$ to a new first order language $\mathcal{L}_\Omega$ by adding a new sort of *ordinal variables* $(\sigma, \tau, \eta, \xi, \delta \ldots)$, new binary relation symbols $<$ and $=$ for the less relation and the equality relation on the ordinals[1] and an $(n+1)$-ary relation symbol $P_{\mathcal{A}}$ for each inductive operator form $\mathcal{A}(P, \vec{x})$ for which P is n-ary.

[1]In general it will be clear from the context whether $<$ and $=$ denote the less and equality relation on the nonnegative integers or on the ordinals.

The *number terms* of $\mathcal{L}_\Omega$ are the number terms of $\mathcal{L}_1$; the *ordinal terms* of $\mathcal{L}_\Omega$ are the ordinal variables of $\mathcal{L}_\Omega$. The *formulas* of $\mathcal{L}_\Omega$ $(A, B, C, \ldots)$ are inductively defined as follows:

1. If R is an n-ary relation symbol of $\mathcal{L}_1$, then $R(s_1, \ldots, s_n)$ is an atomic formula of $\mathcal{L}_\Omega$.

2. The formulas $(\sigma < \tau)$, $(\sigma = \tau)$ and $P_{\mathcal{A}}(\sigma, \vec{s})$ are atomic formulas of $\mathcal{L}_\Omega$.

3. If A and B are $\mathcal{L}_\Omega$ formulas, then so also are $\neg A$, $(A \vee B)$, $(A \wedge B)$ and $(A \rightarrow B)$.

4. If A is an $\mathcal{L}_\Omega$ formula, then so also are $(\exists x)A$ and $(\forall x)A$.

5. If A is an $\mathcal{L}_\Omega$ formula, then so also are $(\exists \xi < \sigma)A$, $(\forall \xi < \sigma)A$, $(\exists \xi)A$ and $(\forall \xi)A$.

For every $\mathcal{L}_\Omega$ formula A we write A^σ to denote the $\mathcal{L}_\Omega$ formula which is obtained by replacing all unbounded ordinal quantifiers $(\mathcal{Q}\xi)$ in A by $(\mathcal{Q}\xi < \sigma)$. Additional abbreviations are:

$$P^\sigma_{\mathcal{A}}(\vec{s}) \;:=\; P_{\mathcal{A}}(\sigma, \vec{s}), \quad P^{<\sigma}_{\mathcal{A}}(\vec{s}) \;:=\; (\exists \xi < \sigma) P^\xi_{\mathcal{A}}(\vec{s}), \quad P_{\mathcal{A}}(\vec{s}) \;:=\; (\exists \xi) P^\xi_{\mathcal{A}}(\vec{s}).$$

We introduce several classes of $\mathcal{L}_\Omega$ formulas, which will be important for the ordinal part of our fixed point theories. The Δ^Ω_0 formulas are the $\mathcal{L}_\Omega$ formulas which do not contain unbounded ordinal quantifiers; the Σ^Ω [Π^Ω] formulas are the $\mathcal{L}_\Omega$ formulas which do not contain positive universal [existential] and negative existential [universal] ordinal quantifiers. The union of Σ^Ω and Π^Ω is denoted by ∇^Ω.

We are now ready to give the exact formulation of Peano arithmetic with ordinals, PA_Ω. It is based on the usual two-sorted predicate calculus with equality and classical logic. The non-logical axioms of PA_Ω are divided into the following six groups:

I. Number-theoretic axioms. The axioms of Peano arithmetic PA with the exception of complete induction on the natural numbers.

II. Inductive operator axioms. For all inductive operator forms $\mathcal{A}(P, \vec{x})$:

$$P^\sigma_{\mathcal{A}}(\vec{s}) \leftrightarrow \mathcal{A}(P^{<\sigma}_{\mathcal{A}}, \vec{s}).$$

III. Σ^Ω Reflection axioms, (Σ^Ω-Ref). For all Σ^Ω formulas A:

$$A \rightarrow (\exists \xi) A^\xi.$$

IV. Linearity axioms.

$$\sigma \not< \sigma \;\wedge\; (\sigma < \tau \wedge \tau < \eta \rightarrow \sigma < \eta) \;\wedge\; (\sigma < \tau \vee \sigma = \tau \vee \tau < \sigma).$$

V. Formula induction on the natural numbers, (F-I_N). For all $\mathcal{L}_\Omega$ formulas $A(x)$:

$$A(0) \wedge (\forall x)(A(x) \rightarrow A(x')) \;\rightarrow\; (\forall x) A(x).$$

VI. Formula induction on the ordinals, $(\mathsf{F}\text{-}\mathsf{I}_\Omega)$. For all $\mathcal{L}_\Omega$ formulas $A(\xi)$:

$$(\forall \xi)[(\forall \eta < \xi)A(\eta) \to A(\xi)] \ \to \ (\forall \xi)A(\xi).$$

This finishes the description of PA_Ω. From the inductive operator and Σ^Ω reflection axioms one can easily deduce that the Σ^Ω formulas $P_\mathcal{A}$ describe fixed points of the inductive operator form $\mathcal{A}(P, \vec{x})$. Moreover, it is well-known that PA_Ω is proof-theoretically equivalent to ID_1 and many of its subsystems are of predicative strength, cf. Jäger [15] and Jäger and Strahm [18] for detailed information.

By the bar or substitution rule we now mean the following rule of inference:

$$(\mathsf{Subst}) \qquad \frac{A(\mathsf{U})}{A(B)},$$

for $A(\mathsf{U})$ in $\mathcal{L}_1$ and $B(x)$ an *arbitrary* $\mathcal{L}_\Omega$ formula. Here $A(B)$ is obtained from $A(\mathsf{U})$ by replacing each subformula $\mathsf{U}(t)$ by $B(t)$.

In the following we will be interested in studying the substitution rule (Subst) together with two fragments of PA_Ω, namely $\mathsf{PA}^\mathsf{w}_\Omega$ and PA^+_Ω. We obtain $\mathsf{PA}^\mathsf{w}_\Omega$ from PA_Ω by allowing induction on the ordinals for Δ^Ω_0 formulas only, and PA^+_Ω is the subsystem of PA_Ω where induction on the ordinals is restricted to Σ^Ω formulas. Summing up, in $\mathsf{PA}^\mathsf{w}_\Omega$ and PA^+_Ω we replace $(\mathsf{F}\text{-}\mathsf{I}_\Omega)$ by $(\Delta^\Omega_0\text{-}\mathsf{I}_\Omega)$ and $(\Sigma^\Omega\text{-}\mathsf{I}_\Omega)$, respectively, so that we have

$$\mathsf{PA}^\mathsf{w}_\Omega \subset \mathsf{PA}^+_\Omega \subset \mathsf{PA}_\Omega.$$

In the sequel we show that the strength of $\mathsf{PA}^+_\Omega + (\mathsf{Subst})$ is bounded by Γ_0 and, hence, $|\mathsf{PA}^\mathsf{w}_\Omega + (\mathsf{Subst})| \leq \Gamma_0$, too. Later we will also see that $\mathsf{AutBON}(\mu)$ can be embedded into $\mathsf{PA}^\mathsf{w}_\Omega + (\mathsf{Subst})$.

We finish this paragraph by mentioning that the theory PA^+_Ω (without the substitution rule) is closely related to the subsystem of Kripke Platek set theory $\mathsf{KPu}^0 + (\mathsf{IND}_\mathsf{N}) + (\Sigma_1\text{-}\mathsf{IND}_\in)$ (cf. Jäger [14]). More precisely, both systems have proof-theoretic ordinal $\varphi(\varphi\varepsilon_0 0)0$ (joint work of M. Rathjen and the author).

4.2 The infinitary system T_∞

In this section we introduce the infinitary system T_∞ which will be used for the proof-theoretic treatment of PA^+_Ω below. It is based on the language $\mathcal{L}_\infty$, which extends $\mathcal{L}_\Omega$ by constants $\bar{\alpha}$ for all ordinals $\alpha < \Gamma_0$. The *ordinal terms* $(\theta, \theta_0, \theta_1, \ldots)$ of $\mathcal{L}_\infty$ are the ordinal variables and the ordinal constants of $\mathcal{L}_\infty$. The *literals* of $\mathcal{L}_\infty$ are the literals of $\mathcal{L}_\Omega$ extended to the language $\mathcal{L}_\infty$. To simplify the notation we often write $A(\alpha)$ instead of $A(\bar{\alpha})$ if α is an ordinal less than Γ_0.

The *formulas* of $\mathcal{L}_\infty$ are inductively generated as follows:

1. Every literal of $\mathcal{L}_\infty$ is an $\mathcal{L}_\infty$ formula.

2. If A and B are $\mathcal{L}_\infty$ formulas, then so also are $(A \vee B)$ and $(A \wedge B)$.

3. If A is an $\mathcal{L}_\infty$ formula, so also are $(\exists x)A$, $(\forall x)A$, $(\exists \xi < \theta)A$ and $(\forall \xi < \theta)A$.

Since T_∞ is a Tait-style system, we assume that the negation $\neg A$ of an $\mathcal{L}_\infty$ formula A is defined as usual by making use of De Morgan's laws and the law of double negation. Notice that $\mathcal{L}_\infty$ formulas do not contain unbounded ordinal quantifiers. The $\mathcal{L}^{\mathsf{c}}_\infty$ formulas are the $\mathcal{L}_\infty$ formulas which do not contain free number and free ordinal variables. Furthermore, a literal of $\mathcal{L}^{\mathsf{c}}_\infty$ is called *primitive* if it is not of the form $\mathsf{U}(s)$, $\neg\mathsf{U}(s)$, $P^\alpha_\mathcal{A}(\vec{s})$ or $\neg P^\alpha_\mathcal{A}(\vec{s})$. Obviously, every primitive literal of $\mathcal{L}^{\mathsf{c}}_\infty$ is either true or false, and in the following we write TRUE for the set of true primitive literals. Finally, two $\mathcal{L}_\infty$ formulas are called *numerically equivalent*, if they differ in closed number terms with identical value only.

In order to measure the complexity of cuts in T_∞ we assign a rank to each $\mathcal{L}^{\mathsf{c}}_\infty$ formula. This definition is tailored so that the process of building up stages of an inductive definition is reflected by the rank of the formulas $P^\alpha_\mathcal{A}(\vec{s})$.

Definition 5 The *rank* $rn(A)$ of a $\mathcal{L}^{\mathsf{c}}_\infty$ formula A is inductively defined as follows:

1. If A is a literal $R(\vec{s})$, $\neg R(\vec{s})$, $\mathsf{U}(s)$, $\neg\mathsf{U}(s)$, $(\alpha < \beta)$, $(\alpha \not< \beta)$, $(\alpha = \beta)$ or $(\alpha \neq \beta)$, then $rn(A) := 0$.

2. If A is a literal $P^\alpha_\mathcal{A}(\vec{s})$ or $\neg P^\alpha_\mathcal{A}(\vec{s})$, then $rn(A) := \omega(\alpha + 1)$.

3. If A is a formula $(B \vee C)$ or $(B \wedge C)$ so that $rn(B) = \beta$ and $rn(C) = \gamma$, then $rn(A) := max(\beta, \gamma) + 1$.

4. If A is a formula $(\exists x)B(x)$ or $(\forall x)B(x)$ so that $rn(B(0)) = \alpha$, then $rn(A) := \alpha + 1$.

5. If A is a formula $(\exists \xi < \alpha)B(\xi)$ or $(\forall \xi < \alpha)B(\xi)$, then

$$rn(A) := sup\{rn(B(\beta)) + 1 : \beta < \alpha\}.$$

We write $oc(B)$ for the set of ordinal constants which occur in the $\mathcal{L}_\infty$ formula B. The proof of the following lemma is a matter of routine (cf. [16, 18]).

Lemma 6 *We have for all inductive operator forms $\mathcal{A}(P, \vec{x})$, all $\mathcal{L}^{\mathsf{c}}_\infty$ formulas A and all ordinals $\alpha < \Gamma_0$:*

1. $rn(\mathcal{A}(P^{<\alpha}_\mathcal{A}, \vec{s})) < rn(P^\alpha_\mathcal{A}(\vec{s}))$.

2. *If $\beta < \alpha$ for all $\beta \in oc(A)$, then $rn(A) < \omega\alpha + \omega$.*

The system T_∞ is formulated as a Tait-style calculus for finite sets $(\Gamma, \Lambda, \ldots)$ of $\mathcal{L}^{\mathsf{c}}_\infty$ formulas (cf. e.g. [23]). If A is an $\mathcal{L}^{\mathsf{c}}_\infty$ formula, then Γ, A is a shorthand for $\Gamma \cup \{A\}$, and similarly for expressions like Γ, A, B. T_∞ contains the following axioms and rules of inference.

I. Axioms. For all finite sets Γ of $\mathcal{L}^{\mathsf{c}}_\infty$ formulas, all closed number terms s and t with identical value, and all literals A in TRUE:

$$\Gamma,\, \neg\mathsf{U}(s),\, \mathsf{U}(t) \quad \text{and} \quad \Gamma,\, A.$$

II. Propositional rules. The usual Tait-style rules for conjunction and disjunction.

III. Number quantifier rules. For all finite sets Γ of $\mathcal{L}^{\mathsf{c}}_\infty$ formulas and all $\mathcal{L}^{\mathsf{c}}_\infty$ formulas $A(s)$:

$$\frac{\Gamma,\, A(s)}{\Gamma,\, (\exists x)A(x)}, \qquad \frac{\Gamma,\, A(t) \text{ for all closed number terms } t}{\Gamma,\, (\forall x)A(x)} \quad (\omega).$$

IV. Ordinal quantifier rules. For all finite sets Γ of $\mathcal{L}^{\mathsf{c}}_\infty$ formulas, all $\mathcal{L}^{\mathsf{c}}_\infty$ formulas $A(\alpha)$ and all ordinals β with $\alpha < \beta < \Gamma_0$:

$$\frac{\Gamma,\, A(\alpha)}{\Gamma,\, (\exists \xi < \beta)A(\xi)}, \qquad \frac{\Gamma,\, A(\gamma) \text{ for all } \gamma < \beta}{\Gamma,\, (\forall \xi < \beta)A(\xi)}.$$

V. Inductive operator rules. For all finite sets Γ of $\mathcal{L}^{\mathsf{c}}_\infty$ formulas, all inductive operator forms $\mathcal{A}(P, \vec{x})$, all closed number terms $\vec{s}$ and all ordinals $\alpha < \Gamma_0$:

$$\frac{\Gamma,\, \mathcal{A}(P^{<\alpha}_{\mathcal{A}}, \vec{s})}{\Gamma,\, P^{\alpha}_{\mathcal{A}}(\vec{s})}, \qquad \frac{\Gamma,\, \neg\mathcal{A}(P^{<\alpha}_{\mathcal{A}}, \vec{s})}{\Gamma,\, \neg P^{\alpha}_{\mathcal{A}}(\vec{s})}.$$

VI. Cut rules. For all finite sets Γ of $\mathcal{L}^{\mathsf{c}}_\infty$ formulas and all $\mathcal{L}^{\mathsf{c}}_\infty$ formulas A:

$$\frac{\Gamma,\, A \qquad \Gamma,\, \neg A}{\Gamma}.$$

The formulas A and $\neg A$ are the cut formulas of this cut; the rank of a cut is the rank of its cut formulas.

As usual, for α and ρ less than Γ_0, we write $\mathsf{T}_\infty \mathrel{\big|\!\frac{\alpha}{\rho}} \Gamma$ if Γ is provable in T_∞ by a proof of depth less than or equal to α so that all cuts in this proof have rank *less* than ρ. Further, we write $\mathsf{T}_\infty \mathrel{\big|\!\frac{<\alpha}{<\rho}} \Gamma$, if there exists $\alpha' < \alpha$ and $\rho' < \rho$ so that $\mathsf{T}_\infty \mathrel{\big|\!\frac{\alpha'}{\rho'}} \Gamma$.

It is easy to check that the assignment of ranks and the rules of inference are tailored so that the methods of predicative proof theory yield full cut elimination for T_∞. Therefore, we omit the proof of the following theorem and refer to Pohlers [20] or Schütte [22].

Theorem 7 (Cut elimination for T_∞) *We have for all finite sets Γ of $\mathcal{L}^{\mathsf{c}}_\infty$ formulas and all ordinals $\alpha, \beta, \rho < \Gamma_0$:*

$$\mathsf{T}_\infty \mathrel{\big|\!\frac{\alpha}{\beta+\omega^\rho}} \Gamma \quad \Longrightarrow \quad \mathsf{T}_\infty \mathrel{\big|\!\frac{\varphi\rho\alpha}{\beta}} \Gamma.$$

We finish this paragraph by mentioning a tautology and a persistency lemma, which we will use in the next section. The proofs proceed as usual.

Lemma 8 (Tautology lemma for T_∞) *We have for all finite sets Γ of $\mathcal{L}^{\mathsf{c}}_\infty$ formulas and all numerically equivalent $\mathcal{L}^{\mathsf{c}}_\infty$ formulas A and B:*

$$\mathsf{T}_\infty \vdash^{2\cdot rn(A)}_{0} \Gamma,\, \neg A,\, B.$$

Lemma 9 (Persistency lemma) *We have for all finite sets Γ of $\mathcal{L}^{\mathsf{c}}_\infty$ formulas, all Σ^Ω formulas $A(\vec{\xi})$ and Π^Ω formulas $B(\vec{\xi})$ of $\mathcal{L}_\Omega$ with free variables among those indicated, all ordinals $\alpha, \beta, \vec{\gamma}, \delta, \rho < \Gamma_0$ so that $\beta \leq \delta$:*

1. $\mathsf{T}_\infty \vdash^{\alpha}_{\rho} \Gamma,\, A^\beta(\vec{\gamma}) \quad \Longrightarrow \quad \mathsf{T}_\infty \vdash^{\alpha}_{\rho} \Gamma,\, A^\delta(\vec{\gamma})$.
2. $\mathsf{T}_\infty \vdash^{\alpha}_{\rho} \Gamma,\, B^\delta(\vec{\gamma}) \quad \Longrightarrow \quad \mathsf{T}_\infty \vdash^{\alpha}_{\rho} \Gamma,\, B^\beta(\vec{\gamma})$.

This finishes our description of the system T_∞. In a next step we want to give the exact upper bound computation of $\mathsf{PA}^+_\Omega + (\mathsf{Subst})$.

4.3 The proof-theoretic reduction of $\mathsf{PA}^+_\Omega + (\mathsf{Subst})$

In this section we show that $|\mathsf{PA}^+_\Omega + (\mathsf{Subst})| \leq \Gamma_0$. We first introduce an infinitary Tait-style version T of PA^+_Ω which is subsequently reduced to T_∞ via an asymmetric interpretation. Suitable iteration of this argument will finally yield the desired upper bound.

In a first step let us describe a semiformal Tait-style reformulation T of PA^+_Ω; essentially, T is PA^+_Ω where full induction on the naturals is replaced by the ω rule. Accordingly, the language $\mathcal{L}^{\mathsf{c}}_\Omega$ of T is $\mathcal{L}_\Omega$ without free number variables, and negation in $\mathcal{L}_\Omega$ is defined. We briefly address the axioms and rules of inference of T.

I. Axioms. For all finite sets Γ of $\mathcal{L}^{\mathsf{c}}_\Omega$ formulas, all numerically equivalent Σ^Ω formulas A_1 and A_2 of $\mathcal{L}^{\mathsf{c}}_\Omega$, all $\mathcal{L}^{\mathsf{c}}_\Omega$ literals B in TRUE and all linearity axioms C:

$$\Gamma,\, \neg A_1,\, A_2 \quad \text{and} \quad \Gamma,\, \sigma \neq \tau,\, \neg A_1(\sigma),\, A_2(\tau) \quad \text{and} \quad \Gamma,\, B \quad \text{and} \quad \Gamma,\, C.$$

II. Propositional and quantifier rules. These include the usual Tait-style inference rules for the propositional connectives and all sorts of quantifiers; a universal number quantifier is introduced via the ω rule.

III. Inductive operator rules. These are formulated as for T_∞, but with ordinal variables instead of constants.

IV. Σ^Ω reflection. For all finite sets Γ of $\mathcal{L}^{\mathsf{c}}_\Omega$ formulas and for all Σ^Ω formulas A:

$$\frac{\Gamma,\, A}{\Gamma,\, (\exists \xi) A^\xi}.$$

V. Σ^Ω induction on the ordinals. For all finite sets Γ of $\mathcal{L}^{\mathsf{c}}_\Omega$ formulas, all Σ^Ω formulas $A(\sigma)$ and all ordinals variables ξ which do not occur in Γ, $A(\sigma)$:

$$\frac{\Gamma,\, \neg(\forall\eta < \xi)A(\eta),\, A(\xi)}{\Gamma,\, A(\sigma)}.$$

VI. Cut rules. These are formulated in the same way as for T_∞.

The *degree* $dg(A)$ of an $\mathcal{L}^{\mathsf{c}}_\Omega$ formula A measures the complexity of A over its Σ^Ω and Π^Ω subformulas. Accordingly, $dg(A) = 0$ for A in ∇^Ω and it is computed in the usual way otherwise. In particular, $dg(A) < \omega$ for all $\mathcal{L}^{\mathsf{c}}_\Omega$ formulas A. Moreover, we have a standard derivability relation $\mathsf{T} \,|\!\frac{\alpha}{k}$ for $\alpha < \Gamma_0$ and $k < \omega$.

Since the main formulas of all non-logical axioms and rules of T are ∇^Ω, we obtain the following partial cut elimination theorem for T; here $2_k(\alpha)$ is given as usual by $2_0(\alpha) = \alpha$ and $2_{k+1}(\alpha) = 2^{2_k(\alpha)}$.

Theorem 10 (Partial cut elimination for T) *We have for all finite sets Γ of $\mathcal{L}^{\mathsf{c}}_\Omega$ formulas, all $\alpha < \Gamma_0$ and $k < \omega$:*

$$\mathsf{T} \,|\!\frac{\alpha}{k}\, \Gamma \quad\Longrightarrow\quad \mathsf{T} \,|\!\frac{2_k(\alpha)}{1}\, \Gamma.$$

The tautology lemma for T reads as usual; as a consequence, we get a substitution lemma which will be used for the reduction of $\mathsf{PA}^+_\Omega + (\mathsf{Subst})$ below.

Lemma 11 (Tautology lemma for T) *We have for all finite sets Γ of $\mathcal{L}^{\mathsf{c}}_\Omega$ formulas and all numerically equivalent $\mathcal{L}^{\mathsf{c}}_\Omega$ formulas A and B:*

$$\mathsf{T}_\infty \,|\!\frac{2\cdot dg(A)}{0}\, \Gamma,\, \neg A,\, B.$$

Lemma 12 (Substitution lemma for T) *Let $\Gamma(\mathsf{U})$ be a finite set of closed $\mathcal{L}_1$ formulas and $B(x)$ an $\mathcal{L}_\Omega$ formula so that $B(0)$ is in $\mathcal{L}^{\mathsf{c}}_\Omega$. Further assume that $\mathsf{T} \,|\!\frac{\alpha}{0}\, \Gamma(\mathsf{U})$ for some infinite ordinal $\alpha < \Gamma_0$. Then we have that $\mathsf{T} \,|\!\frac{\alpha}{0}\, \Gamma(B)$.*

In a next step we want to provide an asymmetric interpretation of the ∇^Ω fragment of T into T_∞. For that purpose, we introduce the crucial notion of a (β, α) instance. Let Γ be a finite set of $\mathcal{L}^{\mathsf{c}}_\Omega$ formulas, Λ a finite set of $\mathcal{L}^{\mathsf{c}}_\infty$ formulas and $\alpha, \beta < \Gamma_0$. Then Λ is called an (β, α) *instance* of Γ if it results from Γ by replacing

(i) each free ordinal variable by an ordinal less than β;

(ii) each universal ordinal quantifier $(\forall\xi)$ in the formulas of Γ by $(\forall\xi < \beta)$;

(iii) each existential ordinal quantifier $(\exists\xi)$ in the formulas of Γ by $(\exists\xi < \alpha)$.

We are ready to state the asymmetric interpretation theorem. For similar asymmetric interpretations, cf. Cantini [2] and Jäger [14].

Theorem 13 (Asymmetric interpretation of T into T_∞) *Assume that Γ is a finite set of ∇^Ω formulas of $\mathcal{L}_\Omega^{\mathsf{c}}$ so that $\mathsf{T} \vdash_1^\alpha \Gamma$ for some ordinal $\alpha < \Gamma_0$. Then we have for all limit ordinals $\beta < \Gamma_0$ and every $(\beta, \varphi\alpha(\beta+\beta))$ instance Λ of Γ:*

$$\mathsf{T}_\infty \vdash_{\varphi\alpha(\beta+\beta)}^{\varphi\alpha(\beta+\beta)} \Lambda.$$

Proof. The theorem is proved by induction on α. In the following let us exemplary discuss the case of Σ^Ω induction on the ordinals. We make tacitly use of Lemma 9.

Let us assume that Γ is the conclusion of Σ^Ω induction on the ordinals. Then there exists a Σ^Ω formula $A(\sigma)$ and an $\alpha_0 < \alpha$ so that

$$\mathsf{T} \vdash_1^{\alpha_0} \Gamma,\, \neg(\forall \eta < \xi) A(\eta),\, A(\xi), \tag{1}$$

for ξ a fresh variable. Now we fix a limit ordinal β and define a sequence of ordinals β_γ $(\gamma < \beta)$ which is given by

$$\beta_0 \;:=\; \varphi\alpha_0(\beta+\beta); \quad \beta_{\gamma+1} \;:=\; \varphi\alpha_0(\beta_\gamma+\beta_\gamma); \quad \beta_\lambda \;:=\; \sup_{\gamma<\lambda} \beta_\gamma, \quad (\gamma \text{ limit}).$$

One easily verifies that (i) (β_γ) is strictly increasing, (ii) β_γ is a limit, and (iii) $\beta_\gamma < \varphi\alpha(\beta+\beta)$. We want to establish by side induction on $\gamma < \beta$ that

$$\mathsf{T}_\infty \vdash_{\varphi\alpha(\beta+\beta)}^{\beta_{\gamma+1}+1} \Lambda,\, A^{\beta_{\gamma+1}}(\gamma).^2 \tag{2}$$

The claim is easily verified in the case of $\gamma = 0$. So assume that $\gamma > 0$ and (2) is true for all $\delta < \gamma < \beta$. Then we immediately obtain by persistency

$$\mathsf{T}_\infty \vdash_{\varphi\alpha(\beta+\beta)}^{\beta_\gamma+1} \Lambda,\, A^{\beta_\gamma}(\delta) \tag{3}$$

for all $\delta < \gamma$. As a consequence we have

$$\mathsf{T}_\infty \vdash_{\varphi\alpha(\beta+\beta)}^{\beta_\gamma+2} \Lambda,\, (\forall \eta < \gamma) A^{\beta_\gamma}(\eta). \tag{4}$$

Now we apply the main induction hypothesis to (1) with the pair $(\beta_\gamma, \beta_{\gamma+1})$ and obtain

$$\mathsf{T}_\infty \vdash_{\varphi\alpha(\beta+\beta)}^{\beta_{\gamma+1}} \Lambda,\, \neg(\forall \eta < \gamma) A^{\beta_\gamma}(\eta),\, A^{\beta_{\gamma+1}}(\gamma). \tag{5}$$

Now we can apply a cut to (4) and (5). One readily verifies by the properties of β_γ and Lemma 6 that the corresponding cut formula has rank less than $\varphi\alpha(\beta+\beta)$ so that we can derive (2) as desired. This finishes our side induction. Moreover, we have $\beta_{\gamma+1} + 1 < \varphi\alpha(\beta+\beta)$ and, hence, our argument is complete. □

We are now ready to put all pieces together in order to yield the upper bound Γ_0 for $\mathsf{PA}_\Omega^+ + (\mathsf{Subst})$. For that purpose, let us write $\mathsf{PA}_\Omega^+ + (\mathsf{Subst})^{\leq n}$ for the subsystem of $\mathsf{PA}_\Omega^+ + (\mathsf{Subst})$ where at most n applications of (Subst) are allowed. Moreover, let $\gamma_0 := \varepsilon_0$ and $\gamma_{n+1} := \varphi\gamma_n 0$. Then we have the following crucial theorem.

[2] To be more precise, we mean the instance of $A^{\beta_{\gamma+1}}(\gamma)$ where all free ordinal variables are replaced according to Λ.

Theorem 14 (Reduction of $\mathsf{PA}^+_\Omega + (\mathsf{Subst})$) *Let C be an $\mathcal{L}^{\mathsf{c}}_\Omega$ formula and A a closed $\mathcal{L}_1$ formula. Then we have for all natural numbers n:*

1. $\mathsf{PA}^+_\Omega + (\mathsf{Subst})^{\le n} \vdash C \quad \Longrightarrow \quad \mathsf{T} \vdash^{<\gamma_{2n}}_{1} C.$

2. $\mathsf{PA}^+_\Omega + (\mathsf{Subst})^{\le n} \vdash A \quad \Longrightarrow \quad \mathsf{T}_\infty \vdash^{<\gamma_{2n+2}}_{0} A.$

Proof. We proof 1. and 2. together by induction on n. Let us first establish the theorem for $n = 0$. In the case of 1. we have by a standard embedding that $\mathsf{T} \vdash^{<\omega+\omega}_{<\omega} C$ whenever $\mathsf{PA}^+_\Omega \vdash C$ and, hence, we get by Theorem 10 that $\mathsf{T} \vdash^{<\gamma_0}_{1} C$. For 2., assume that $\mathsf{PA}^+_\Omega \vdash A$ for an arithmetic A, which yields $\mathsf{T} \vdash^{<\gamma_0}_{1} A$ by 1. Thus we obtain by the above asymmetric interpretation theorem $\mathsf{T}_\infty \vdash^{<\gamma_1}_{<\gamma_1} A$, and by full cut elimination for T_∞ (Theorem 7) $\mathsf{T}_\infty \vdash^{<\gamma_2}_{0} A$.

Let us now establish the claims of the theorem for $n+1$ assuming that they are true for n. Again we first treat 1. and want to establish an embedding of $\mathsf{PA}^+_\Omega + (\mathsf{Subst})^{\le n+1}$ into T. Here the crucial case occurs when we derive $A(B)$ from $A(\mathsf{U})$ for A arithmetic and B arbitrary, assuming that $\mathsf{PA}^+_\Omega + (\mathsf{Subst})^{\le n} \vdash A(\mathsf{U})$. But then we know from the induction hypothesis of 2. that $\mathsf{T}_\infty \vdash^{<\gamma_{2n+2}}_{0} A(\mathsf{U})$, and hence also $\mathsf{T} \vdash^{<\gamma_{2n+2}}_{0} A(\mathsf{U})$, which by the substitution lemma (Lemma 12) yields $\mathsf{T} \vdash^{<\gamma_{2n+2}}_{0} A(B)$. All together we see that we get a (relativized) standard embedding so that $\mathsf{T} \vdash^{<\gamma_{2n+2}}_{<\omega} C$ whenever $\mathsf{PA}^+_\Omega + (\mathsf{Subst})^{\le n+1} \vdash C$, and by partial cut elimination for T this yields $\mathsf{T} \vdash^{<\gamma_{2n+2}}_{1} C$ as desired. For the verification of the induction step of 2. let $\mathsf{PA}^+_\Omega + (\mathsf{Subst})^{\le n+1} \vdash A$ for an arithmetic A. We have just shown that this yields $\mathsf{T} \vdash^{<\gamma_{2n+2}}_{1} A$, so that we obtain $\mathsf{T}_\infty \vdash^{<\gamma_{2n+3}}_{<\gamma_{2n+3}} A$ by the asymmetric interpretation theorem. Finally, we get $\mathsf{T}_\infty \vdash^{<\gamma_{2n+4}}_{0} A$ by full cut elimination for T_∞. This finishes our argument. $\Box$

As usual, lengths of cut free derivations give rise to upper bounds of the proof-theoretic ordinal (cf. [20, 22]), so that we are able to state the following corollary.

Corollary 15 $|\mathsf{PA}^+_\Omega + (\mathsf{Subst})| \le \Gamma_0$.

4.4 A remark on Π^Ω induction on the ordinals

In the following let us briefly indicate how to extend the upper bound computation for $\mathsf{PA}^+_\Omega + (\mathsf{Subst})$ in the presence of Π^Ω induction on the ordinals. We show that Π^Ω induction in the ordinals, $(\Pi^\Omega\text{-}\mathsf{I}_\Omega)$, follows from the principle $(\Pi^\Omega_2\text{-}\mathsf{Ref})$, so-called Π^Ω_2 reflection on the ordinals, and we argue that $(\Pi^\Omega_2\text{-}\mathsf{Ref})$ is apt for a proof-theoretic treatment by means of asymmetric interpretation as presented above.

By $(\Pi^\Omega_2\text{-}\mathsf{Ref})$ we mean the collection of statements

$$(\forall \vec{\xi})(\exists \vec{\eta}) A(\vec{\xi}, \vec{\eta}) \;\rightarrow\; (\forall \sigma)(\exists \tau > \sigma)(\forall \vec{\xi} < \tau)(\exists \vec{\eta} < \tau) A(\vec{\xi}, \vec{\eta})$$

for each Δ^Ω_0 formula $A(\vec{\xi}, \vec{\eta})$. The following lemma shows that indeed Π^Ω induction on the ordinals follows from Δ^Ω_0 induction on the ordinals plus $(\Pi^\Omega_2\text{-}\mathsf{Ref})$.

Lemma 16 *We have that* $\mathsf{PA}^{\mathsf{w}}_{\Omega} + (\Pi^{\Omega}_2\text{-}\mathsf{Ref})$ *proves each instance of* $(\Pi^{\Omega}\text{-}\mathsf{I}_{\Omega})$.

Proof. By $(\Sigma\text{-}\mathsf{Ref})$ we may assume that a given Π^{Ω} formulas $A(\eta)$ has the form $(\forall\sigma)B(\sigma,\eta)$ for $B(\sigma,\eta)$ a Δ^{Ω}_0 formula. In the following we work informally in the theory $\mathsf{PA}^{\mathsf{w}}_{\Omega} + (\Pi^{\Omega}_2\text{-}\mathsf{Ref})$ and assume

$$(\forall\xi)[(\forall\eta < \xi)A(\eta) \rightarrow A(\xi)], \tag{1}$$

which is spelled out as

$$(\forall\xi)[(\forall\eta < \xi)(\forall\sigma)B(\sigma,\eta) \rightarrow (\forall\sigma)B(\sigma,\xi)]. \tag{2}$$

Given arbitrary ordinals ξ_0, σ_0, we must derive $B(\xi_0, \sigma_0)$. First observe that (2) is equivalent to

$$(\forall\xi,\tau)(\exists\sigma)[(\forall\eta < \xi)B(\sigma,\eta) \rightarrow B(\tau,\xi)]. \tag{3}$$

Applying $(\Pi^{\Omega}_2\text{-}\mathsf{Ref})$ to (3) yields an ordinal $\delta > \sigma_0, \xi_0$ so that we have

$$(\forall\xi,\tau < \delta)(\exists\sigma < \delta)[(\forall\eta < \xi)B(\sigma,\eta) \rightarrow B(\tau,\xi)], \tag{4}$$

which in turn is equivalent to

$$(\forall\xi < \delta)[(\forall\eta < \xi)(\forall\sigma < \delta)B(\sigma,\eta) \rightarrow (\forall\sigma < \delta)B(\sigma,\xi)]. \tag{5}$$

Applying Δ^{Ω}_0 induction on the ordinals to (5) yields

$$(\forall\xi < \delta)(\forall\sigma < \delta)B(\sigma,\xi). \tag{6}$$

In particular, we have $B(\sigma_0, \xi_0)$ as desired. $\Box$

As we have mentioned above, $(\Pi^{\Omega}_2\text{-}\mathsf{Ref})$ enjoys a very straightforward asymmetric interpretation. Basically, Theorem 13 just carries over to the presence of $(\Pi^{\Omega}_2\text{-}\mathsf{Ref})$; the details are left to the reader. Hence, $\mathsf{PA}^{+}_{\Omega} + (\Pi^{\Omega}_2\text{-}\mathsf{Ref}) + (\mathsf{Subst})$ is not stronger than $\mathsf{PA}^{+}_{\Omega} + (\mathsf{Subst})$, so that we can state the following theorem.

Theorem 17 $|\mathsf{PA}^{+}_{\Omega} + (\Pi^{\Omega}_2\text{-}\mathsf{Ref}) + (\mathsf{Subst})| \leq \Gamma_0$.

From the previous lemma we can thus conclude:

Corollary 18 $|\mathsf{PA}^{+}_{\Omega} + (\Pi^{\Omega}\text{-}\mathsf{I}_{\Omega}) + (\mathsf{Subst})| \leq \Gamma_0$.

This finishes our short addendum concerning Π^{Ω} induction on the ordinals.

5 The upper bound of $\mathsf{AutBON}(\mu)$

In this section we establish the upper bound Γ_0 for the theory $\mathsf{AutBON}(\mu)$ by sketching an interpretation into the fixed point theory with ordinals $\mathsf{PA}^{\mathsf{w}}_{\Omega} + (\mathsf{Subst})$. As this embedding is very similar to the one given in Feferman and Jäger [9], we sketch the main lines of the argument only and indicate the modifications which arise in the presence of the bar rule.

In [9] the application operation with μ is treated by a so-called generated model construction in the framework of Peano arithmetic with ordinals. More specifically, application is modeled as a fixed point of a suitable ternary operator form $\mathcal{A}(P, x, y, z)$ so that the $\mathcal{L}$ formula $(xy \simeq z)$ is translated as $P_{\mathcal{A}}(x, y, z)$; this interpretation is lifted to a translation $(\cdot)^*$ of $\mathcal{L}$ into $\mathcal{L}_\Omega$ in a straightforward manner. For the treatment of $\mathsf{AutBON}(\mu)$ we can work with essentially the same operator form $\mathcal{A}(P, x, y, z)$ as in [9], p. 258, except for the following modifications: (i) the clauses for primitive recursion $\mathsf{r_N}$ can be omitted; (ii) the clauses for μ have to be modified in a straightforward manner in order to validate our slight strengthening of the axiomatization of μ; (iii) we have to add two clauses for the characteristic function $\mathsf{c_U}$ of U, namely:

$$x = \hat{\mathsf{c}}_{\mathsf{U}} \wedge z = 0 \wedge \mathsf{U}(y); \qquad x = \hat{\mathsf{c}}_{\mathsf{U}} \wedge z = 1 \wedge \neg\mathsf{U}(y).$$

Here $\hat{\mathsf{c}}_{\mathsf{U}}$ denotes a suitable code for $\mathsf{c_U}$. Again the presence of Δ^{Ω}_0 induction on the ordinals is crucial in order to verify that the so-obtained application operation $P_{\mathcal{A}}$ is functional in its third argument.

We have that the bar rule (BR) in $\mathcal{L}$ can easily be validated by an instance of (Subst) in $\mathcal{L}_\Omega$ and, hence, we are able to formulate the following embedding theorem.

Theorem 19 *We have for every $\mathcal{L}$ formula A:*

$$\mathsf{AutBON}(\mu) \vdash A \quad \Longrightarrow \quad \mathsf{PA}^{\mathsf{w}}_{\Omega} + (\mathsf{Subst}) \vdash A^*.$$

From Theorem 3 and Corollary 15 we are now able to derive the following proof-theoretic equivalences.

Corollary 20 *We have the following proof-theoretic equivalences:*

$$\mathsf{AutBON}(\mu) \equiv \mathsf{Aut}(\Pi^1_0) \equiv \mathsf{PA}^{\mathsf{w}}_{\Omega} + (\mathsf{Subst}) \equiv (\Pi^1_0\text{-}\mathsf{CA})_{<\Gamma_0} \equiv \mathsf{RA}_{<\Gamma_0}.$$

The proof-theoretic ordinal of all these theories is Γ_0.

We can further strengthen our applicative axioms by assuming that application is always total, (Tot), and that operations are extensional, (Ext). It is established in Jäger and Strahm [17] that the presence of (Tot) and (Ext) does not raise the proof-theoretic strength of various applicative theories including μ, and one readily verifies that these methods carry over to the present situation. Hence, the system $\mathsf{AutBON}(\mu) + (\mathsf{Tot}) + (\mathsf{Ext})$ is not stronger than $\mathsf{AutBON}(\mu)$.

Theorem 21 *We have the following proof-theoretic equivalence:*

$$\mathsf{AutBON}(\mu) + (\mathsf{Tot}) + (\mathsf{Ext}) \equiv \mathsf{AutBON}(\mu).$$

6 $\mathsf{AutBON}(\mu)$ based on primitive recursive operations

In this section we shortly address the effect of replacing the axioms for a partial combinatory algebra in BON by weaker axioms that allow an interpretation in terms of the primitive recursive indices. We sketch that the corresponding modification $\mathsf{AutPRON}(\mu)$ of $\mathsf{AutBON}(\mu)$ has the same proof-theoretic strength as $(\Pi^1_0\text{-}\mathsf{CA})+(\mathsf{BR})$ and, hence, its proof-theoretic ordinal is exactly $\varphi 20$.

Applicative systems allowing for an interpretation in the primitive recursive functions go back to Schlüter. In [21] he introduced an abstract theory of rules for enumerated classes of functions with the primitive recursive ones as a guiding example. Instead of the axioms of a partial combinatory algebra, we have axioms for a so-called *partial enumerative algebra*, where it is assumed that i, a and b are new constants of our language:[3]

$$\begin{aligned} &\mathsf{k}xy = x; \qquad \mathsf{i}x = x; \\ &\mathsf{a}(x,y)\downarrow \,\wedge\, \mathsf{a}(x,y)z \simeq (xz, yz); \\ &\mathsf{b}(x,y)\downarrow \,\wedge\, \mathsf{b}(x,y)z \simeq x(yz). \end{aligned}$$

It is shown in [21] that the axioms of a partial enumerative algebra allow one to define a careful concept of lambda abstraction as well as to prove a weak form of the recursion theorem.

The theory PRON of primitive recursive operations and numbers is now obtained from BON by replacing the axioms for a partial combinatory algebra by those for a partial enumerative algebra, and by adding an operation $\mathsf{r_N}$ which axiomatizes closure under primitive recursion, cf. [21] for details. The system $\mathsf{AutPRON}(\mu)$ extends $\mathsf{PRON}(\mu)$ by the axioms about μ, (IND) and (BR).

Let us now briefly address the proof-theoretic strength of $\mathsf{AutPRON}(\mu)$. For that purpose, it is helpful to make a few comments on how to model application in $\mathsf{PRON}(\mu)$. It is possible to obtain a standard recursion-theoretic model of $\mathsf{PRON}(\mu)$ in terms of arithmetic recursion theory. In particular, a suitable application relation capturing "primitive recursive in μ" can be defined so that the sets in the sense of $\mathcal{P}(\mathsf{N})$ with respect to this interpretation are exactly the *arithmetic* sets. This is in sharp contrast to the recursion-theoretic model of $\mathsf{BON}(\mu)$, which is based on Π^1_1 recursion theory: here $\mathcal{P}(\mathsf{N})$ coincides with the *hyperarithmetic* sets.

Formalization of the standard model of $\mathsf{PRON}(\mu)$ easily yields an embedding of $\mathsf{PRON}(\mu)$ plus full induction on the naturals into $(\Pi^1_0\text{-}\mathsf{CA})$. Further, if the model is relativized to the relation U with its characteristic function $\mathsf{c_U}$, one can establish a reduction of $\mathsf{AutPRON}(\mu)$ to $(\Pi^1_0\text{-}\mathsf{CA}) + (\mathsf{BR})$. Moreover, it is straightforward to

[3] To be precise, terms are now defined from constants by closing against pairs and application, cf. [21] for details.

show that $(\Pi^1_0\text{-}\mathsf{CA}) + (\mathsf{BR})$ is contained in $\mathsf{AutPRON}(\mu)$ via the embedding $(\cdot)^{\mathsf{N}}$. Finally, standard methods of predicative proof theory serve to determine $\varphi 20$ as the proof-theoretic ordinal of $(\Pi^1_0\text{-}\mathsf{CA}) + (\mathsf{BR})$; $\varphi 20$ is also the proof-theoretic ordinal of ramified analysis in all finite levels, $\mathsf{RA}_{<\omega}$.

Theorem 22 *We have the following proof-theoretic equivalences:*

$$\mathsf{AutPRON}(\mu) \equiv (\Pi^1_0\text{-}\mathsf{CA}) + (\mathsf{BR}) \equiv \mathsf{RA}_{<\omega}.$$

The proof-theoretic ordinal of all these theories is $\varphi 20$.

This finishes our sketchy remarks on an applicative theory with primitive recursive operations, μ operator, and bar rule.

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